The fourth Homework

April 14, 2025

1. Prove the Unique Readability Theorem for $\mathcal{L}$ (Theorem 1.4).

   SOLUTION: Readability theorem is just [FL01-Theorem 1.3], we only prove uniqueness. We need the following claim:

   Claim. If $\varphi \in \mathcal{L}$, then no proper initial segment of φ is an element of $\mathcal{L}$.

   We ignore its proof since you can find it in the textbook [P33-Lemma 2.11]. Then we can get the conclusion by indcution on the length of formulas. The case of $|\varphi| = 1$ vacuously holds since there isn't such formula. As for the indcution step, if φ is an atomic formula, the conclusion follows from the unique readability of terms [FL01-Theorem 1.2], if φ is a compound formula, the conclusion follows from above claim. ■

2. Consider the set of sequences defined as in Definition 1.2 except that the last clause is changed to read,

   $$\text{“If } \varphi \in L \text{ and } x_i \in \mathfrak{X}, \text{ then } \forall x_i \varphi \in L\text{”}$$

   in which the parenthese are omitted. Is this set uniquely readable?

   SOLUTION: Yes, uniquely readable.

   - Readability follows from the following facts:
     - $\varphi = P_i(\tau_1 \dots \tau_n)$ — P_i
     - $\varphi = (\tau_1 \hat{=} \tau_2)$ — $(x_i, (c_i$ or $(F_i$
     - $\varphi = (\neg\psi)$ — $(\neg$
     - $\varphi = (\psi_1 \to \psi_2)$ — $(P_i, (\neg$ or $(\forall$
     - $\varphi = \forall x_i \psi$ — $\forall$
   - Then we can verify the claim in problem 5 still holds. And the change have no effect on unique readability of terms. These prove uniqueness.

3. Consider the set of sequences defined as in Definition 1.2 except that the last clause is changed to read,

   $$\text{“If } \varphi_1, \varphi_2 \in L, \text{ then } \varphi_1 \to \varphi_2 \in L\text{”}$$

   in which the parenthese are omitted, Is this set uniquely readable?

   SOLUTION: No, since $\varphi_1 \to \varphi_2 \to \varphi_3$ can be read as $(\varphi_1 \to \varphi_2) \to \varphi_3$ or $\varphi_1 \to (\varphi_2 \to \varphi_3)$.