The Third Homework

March 24, 2025

1. Find out which of the following formulas is a tautology without using Truth table.

   1). $(((A_1 \to A_1) \to A_2) \to A_2)$.

   2). $((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$.

   <u>SOLUTION</u>:

   - For 1),
     (a) By Δ_0-I axioms, we have $\varnothing \vdash (A_1 \to A_1)$;
     (b) By (a) and Inference, we have $\{((A_1 \to A_1) \to A_2)\} \vdash A_2$;
     (c) By (b) and Deduction, we have $\varnothing \vdash (((A_1 \to A_1) \to A_2) \to A_2)$;
     (d) By (c) and Soundness theorem (Version II), we have $\varnothing \vDash (((A_1 \to A_1) \to A_2) \to A_2)$, so it is a tautology.
   - For 2),
     (a) By Δ_0-IV axioms, we have $\varnothing \vdash \neg A_1 \to (A_1 \to A_2)$;
     (b) By (a) and Inference, we have $\{\neg A_1\} \vdash A_1 \to A_2$;
     (c) Let $\varphi_1 = (A_1 \to A_2)$, by Δ_0-IV axioms, we have $\varnothing \vdash \varphi_1 \to (\neg A_2 \to \neg(\varphi_1 \to A_2))$;
     (d) By (b), (c) and Inference, we have $\{\neg A_1, \neg A_2\} \vdash \neg(\varphi_1 \to A_2)$;
     (e) Let $\varphi_2 = (\varphi_1 \to A_2)$, by Δ_0-IV axioms, we have $\varnothing \vdash \neg\varphi_2 \to (\varphi_2 \to A_2)$;
     (f) By (d), (e) and Inference, we have $\{\neg A_1, \neg A_2\} \vdash \varphi_2 \to A_2$;
     (g) Let $\varphi_3 = (\varphi_2 \to A_2)$, by Δ_0-IV axioms, we have $\varnothing \vdash \varphi_3 \to (\neg A_2 \to \neg(\varphi_3 \to A_2))$;
     (h) By (f), (g) and Inference, we have $\{\neg A_1, \neg A_2\} \vdash \neg(\varphi_3 \to A_2)$, i.e,
     $$\{\neg A_1, \neg A_2\} \vdash \neg((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$$
     (i) By Soundness theorem (Version II), we have
     $$\{\neg A_1, \neg A_2\} \vDash \neg((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$$
     This implies original formula isn't a tautology. ■

2. For $\Gamma \subseteq \mathcal{L}_0$ and ψ in $\mathcal{L}_0$, show that

$$\Gamma \cup \{\varphi\} \models \psi \qquad \text{if and only if} \qquad \Gamma \models (\varphi \to \psi).$$

<u>SOLUTION</u>:

Method 1: By Soundness theorem (Version II) and Completeness theorem (Version II), it is equivalent to show that

$$\Gamma \cup \{\varphi\} \vdash \psi \qquad \text{if and only if} \qquad \Gamma \vdash (\varphi \to \psi).$$

For "$\Leftarrow$", it follows from Inference. For "$\Rightarrow$", it follows from Deduction.

Method 2: For "$\Rightarrow$", Suppose not, there exists a truth assignment ν such that $\nu \models \Gamma \cup \{\varphi\}$ and $\nu \not\models \varphi \to \psi$. Then $\nu \models \varphi$ and $\nu \not\models \psi$, contradict to the condition. For "$\Leftarrow$", Suppose that ν is any truth assignment such that $\nu \models \Gamma \cup \{\varphi\}$, then $\nu \models \varphi \to \psi$ by the condition. If $\nu \not\models \psi$, then contradict to the definition of ν. ■

3. Two physicists, A and B, and a logician C, are wearing hats, which they know are either black or white but not all white. A can see the hats of B and C; B can see the hats of A and C; C is blind. Each is asked in turn if they know the color of their own hat. The answers are: A: "No", B: "No", C: "Yes". What color is C's hat and how does C know.

<u>SOLUTION</u>: C's hat is black.

We use 0 for the white hat and 1 for the black hat. Then the table below shows all the possibilities for the color of three people's hats.

case	A	B	C	
1	0	0	0	✗
2	0	0	1	
3	0	1	0	✗
4	0	1	1	
5	1	0	0	✗
6	1	0	1	
7	1	1	0	✗
8	1	1	1	

- Since not all hats are white. so case 1 doesn't hold.
- Since A don't know the color of his own hat, so at least one of B and C has the white hat, then case 5 doesn't hold.
- Since B don't know the color of his own hat, so C's hat can't be white (or else B's hat must be black). Thus case 3 and case 7 don't hold.
- Now C can deduce that its hat is black. ■

4. Use Compactness to show that every partial order $\prec_0$ on a set X can be extended to a total order $\prec$ on X.

SOLUTION: Assign p_{ab} for each $(a, b) \in X \times X$. Consider a Σ_X such that:

1) p_{ab}, for $a, b \in X$ and $a \prec_0 b$.
2) $p_{ab} \to \neg p_{ba}$, for $a, b \in X$.
3) $p_{ab} \wedge p_{bc} \to p_{ac}$, for $a, b, c \in X$.
4) $p_{ab} \vee p_{ba}$, for $a, b \in X$ and $a \neq b$.

If an assignment $\nu \vDash \Sigma_X$, then the set $\{(a, b) \mid \nu(p_{ab}) = T\}$ is a total order on X which is an extension of $\prec_0$.

Therefore it is enough to show that Σ_X is satisfiable. By Compactness, it is enough to show that every finite $\Sigma \subset \Sigma_X$ is satisfiable.

Let $K = \{a \in X \mid \text{there exists } b \text{ s.t. } p_{ab} \text{ or } p_{ba} \in \Sigma\}$, it is a finite set since Σ is finite. Let $\Sigma_K = \Sigma_X \restriction K$, i.e., all formulas in 1) – 4) where $a, b, c \in K$. Since $\Sigma \subset \Sigma_K$, it is enough to show that Σ_K is satisfiable.

Claim. The above Σ_K is satisfiable.

Induction on $|K|$. The case $|K| = 1$ is trivial. Suppose the claim holds for all K' of size $< |K|$, and now consider K.

Pick $u \in K$. Let $K_1 = \{v \in K \mid \Sigma_K \vdash p_{vu}\}$, $K_4 = \{v \in K \vdash \Sigma_k \vdash p_{uv}\}$, $K_3 = \{u\}$ and $K_2 = k \setminus (K_1 \cup K_3 \cup K_4)$. Since $\neg p_{uu} \in \Sigma_K$, $|K_i| < |K|$, $i = 1, 2, 3, 4$. So Σ_{K_i}, $i = 1, 2, 3, 4$, are satisfiable. Suppose $v_i \vDash \Sigma_{K_i}$, $i = 1, 2, 3, 4$. Define ν as follows:

- for $i = 1, 2, 3, 4$, set $\nu(p_{ab}) = v_i(p_{ab})$ if $a, b \in K_i$;
- set $\nu(p_{ab}) = 1$ if $a \in K_i$, $b \in K_j$ and $i < j$;
- set $\nu(p_{ab}) = 0$ if $a \in K_i$, $b \in k_j$ and $i > j$.

It is easy to verify $v \vDash \Sigma_K$. ■