The Sixth Homework

April 28, 2025

1. 1) Consider a fixed structure $\mathcal{M}$. Show that $\{\mathcal{N} \mid \mathcal{N} \equiv \mathcal{M}\}$ is an EC_Δ. (Hint: show that it is $\mathfrak{M}(\mathrm{Th}(\mathcal{M}))$.)

   2) A class $\mathfrak{A}$ of structures is **elementarily closed** if

   $$\mathcal{M} \in \mathfrak{A} \wedge \mathcal{N} \equiv \mathcal{M} \Rightarrow \mathcal{N} \in \mathfrak{A}.$$

   Show that any such class is a union of EC_Δ classes.

   3) Conversely, show that any class that is the union of EC_Δ classes is elementarily closed.

   <u>Solution</u>: Fix a language $\mathcal{L}_\mathcal{A}$,

   (a) for any structure $\mathcal{M}$, we define the **theory of** $\mathcal{M}$, written $\mathrm{Th}(\mathcal{M})$ to be the set of all sentences true in $\mathcal{M}$, i.e., $\mathrm{Th}(\mathcal{M}) = \{\varphi \in \mathcal{L}_\mathcal{A} \mid \varphi \text{ is a sentence and } \mathcal{M} \vDash \varphi\}$;

   (b) for any set of Σ of sentences, $\mathfrak{M}(\Sigma)$ denotes the class of all models of Σ, i.e., $\mathfrak{M}(\Sigma) = \{\mathcal{M} \mid \mathcal{M} \vDash \sigma \in \Sigma, \text{ for every } \sigma \in \Sigma\}$.

   Now we give the solution:

   1) We show that the set $\{\mathcal{N} \mid \mathcal{N} \equiv \mathcal{M}\}$ is the set $\{\mathcal{N} \mid \mathcal{N} \vDash \mathrm{Th}(\mathcal{M})\}$ and the conclusion follows from the latter is an EC_Δ.

   $$\begin{aligned}\mathcal{N} \equiv \mathcal{M} &\Leftrightarrow \text{ for any } \mathcal{L}_\mathcal{A}\text{-sentence } \varphi, \mathcal{M} \vDash \varphi \text{ iff } \mathcal{N} \vDash \varphi\\ &\Leftrightarrow \text{ for any } \mathcal{L}_\mathcal{A}\text{-sentence } \varphi, \mathcal{M} \vDash \varphi \text{ implies } \mathcal{N} \vDash \varphi \text{ and } \mathcal{M} \nvDash \varphi \text{ implies } \mathcal{N} \nvDash \varphi\\ &\Leftrightarrow \text{ for any } \mathcal{L}_\mathcal{A}\text{-sentence } \varphi, \mathcal{M} \vDash \varphi \text{ implies } \mathcal{N} \vDash \varphi \text{ and } \mathcal{M} \vDash (\neg\varphi) \text{ implies } \mathcal{N} \vDash (\neg\varphi)\\ &\Leftrightarrow \text{ for any } \varphi \in \mathrm{Th}(\mathcal{M}), \mathcal{N} \vDash \varphi\\ &\Leftrightarrow \mathcal{N} \in \mathfrak{M}(\mathrm{Th}(\mathcal{M})).\end{aligned}$$

   2) Since $\mathfrak{A}$ is elementarily closed, then for each $\mathcal{M} \in \mathfrak{A}$, $\{\mathcal{N} \mid \mathcal{N} \equiv \mathcal{M}\} \subseteq \mathfrak{A}$, i.e., $\mathfrak{M}(\mathrm{Th}(\mathcal{M})) \subseteq \mathfrak{A}$. Thus $\mathfrak{A} \subseteq \bigcup_{\mathcal{M}\in\mathfrak{A}} \mathfrak{M}(\mathrm{Th}(\mathcal{M})) \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is an union of EC_Δ classes.

   3) Suppose that $\mathfrak{A}$ is an union of EC_Δ classes, for any $\mathcal{M} \in \mathfrak{A}$, we can pick a set Σ of $\mathcal{L}_\mathcal{A}$-sentences, such that $\mathcal{M} \in \mathfrak{M}(\Sigma) \subseteq \mathfrak{A}$. For any $\mathcal{N}$ such that $\mathcal{N} \equiv \mathcal{M}$, we have $\mathcal{N} \vDash \Sigma$ and thus $\mathcal{N} \in \mathfrak{M}(\Sigma)$, so $\mathcal{N} \in \mathfrak{A}$. ■

2. Suppose that A is finite and that M is a finite $\mathcal{L}_A$-structure. Prove that there is an $\mathcal{L}_A$-sentence φ such that for every $\mathcal{L}_A$-structure N, if $N \vDash \varphi$ then $N \cong M$.

   <u>Solution</u>: Let $\mathfrak{C}$, $\mathfrak{F}$ and $\mathfrak{P}$ be the set of constant, function and predicate symbols in A. These set is finite since A is finite. Suppose $M = \{a_1, \ldots, a_n\}$ and I be the interpretation of symbols in M. Now we define a series of formula as follow:

(a) let $\varphi_n \equiv \bigwedge_{1\le i<j\le n}(\neg(x_i \hat{=} x_j))$;

(b) let $\psi_n = (\forall x_{n+1})(\bigvee_{i=1}^{n}(x_i \hat{=} x_{n+1}))$;

(c) for each constant symbol $c \in \mathfrak{C}$, if $I(c) = a_i$, let $\varphi_c \equiv (x_i \hat{=} c)$;

(d) for each function symbol $F \in \mathfrak{F}$, written $m = \pi(F)$, let

$$\varphi_F = \bigwedge_{((x_{i_1},\dots,x_{i_m}),x_{i_0})\in I(F)} F(x_{i_1},\dots,x_{i_m}) \hat{=} x_{i_0};$$

(e) for each predicate symbol P in $\mathfrak{P}$, written $m = \pi(P)$, let

$$\varphi_P = \bigwedge_{(x_{i_1},\dots,x_{i_m})\in I(P)} P(x_{i_1},\dots,x_{i_m}).$$

Finally, we define $\varphi \equiv (\exists x_1)\dots(\exists x_n)(\varphi_n \wedge \psi_n \wedge (\bigwedge_{c\in\mathfrak{C}}\varphi_c) \wedge (\bigwedge_{F\in\mathfrak{F}}\varphi_F) \wedge (\bigwedge_{P\in\mathfrak{P}}\varphi_P))$, it is obvious that φ is a sentence.

If $N \vDash \varphi$, then by φ_n and ψ_n, we can know that there are exactly n elements in N. Assume that $N = \{b_1,\dots,b_n\}$, ν and μ are respectively assignment of M and N such that for each $1 \le i \le n$, $\nu(x_i) = a_i$, $\mu(x_i) = b_i$. Let $e : M \to N, a_i \mapsto b_i$ is a bijection. For any constant symbol $c \in \mathfrak{C}$ such that $I(c) = a_i$, since $N \vDash \varphi$, then $b_i = c^N$ and $e(c^M) = e(I(c)) = e(a_i) = b_i = c^N$. Similarly, we can verify the other conditions of homomorphisms between structures [FL03 Definition 4.1]. Thus $N \cong M$. ■

3. Fix $A = \{P\}$, where P is a binary function symbol. For each of the following pairs of $\mathcal{L}_A$-structures, show that they are not elementarily equivalent, by giving a sentence true in one and false in the others.

   1) $(\mathbb{R}; \times)$ and $(\mathbb{R}^*; \times^*)$, where $\times$ is the usual multiplication operation on the real numbers, $\mathbb{R}^*$ is the set of the non-zero reals, and $\times^*$ is $\times$ restricted to $\mathbb{R}^*$.
   2) $(\mathbb{N}; +)$ and $(\mathbb{Z}^+; +^*)$, where $+^*$ is $+$ restricted to the set $\mathbb{Z}^*$ of positive integers.
   3) For each of the above structures, give a sentence true in that structure and false in the other.

<u>SOLUTION</u>:

1) Let φ_1 be the sentence $\exists x \forall y (x \hat{\times} y \hat{=} x)$. Then $\mathbb{R} \vDash \varphi_1$ and $\mathbb{R}^* \nvDash \varphi_1$.
2) Let φ_2 be the sentence $\exists x \forall y (x \hat{+} y \hat{=} y)$. Then $\mathbb{N} \vDash \varphi_2$ and $\mathbb{Z}^+ \nvDash \varphi_2$.
3) $\mathbb{R}^* \vDash \neg\varphi_1$ and $\mathbb{R} \nvDash \neg\varphi_1$. $\mathbb{Z}^+ \vDash \neg\varphi_2$ and $\mathbb{N} \nvDash \neg\varphi_2$. ■

4. Let $A = \varnothing$ and $\mathcal{N}$ be the $\mathcal{L}_A$-structure whose universe is $\mathbb{N}$. Show that for every infinite $S \subseteq \mathbb{N}$, the $\mathcal{L}_A$-structure with S being its universe is an elementary substructure of $\mathcal{N}$.

<u>SOLUTION</u>: we write the $\mathcal{L}_A$-structure with S being its universe as $\mathcal{S}$.

1) $\mathcal{S}$ is a substructure of $\mathcal{N}$ ($\mathcal{S} \subseteq \mathcal{N}$): since $A = \varnothing$, it follows from $S \subseteq N$.

2) $\mathcal{S}$ is an elementary substructure of $\mathcal{N}$ ($\mathcal{S} \preccurlyeq \mathcal{N}$): we show that for all $\mathcal{L}_{\mathcal{A}}$-formulas φ, and for all $\mathcal{S}$-assignment ν, $(\mathcal{S}, \nu) \vDash \varphi \Leftrightarrow (\mathcal{N}, \nu) \vDash \varphi$ by induction on the length of φ. There are no formulas of length 1, and so the conclusion is trivially true for all length 1 formulas. By Readability for Formulas, we analyze φ by considering the following various cases:

i. If φ is atomic and $\varphi \equiv (x_i \hat{=} x_j)$, $i, j \in \mathbb{N}$, then

$$(\mathcal{S}, \nu) \vDash \varphi \Leftrightarrow \nu(x_i) = \nu(x_j) \in S \Leftrightarrow (\mathcal{N}, \nu) \vDash x_i = x_j.$$

ii. If there is θ such that $\varphi \equiv (\neg\theta)$, then

$$(\mathcal{S}, \nu) \vDash \varphi \Leftrightarrow (\mathcal{S}, \nu) \nvDash \theta \Leftrightarrow (\mathcal{N}, \nu) \nvDash \theta \Leftrightarrow (\mathcal{N}, \nu) \vDash \varphi$$

iii. If there are ψ_1 and ψ_2 such that $\varphi \equiv (\psi_1 \to \psi_2)$, then

$$\begin{aligned} (\mathcal{S}, \nu) \vDash \varphi \Leftrightarrow & \text{ either } (\mathcal{S}, \nu) \nvDash \psi_1 \text{ or } (\mathcal{S}, \nu) \vDash \psi_2 \\ \Leftrightarrow & \text{ either } (\mathcal{N}, \nu) \nvDash \psi_1 \text{ or } (\mathcal{N}, \nu) \vDash \psi_2 \\ \Leftrightarrow & (\mathcal{N}, \nu) \vDash \varphi. \end{aligned}$$

iv. If there are ψ and x_i such that $\varphi \equiv (\forall x_i \psi)$ then

$$\begin{aligned} (\mathcal{S}, \nu) \vDash \varphi \Leftrightarrow & \text{ for all } \mathcal{S}\text{-assignment } \mu, \text{ if } \nu \sim_{\varphi} \mu, \text{ then } (\mathcal{S}, \mu) \vDash \psi \\ \Leftrightarrow & \text{ for all } \mathcal{S}\text{-assignment } \mu, \text{ if } \nu \sim_{\varphi} \mu, \text{ then } (\mathcal{N}, \mu) \vDash \psi \\ & (\mathcal{N}, \nu) \vDash \varphi. \end{aligned}$$

■

5. Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times)$. Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

SOLUTION: If $\mathcal{M} \subseteq \mathcal{N}$, then $M \subseteq \mathbb{N}$ and $0^{\mathcal{M}} = 0^{\mathcal{N}}, 1^{\mathcal{M}} = 1^{\mathcal{N}}, +^{\mathcal{M}} = +^{\mathcal{N}} \restriction M^2, \times^{\mathcal{M}} = \times^{\mathcal{N}} \restriction M^2$. If $n \in \mathcal{M}$, then $n+1 \in \mathcal{M}$, thus by induction, $M = \mathbb{N}$. Therefore $+^{\mathcal{M}} = +^{\mathcal{N}}$, $\times^{\mathcal{M}} = \times^{\mathcal{N}}$ and $\mathcal{M} = \mathcal{N}$. ■

6. Let $A = \{P\}$ where P is an unary predicate symbol. Let $\mathcal{M} = (M, I)$ be the finite $\mathcal{L}_{\mathcal{A}}$-structure with $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$.

- Which subsets of M are definable in M without parameters.
- Which subsets of M are definable in M with parameters.

SOLUTION:

1) Note that $f : M \to M$ is an automorphism iff $f(\{a, b\}) = \{a, b\}$ and $f(\{c, d, e\}) = \{c, d, e\}$. The definable subsets without parameters are as follows

$$\begin{aligned} \varnothing &= \{x \in M \mid \mathcal{M} \vDash \neg(x \hat{=} x)\} \\ \{a, b\} &= \{x \in M \mid \mathcal{M} \vDash P(x)\} \\ \{c, d, e\} &= \{x \in M \mid \mathcal{M} \vDash (\neg P(x))\} \\ M &= \{x \in M \mid \mathcal{M} \vDash (x \hat{=} x)\}. \end{aligned}$$

For other subset X, there must be some automorphism f s.t. $f(X) \neq X$.

2) Every subset of M are definable in M with parameters since M is finite, for example,

$$\{a, c, e\} = \{x \in M \mid \mathcal{M} \vDash (x \hat{=} a \vee x \hat{=} c \vee x \hat{=} e)\}. \qquad \blacksquare$$

7. Prove the second claim of Example 4.6.

Example 4.6. Suppose that $A = \varnothing$ and $\mathcal{M} = (M, \varnothing)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. Note that any bijection $e : M \to M$ defines an automorphism of $\mathcal{M}$. The following claims follow from the Definability Theorem. Suppose that $D \subseteq M$, then

1) D is definable in M without parameters iff $D = \varnothing$ or $D = M$.
2) D is definable in M from parameters iff D is finite or $M \setminus D$ is finite.

<u>Solution</u>:

1) Since $\varnothing = \{x \in M \mid \mathcal{M} \vDash \neg(x \hat{=} x)\}$ and $M = \{x \in M \mid \mathcal{M} \vDash (x \hat{=} x)\}$, D is definable in M without parameters. For any m_i and m_j in M, we can define a bijection $e : M \to M$ as follow

$$e(x) = \begin{cases} m_j & \text{if } x = m_i; \\ m_i & \text{if } x = m_j; \\ x & \text{otherwise.} \end{cases}$$

Thus for any nonempty proper subset D, we can find a bijection (i.e. an automorphism of $\mathcal{M}$) and it doesn't fix D. By [FL04 Theorem 4.6], we can know D isn't definable in M without parameters.

2) For any finite set $D = \{d_1, \ldots, d_n\}$, we have $D = \{x \in M \mid \mathcal{M} \vDash (x \hat{=} d_1) \vee \cdots \vee (x \hat{=} d_n)\}$ and $M \setminus D = \{x \in M \vDash \mathcal{M} \vDash (\neg(x \hat{=} d_1)) \wedge \cdots \wedge (\neg(x \hat{=} d_n))\}$, thus any finite set and its complementary set is definable in M from parameters. Converse, assume that D is definable in M from parameters, i.e., there exists a $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in M^k$ such that $D = \{x \in M \mid \mathcal{M} \vDash \varphi[x, \bar{b}]\}$, and both D and $M \setminus D$ are infinite set, then we can find $m_i \in D$, $m_j \in M \setminus D$ (with $m, n \notin \bar{b}$)[1] and a bijection $e : M \to M$ as in 1). Then e fixes $\bar{b}$ but doesn't fix D, contradict to [FL04 Theorem 4.6]. $\blacksquare$

[1] here we see $\bar{b}$ as a subset of M irregularly.