The First Homework

March 24, 2025

1. For which natural numbers n are there elements of $\mathcal{L}_0$ of length n? Provide detailed argument.

   SOLUTION: Exactly, for all $n \in \mathbb{N}^+ \setminus \{2, 3, 6\}$, there are elements of $\mathcal{L}_0$ of length n. We discuss the problem in three steps.

   (a) $|\varphi|$ can't be 2, 3 and 6.

   - Assume there exists $\varphi \in \mathcal{L}_0$ such that $|\varphi| = 2$, then φ is not the form $\langle A_n \rangle$. By Unique Readability Theorem, either there is $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$, which means $|\varphi| \geq 4$, or there are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$, which means $|\varphi| \geq 5$, a contradiction.
   - The same discussion follows for $|\varphi| = 3$.
   - Assume there exist $\varphi \in \mathcal{L}_0$ such tha t $|\varphi| = 6$. Also according to Unique Readability Theorem, either there is $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$, which means $|\psi| = 3$, or there are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$, which means $|\psi_1| + |\psi_2| = 3$, both are impossible.

   (b) Give examples for the case $|\varphi|$ is 1, 4 and 5.

   - let $\varphi = \langle A_1 \rangle$, then $|\varphi| = 1$.
   - let $\varphi = (\neg A_1)$, then $|\varphi| = 4$.
   - let $\varphi = (A_1 \to A_2)$, then $|\varphi| = 5$.

   (c) Prove $|\varphi|$ can be any natural numbers of length ≥ 7, i.e. length in set

   $$\{\ 3k + i \mid k \in \mathbb{N}, k \geq 2, i = 1, 2, 3\ \}.$$

   Prove by indcution on k:

   - Base case ($k = 2$): pick $\psi \in \mathcal{L}_0$ with $|\psi| = 4$ (or 5) and let $\varphi = (\neg\psi)$ for $i = 1$ (or 2), pick $\psi_1, \psi_2 \in \mathcal{L}_0$ with $|\psi_1| = 1, |\psi_2| = 5$ and let $\varphi = (\psi_1 \to \psi_2)$ for $i = 3$.
   - Inductive Step: assume the conclusion holds for some $k \in \mathbb{N}$ with $k \geq 2$, i.e., for any $i =$ 1, 2, 3, there is $\varphi' \in \mathcal{L}_0$ such that $|\varphi'| = 3k + i$, then let $\varphi = (\neg\varphi')$ and it witness the conclusion holds for $k + 1$ since $|\varphi| = 3k + i + 3 = 3(k + 1) + i$. ■

2. Show that a sequence φ is an element of $\mathcal{L}_0$ if and only if there is a finite sequence of sequences $\langle \varphi_1, \ldots, \varphi_n \rangle$ such that $\varphi_n = \varphi$, and for each $i \leq n$,

   - either there is an m such that $\varphi_1 = \langle A_m \rangle$,
   - or there is a $j < i$ such that $\varphi_i = (\neg\varphi_j)$,

- or there are $j_1, j_2 < i$ such that $\varphi_i = (\varphi_{j_1} \to \varphi_{j_2})$.

<u>SOLUTION</u>:

- " $\Rightarrow$ ": Build the corresponding finite sequence by induction on the construction of formulas via the Unique Readability. That is to say, for arbitrary $\varphi \in \mathcal{L}_0$, suppose conclusion holds for its all proper subformulas.
  - If $\varphi = \langle A_n \rangle$ for some $n \in \mathbb{N}$, the sequence $\langle \varphi \rangle$ work.
  - If there is $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$, let $\langle \psi_1, \ldots, \psi_k \rangle$ be the sequence which corresponds to ψ, then $\langle \psi_1, \ldots \psi_k, \varphi \rangle$ works for φ.
  - If there are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$, let $\langle \psi_{i,1}, \ldots, \psi_{i,k_i} \rangle$ be the sequence which corresponds to ψ_i, $i = 0, 1$, respectively. Then $\langle \psi_{1,1} \ldots, \psi_{1,k_1}, \psi_{2,1}, \ldots, \psi_{2,k_2}, \varphi \rangle$ works for φ.
- " $\Leftarrow$ ": Assume the sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$, in face, we can show that $\varphi_i \in \mathcal{L}_0$ for each $i \leq n$ by inducton.
  - Base Case (i = 1): by the property of $\langle \varphi_1, \ldots, \varphi_n \rangle$, there is an m such that $\varphi_1 = \langle A_m \rangle$, so $\varphi_1 \in \mathcal{L}_0$.
  - Indcutive Step: Assume $\varphi_i \in \mathcal{L}_0$ for some $1 \leq k < n$, consider φ_{k+1},
    - there is an l such that $\varphi_{k+1} = \langle A_l \rangle$, same as base case.
    - there is a $j < k+1$ such that $\varphi_{k+1} = (\neg\varphi_j)$, $\varphi_{k+1} \in \mathcal{L}_0$ by definition of $\mathcal{L}_0$, completely similar for the case of there are $j_1, j_2 < k + 1$ such that $\varphi_{k+1} = (\varphi_{j_1} \to \varphi_{j_2})$. ■