# The Fifth Homework

April 14, 2025

1. Let $\mathcal{L}_{\mathcal{A}}$ be a language with one binary relation symbol.Give an example of a sentence $\varphi \in \mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$-structures $M_1 \vDash \varphi$ and $M_2 \vDash \varphi$

   SOLUTION: Let $\varphi \equiv (\exists x)(\forall y)(\neg(y < x))$, $\mathcal{M}_1$ is $(\mathbb{N}, <)$ and $\mathcal{M}_2$ is $(\mathbb{N}, >)$. ■

2. Do there exist an alphabet $\mathcal{A}$, an $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}$, an $\mathcal{M}$-assignment $\nu$, and an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi$ such that $(\mathcal{M}, \nu) \vDash \varphi$ and $(\mathcal{M}, \nu) \vDash (\neg\varphi)$? Do there exist such $\mathcal{M}$ and $\nu$ such that $(\mathcal{M}, \nu) \nvDash \varphi$ and $(\mathcal{M}, \nu) \nvDash (\neg\varphi)$?

   SOLUTION: No for both question, since we have $(\mathcal{M}, \nu) \vDash (\neg\varphi)$ if and only if $(\mathcal{M}, \nu) \nvDash \varphi$ by definition. ■

3. Suppose that $A_1, \ldots, A_n$ are propositional symbols, that $\theta$ is a propositional tautology, and that $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}_{\mathcal{A}}$-formulas. Let $\psi$ be the result of substituting for each $i$, the formula $\varphi_i$ for each occurrence of the propositional symbol $A_i$ in $\theta$. Prove that for every $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}$ and every $\mathcal{M}$-assignment $\nu$, $(\mathcal{M}, \nu) \vDash \psi$.

   SOLUTION: The conclusion is derived from the following claim since $\theta$ is a propositional tautology. Thus we just need to prove that it holds.

   *Claim.* $(\mathcal{M}, \nu \vDash \psi)$ if and only if $\theta(\bar{A}; \overline{(M, \nu) \vDash \varphi_i})$ is true, where $\overline{(M, \nu) \vDash \varphi_i}$ is a truth assignment for $\{A_1, \ldots, A_n\}$.

   We show that it holds by induction on the length of $\theta$. If $|\theta| = 1$, then there is a $1 \leq i \leq n$ such that $\theta = \langle A_i \rangle$ and $\psi = \varphi_i$, therefore $(\mathcal{M}, \nu) \vDash \psi$ iff $(\mathcal{M}, \nu) \vDash \varphi_i$ iff $\theta(\bar{A}; \overline{(M, \nu) \vDash \varphi_i})$. Now suppose that $|\theta| = n > 1$ and the claim holds for all $m$ less than $n$. By readability, $\theta$ has two forms:

   (a) there is a $\eta$ such that $\theta = (\neg\eta)$, then

   $$\theta(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_i}) = (\neg\eta)(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_i}) = (\neg\eta(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_i}))$$

   and $(\mathcal{M}, \nu) \vDash \psi$ iff $(\mathcal{M}, \nu) \nvDash \eta(\bar{A}; \bar{\varphi})$ iff $\eta(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_1})$ is false iff $\theta(\bar{A}; \overline{(M, \nu) \vDash \varphi_i})$ is true.

   (b) there are $\eta_1$ and $\eta_2$ such that $\theta = (\eta_1 \to \eta_2)$, then

   $$\begin{aligned}
   &(\mathcal{M}, \nu) \vDash \psi \\
   \text{iff } &(\mathcal{M}, \nu) \vDash (\eta_1(\bar{A}; \bar{\varphi}) \to \eta_2(\bar{A}; \bar{\varphi})) \\
   \text{iff } &\text{ either } (\mathcal{M}, \nu) \nvDash \eta_1(\bar{A}; \bar{\varphi}) \text{ or } (\mathcal{M}, \nu) \vDash \eta_2(\bar{A}; \bar{\varphi}) \\
   \text{iff } &\text{ either } \eta_1(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_i}) \text{ is false or } \eta_2(\bar{A}; \overline{(\mathcal{M}, \nu) \vDash \varphi_i}) \text{ is true} \\
   \text{iff } &\theta(\bar{A}; \overline{(M, \nu) \vDash \varphi_i}) \text{ is true.}
   \end{aligned}$$

   ■

4. Give an example of an $A$ and an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi$ such that

   (a) $\varphi$ is a sentence,

   (b) there is at least one $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$,

   (c) and for all $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$, if $\mathcal{M} \vDash \varphi$, then the universe of $M$ is infinite.

   <ins>Solution</ins>: Let $A = \{P\}$ where $P$ is a binary predicate and $\varphi$ is an $\mathcal{L}_{\mathcal{A}}$-formula which express that $P$ is a linear order without grestest point. i.e., $\varphi$ is the conjunction of the following formulas:

   1) $(\forall x)(\forall y)(P(x, y) \to \neg P(y, x))$;

   2) $(\forall x)(\forall y)(\forall z)(P(x, y) \wedge P(y, z) \to P(x, z))$;

   3) $(\forall x)(\forall y)(P(x, y) \vee P(y, z))$;

   4) $(\forall x)(\exists y)P(x, y)$.

   It is obvious that $\varphi$ is a sentence, $(\mathbb{N}, <) \vDash \varphi$, and no finite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$. ■

5. For each of the following relations, give a formula which defines it in $(\mathbb{N}; +, \cdot)$.

   (a) $\{0\}$.

   (b) $\{1\}$.

   (c) $\{(m, n) \mid n \text{ is the successor of } m \text{ in } \mathbb{N}\}$.

   (d) $\{(m, n) \mid m < n \text{ in } \mathbb{N}\}$.

   <ins>Solution</ins>:

   (a) $(\forall y)(x + y = y)$;

   (b) $(\forall y)(x \cdot y = x)$;

   (c) $(\exists z)(x + z = y \wedge (\forall w)(z \cdot w = w))$;

   (d) $(\neg(x = y)) \wedge (\exists z)(x + z = y)$. ■