Equivalent versions of three theorems in the first order logic

Soundness（可靠性定理）:

Theorem 0.1. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

(2) Any satisfiable set of formulas is consistent.

Proof. (1) $\Rightarrow$ (2). Let Γ be a set of formulas satisfied by a model $\mathcal{M}$ and an assignment $s : V \to |\mathcal{M}|$. We want to show that Γ is consistent. Suppose NOT. Then there is a formula φ such that

$$\Gamma \vdash \varphi \quad \text{and} \quad \Gamma \vdash \neg\varphi.$$

By (1), we have

$$\Gamma \models \varphi \quad \text{and} \quad \Gamma \models \neg\varphi.$$

Hence,

$$(\mathcal{M}, s) \models \varphi \quad \text{and} \quad (\mathcal{M}, s) \models \neg\varphi.$$

But this is impossible! Hence Γ must be consistent.

(2) $\Rightarrow$ (1). Suppose $\Gamma \vdash \varphi$, we want to show that $\Gamma \models \varphi$. We prove by contradiction. Suppose there exist a model $\mathcal{M}$ and an assignment $s : V \to |\mathcal{M}|$ such that

- $(\mathcal{M}, s) \models \gamma$, for every $\gamma \in \Gamma$; and
- $(\mathcal{M}, s) \not\models \varphi$, i.e., $(\mathcal{M}, s) \vDash \neg\varphi$.

This means that $(\mathcal{M}, s)$ witnesses that $\Gamma \cup \{\neg\varphi\}$ is satisfiable. By (2), $\Gamma \cup \{\neg\varphi\}$ is consistent. But this contradicts the assumption $\Gamma \vdash \varphi$. So it must be that $\Gamma \models \varphi$. □

Completeness （完全性定理）:

Theorem 0.2. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

(2) Any consistent set of formulas is satisfiable.

Proof. (1) $\Rightarrow$ (2). Let Γ be a consistent set of formulas. Note that $\Gamma \models \varphi$ means that for any model $\mathcal{M}$ and any assignment $s : V \to |\mathcal{M}|$, if $(\mathcal{M}, s)$ satisfies every formula in Γ, then $(\mathcal{M}, s)$ satisfies φ as well. If Γ is unsatisfiable, since Γ is satisfied by no models and assignments, $\Gamma \models \varphi$ holds vacuously for any formula φ, in particular for $\varphi \equiv \neg(x \hat{=} x)$. But then by (1), $\Gamma \vdash \neg(x \hat{=} x)$, contradicting the assumption that Γ is consistent.

(2) $\Rightarrow$ (1). Suppose $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent. By (2), $\Gamma \cup \{\neg\varphi\}$ is satisfied by some model $\mathcal{M}$ and some assignment $s : V \to |\mathcal{M}|$. In particular, $(\mathcal{M}, s) \models \neg\varphi$. But from $\Gamma \models \varphi$, we have $(\mathcal{M}, s) \models \varphi$. Contradiction! So if $\Gamma \models \varphi$, it must be that $\Gamma \vdash \varphi$. □

Compactness （紧致性定理）:

Theorem 0.3. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \models \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \models \varphi$.

(2) For any set of formula Γ, if every finite subset Γ_0 of Γ is satisfiable, then Γ is satisfiable.

Proof. (1) $\Rightarrow$ (2). Suppose that every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable. Consider the formula $\varphi \equiv \neg(x \mathrel{\hat{=}} x)$. If Γ is unsatisfiable, then $\Gamma \models \varphi$, since Γ is satisfied by no models and assignments. By (1), for for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \models \varphi$. As φ is false, there is no model and assignment that can satisfy Γ_0, contradicting the assumption that every finite subset of Γ is satisfiable.

(2) $\Rightarrow$ (1). Suppose $\Gamma \models \varphi$. Assume that for any finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \models \varphi$ fails. This means that for each Γ_0, there exists $(\mathcal{M}_0, s_0)$ such that

$$(\mathcal{M}_0, s) \models \gamma, \text{ for every } \gamma \in \Gamma_0 \cup \{\neg\varphi\}.$$

In other word, every subset of $\Gamma \cup \{\neg\varphi\}$ of the form $\Gamma_0 \cup \{\neg\varphi\}$, $\Gamma_0 \subset \Gamma$ finite, is satisfiable. It follows immediately that every finite subsets of $\Gamma \cup \{\neg\varphi\}$ is satisfiable. By (2), $\Gamma \cup \{\neg\varphi\}$ is satisfiable, contradicting that $\Gamma \models \varphi$. This proves (2) $\Rightarrow$ (1). $\square$