Mathematical Logic

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Compactness

Next

The Compactness Theorem

Theorem 6.1 (Compactness, Version I)

Suppose that $\Gamma \subseteq \mathcal{L}$. *If for every finite* $\Gamma_0 \subseteq \Gamma$, Γ_0 *is satisfiable, then* Γ *is satisfiable.*

Theorem 6.2 (Compactness, Version II)

Suppose that $\Gamma \subseteq \mathcal{L}$ *and* $\varphi \in \mathcal{L}$. *If* $\Gamma \vDash \varphi$, *then there exists some finite* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \vDash \varphi$.

Proof of the Compactness Theorem

Proof of Verion I.

Suppose Γ is not satisfiable, by Completeness, $\Gamma \vdash \neg(x_1 \hat{=} x_1)$. The deduction of $\neg(x_1 \hat{=} x_1)$ from Γ uses only finitely many formulas in Γ. Let Γ_0 be such finite subset of Γ. Then $\Gamma_0 \vdash \neg(x_1 \hat{=} x_1)$. By Completeness again, Γ_0 is not satisfiable. Contradiction! □

Remark

The above proof of Compactness Theorem uses Gödel's Completeness Theorem. It can also be proved by using the method of ultraproduct.

Theory

Definition 6.1

- A **theory** is a set of sentences that is consistent and closed under logical implication.
- A theory T is **complete** iff for every sentences φ in the language of T, either $\varphi \in T$ or $\neg\varphi \in T$.

Definition 6.2

- Suppose that T_0, T_1 are two theories, $T_0 \subseteq T_1$ and T_1 is complete, T_1 is called a **completion** of T_0.
- Suppose $\mathcal{M}$ is a $\mathcal{L}_A$-structure. Then

$$\mathsf{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi,\ \varphi \text{ is a } \mathcal{L}_A\text{-sentence}\}$$

  is a complete theory in $\mathcal{L}_A$.

Theorem 6.3 (Overflow/Overspill)

Suppose that Γ *is a theory such that for every* $n \in \mathbb{N}$, Γ *has a model of size* $\geqslant n$. *Then* Γ *has an infinite model.*[a]

[a] One can replace "model" by "Σ-model" for any Σ.

Proof.

- Consider $\Delta = \Gamma \cup \{\neg(x_i \hat{=} x_j) \mid i < j,\ i, j \in \mathbb{N}\}$.
- Let Δ_0 be a (any) finite subset of Δ. Pick n such that
$$\Delta_0 \subseteq \Gamma \cup \{\neg(x_i = x_j) \mid i < j < n\}$$
- Let $\mathcal{M}_0 = \{m_i \mid i < k\}$ be a model of Γ with $k \geqslant n$.
- Take $\nu_0 : \mathfrak{X} \to M_0$ such that $\nu_0(x_i) = m_i$, $i < n$. Then $(\mathcal{M}_0, \nu_0) \models \Delta_0$.
- By Compactness, there are $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \models \Delta$. This $\mathcal{M}$ is necessarily an infinite model, as $\operatorname{ran}(\nu) \subset M$ and $\nu(x_i) \neq \nu(x_j)$ whenever $i \neq j$. □

Underflow

Corollary 6.4 (Underflow/Underspill)

Suppose Σ is a set of $\mathcal{L}$-sentences. Suppose φ is an $\mathcal{L}$-sentence satisfied by all infinite Σ-models. Then there is an $n \in \mathbb{N}$ such that φ is satisfied by every Σ-model of size $\geqslant n$.

Proof.

- Assume towards a contradiction that for every $n \in \mathbb{N}$, $\Gamma = \{\neg\varphi\}$ has a Σ-model of size $\geqslant n$.
- Then by Overflow, $\Gamma = \{\neg\varphi\}$ is satisfied by an infinite Σ-model $\mathcal{M}$.
- But then, $\mathcal{M}$ does not satisfy φ. Contradiction! □

Elementary classes of models

Corollary 6.5

Let Σ be a set of sentences and T be a theory in $\mathcal{L}_A$. Suppose that for any $n \in \mathbb{N}$, T has a finite Σ-model of size $\geqslant n$. Then

- *the class of all finite Σ-models of T is not $\mathbf{EC}_\Delta$.*
- *the class of all infinite Σ-models of T is not $\mathbf{EC}$.*

Proof.

- Suppose NOT, i.e. $\mathfrak{M}(\Gamma)$ = the class of all finite $(T \cup \Sigma)$-models. By Overflow, $T \cup \Sigma$ has an infinite model. But then the class of $(T \cup \Sigma)$-models contains a non-finite model.
- Suppose NOT, i.e. $\mathfrak{M}(\{\varphi\})$ = the class of all infinite $(T \cup \Sigma)$-models. By Underflow, there is an $n \in \mathbb{N}$ such that $\{\varphi\}$ is satisfied by every $(T \cup \Sigma)$-model of size $\geqslant n$. Thus $\mathfrak{M}(\{\varphi\})$ contains a finite $(T \cup \Sigma)$-model. Contradiction! □

Wellordered sets

Definition 6.3

Suppose that $\prec$ is a total ordering on a set X. Then $\prec$ is a **wellorder** (a **wellfounded** total order) of X iff there is no infinite $\prec$-descending chain, i.e. no sequence $\langle a_n : n \in \mathbb{N} \rangle$ s.t. $a_{n+1} \prec a_n$ for every n.

Theorem 6.6

Suppose that Γ is a theory in the language $\mathcal{L}_{\{<\}}$ and Γ is satisfied by an infinite total order. Then Γ is satisfied by a total order which is not a wellorder.

This implies that the class of well orders is not $\mathbf{EC}_\Delta$.

Proof.

- Consider
$$\Delta = \Gamma \cup \{x_j \prec x_i \mid i < j \in \mathbb{N}\} \cup \mathrm{Total}\,.$$
- Suppose $\Delta_0 \subseteq \Delta$ is finite. Let $\mathcal{M}_0$ be the infinite total order assumed in the hypothesis. Let ν_0 be such that for each x_i that appears freely in Δ_0, the values of $\nu(x_i)$'s fit the configuration specified by Δ_0.
- Then $(\mathcal{M}_0, \nu_0) \models \Delta_0$.
- By Compactness, there is a pair $(\mathcal{M}, \nu) \models \Delta$. Then $\{\nu(x_i) \mid i \in \mathbb{N}\}$ gives an infinite $\prec$-decreasing sequence in $\mathcal{M}$, thus $\mathcal{M}$ is total but not wellfounded. □

Comment on Language

Our proof of Compactness (Completeness, essentially) uses **Axiom of Countable Choice**. Assuming full **AC**, Compactness is applicable to languages of any size, and the use of Compactness is more flexible.

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Theorem 6.7 (Upward Löwenheim-Skolem)

Suppose $T \subset \mathcal{L}_A$ has an infinite model $\mathcal{M}$. Then for every set X, there is a model $\mathcal{M}_X = (M_X, I_X)$ of T s.t. $M \cup X \subset M_X$ and $\mathcal{M} \preccurlyeq \mathcal{M}_X$, moreover, $|M_X| = \max(|M \cup X|, |\mathcal{L}_A|)$.

Hint: Expand $\mathcal{L}_A$ to $\mathcal{L}_M = \mathcal{L}_A \cup \{c_m \mid m \in M\}$, and expand $\mathfrak{X}$ to $\mathfrak{X}^* = \mathfrak{X} \cup \{t_x \mid x \in X\}$ (t_x are new variables). Consider

$$\Gamma = \mathrm{Th}_{\mathcal{L}_M}(\mathcal{M}) \cup \{\neg(t_x = t_{x'}) \mid x \neq x'\} \cup \{t_x = c_x \mid x \in M \cap X\}$$

The existence of nonstandard model

Exercise 6.1

1. Suppose $\mathcal{M}$ is an $\mathcal{L}_A$-structure. Let $A^* = A \cup \{c_m \mid m \in M\}$. Identify $\mathcal{M}$ as an $\mathcal{L}_{A*}$-structure. Show that for any $\mathcal{L}_{A*}$-structure $\mathcal{N}$, if $\mathcal{N} \models \mathrm{Th}_{\mathcal{L}_{A*}(\mathcal{M})}$, then $\mathcal{M} \preccurlyeq \mathcal{N}$.
2. Suppose that $\mathcal{M}$ is an infinite $\mathcal{L}_A$-structure. Show that there is an $\mathcal{M}_1$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are elementarily equivalent and $\mathcal{M}_1$ has an element which is not the interpretation of any constant symbol.

HINT (for (2)): Apply Compactness to

$$\Gamma = \mathrm{Th}_{\mathcal{L}_A}(\mathcal{M}) \cup \{\neg(x_1 \hat{=} c) \mid c \in \mathcal{C} \cap A\}.$$

Number Theory

Let $A = \{\hat{0}, \hat{S}, \hat{+}, \hat{\cdot}\}$. **Peano arithmetic**, (**PA**) has the following list of axioms:

1. $\hat{0} \mathrel{\hat{\neq}} \hat{S}x$
2. $\hat{S}(x) \mathrel{\hat{=}} \hat{S}(y) \rightarrow x \mathrel{\hat{=}} y$
3. $x \mathbin{\hat{+}} \hat{0} \mathrel{\hat{=}} x$
4. $x \mathbin{\hat{+}} \hat{S}(y) \mathrel{\hat{=}} \hat{S}(x \mathbin{\hat{+}} y)$
5. $x \mathbin{\hat{\cdot}} \hat{0} \mathrel{\hat{=}} \hat{0}$
6. $x \mathbin{\hat{\cdot}} \hat{S}(y) \mathrel{\hat{=}} (x \mathbin{\hat{\cdot}} y) \mathbin{\hat{+}} x$

And finally for each formula $\varphi(v, \bar{x}) \in \mathcal{L}_A$,

7_φ. $\varphi(\hat{0}, \bar{x}) \wedge \forall v\,(\varphi(v, \bar{x}) \to \varphi(\hat{S}(v), \bar{x})) \to \forall v\,\varphi(v, \bar{x})$

The whole list of axioms 7_φ, one for each φ, is called the **axiom schema of induction**.

Peano Arithmetic is often referred as the axiom system of Number theory.

Models of Number Theory

- The **standard model** of Number Theory is
$$\mathcal{N} = (\mathbb{N}; 0, S, +, \cdot).$$
- **Complete number theory** is the set $\text{Th}(\mathcal{N})$.
- $\mathcal{N}$ is not the only model of $\mathrm{Th}(\mathcal{N})$. (see next slide)
- All other (non-isomorphic) models are called **nonstandard**.
- $\mathcal{N} \models$ PA, thus $\text{Th}(\mathcal{N})$ is a completion of PA.

Theorem 6.8

Nonstandard models of Th($\mathcal{N}$) *exists.*

This is an instance of Exercise 6.1.

Proof.

Consider $\Gamma = \text{Th}(\mathcal{N}) \cup \{\neg(x_0 \hat{=} \hat{S}^n(0)) \mid n \in \mathbb{N}\}$. Apply Compactness to show that Γ is consistent. □

Nonstandard models of Arithmetics

We shall discuss the nonstandard models of the theory of the following reduced structures of number theory:

$$\mathcal{N}_S = (\mathbb{N}; 0, S)$$
$$\mathcal{N}_L = (\mathbb{N}; 0, S, <)$$
$$\mathcal{N}_A = (\mathbb{N}; 0, S, <, +)$$
$$\mathcal{N}_M = (\mathbb{N}; 0, S, <, +, \cdot)$$

$\mathcal{N}_S = (\mathbb{N}; 0, S)$

Th($\mathcal{N}_S$) can be axiomatized as follows, denoted as A_S,

$S1.$ $\forall x \hat{S}x \neq \hat{0}$

$S2.$ $\forall x \forall y\, (\hat{S}x = \hat{S}y \to x = y)$

$S3.$ $\forall y (y \neq \hat{0} \to \exists x (y = \hat{S}x))$

$S4\text{-}n.$ (for each $n \in \mathbb{N} - \{0\}$). $\forall x (\hat{S}^n x \neq 0)$.

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$S3.$ $\forall y (y \neq \hat{0} \to \exists x (y = \hat{S}x))$

$S4\text{-}n.$ (for each $n \in \mathbb{N} - \{0\}$). $\forall x (\hat{S}^n x \neq 0)$.

- A model of Th($\mathcal{N}_S$) consists of a standard part, $\mathbb{N}$, plus a certain number of "$\mathbb{Z}$-chains". (no order!)
- Any two models of Th($\mathcal{N}_S$) with the same number of "$\mathbb{Z}$-chains" are isomorphic.

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$S2.$ $\forall x \forall y\, (\hat{S}x = \hat{S}y \to x = y)$

$S3.$ $\forall y (y \neq \hat{0} \to \exists x (y = \hat{S}x))$

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Question

How many countable models of Th($\mathcal{N}_S$) are there?

$\mathcal{N}_L = (\mathbb{N}; 0, S, <)$

Th$(\mathcal{N}_L)$ can be axiomatized as follows, denoted as A_L,

$S3.$ $\forall y(y \neq \hat{0} \to \exists x(y = \hat{S}x))$

$L1.$ $\forall x \forall y\,(x \mathrel{\hat{<}} \hat{S}y \Leftrightarrow x \mathrel{\hat{\leqslant}} y)$

$L2.$ $\forall x\, \neg(x \mathrel{\hat{<}} \hat{0})$

$L3.$ $\forall x \forall y\,(x \mathrel{\hat{<}} y \vee x \mathrel{\hat{=}} y \vee y \mathrel{\hat{<}} x)$

$L4.$ $\forall x \forall y\,(x \mathrel{\hat{<}} y \to \neg(y \mathrel{\hat{<}} x))$

$L5.$ $\forall x \forall y \forall z\,(x \mathrel{\hat{<}} y \wedge y \mathrel{\hat{<}} z \to x \mathrel{\hat{<}} z)$

- A model of Th$(\mathcal{N}_L)$ consists of a standard part, $\mathbb{N}$, followed a certain number of linearly ordered "$\mathbb{Z}$-chains" (with order).

- Consider $\mathcal{N}_A = (\mathbb{N}; 0, S, <, +)$ and $\mathcal{N}_M = (\mathbb{N}; 0, S, <, +, \cdot)$
- Note that the relation $<$ on $\mathbb{N}$ can be defined from $\{0, S, +\}$.
- Nonstandard models of Th($\mathcal{N}_A$) and Th($\mathcal{N}_M$) are also models of Th($\mathcal{N}_\mathcal{L}$), but further require the "ordertype" of the $\mathbb{Z}$-chains to be a "dense linear order without endpoints".

Question

1. What are the (countable) models of Th($\mathcal{N}_L$)?
2. How about Th($\mathcal{N}_A$), Th($\mathcal{N}_M$)?

Nonstandard models of analysis

We use a first-order language with uncountable symbols. In addition to $+, \cdot, <$, we include

- $\hat{c}_r$, for every $r \in \mathbb{R}$;
- $\hat{F}_f$, for every n-ary function over $\mathbb{R}$;
- $\hat{P}_p$, for every n-ary relation over $\mathbb{R}$.

For this language, we have a standard structure with: $\mathfrak{R} = (\mathbb{R}, I)$, where $I(\hat{c}_r) = r$, $I(\hat{F}_f) = f$, $I(\hat{P}_p) = p$.

Theorem 6.9

$\mathrm{Th}(\mathfrak{R})$ *has a nonstandard model.*

This is another instance of Exercise 6.1.

Proof.

Consider $\Sigma = \mathrm{Th}(\mathfrak{R}) \cup \{\hat{0} < x_1 \wedge x_1 < \hat{c}_r \mid r \in \mathbb{R}^+\}$. Apply Compactness to show that Σ is consistent. [a] □

[a](Abraham Robinson, 1961) The interpretation of x_1 is a infinitesimal. It gives the intuition of Leibniz's dx.

Since the language of $\mathfrak{R}$ contains $\{c_r \mid r \in \mathbb{R}\}$, the model of Σ is in fact an elementary extension of $\mathfrak{R}$.

Let $\mathfrak{R}^*$ be a nonstandard model of Th($\mathfrak{R}$). There is a natural elementary embedding of $\mathfrak{R}$ into $\mathfrak{R}^*$:

$$h(r) = I^*(\hat{c}_r), \quad \text{for each } r \in \mathbb{R}.$$

So we may view $\mathfrak{R} \preccurlyeq \mathfrak{R}^*$ as an elementary substructure. We call elements of $\mathbb{R}^*$ **hyper-reals**, those in $\mathbb{R}$ **standard reals**.

Let $\mathbb{R}^*$ denote the universe of $\mathfrak{R}^*$, and f^*, p^* abbreviates $I^*(\hat{F}_f), I^*(\hat{P}_p)$ respectively. In general, for any definable object A in $\mathfrak{R}$, A^* denotes the corresponding object in $\mathfrak{R}^*$.

1st-order Properties of $\mathfrak{R}^*$

- The binary relation $<^*$ on $\mathbb{R}^*$ is a linear order.

  For instance, the property "transitivity" can be expressed by the sentence:

  $$\forall x \forall y \forall z \, (x \mathrel{\hat{<}} y \wedge y \mathrel{\hat{<}} z \to x \mathrel{\hat{<}} z)$$

- The binary operation $+^*$ on $\mathbb{R}^*$ is commutative.
  … *Applying to each of the field axioms, we conclude that*

  $$(\mathbb{R}^*; 0, 1, +^*, \cdot^*) \text{ is a field.}$$

- $|a +^* b|^* \leqslant^* |a|^* +^* |b|^*$, for all $a, b \in \mathbb{R}^*$.

Properties of $\mathfrak{R}$ that $\text{C}\textsc{annot}$ be expressed in the first order language are likely to fail in $\mathfrak{R}^*$.

- There is an $b \in \mathbb{R}^* - \mathbb{R}$ that is infinitely large, i.e., $r <^* b$ for all $r \in \mathbb{R}$. And $1/^*b$ is a sample **infinitesimal**.

- The least-upper-bound property fails in $\mathfrak{R}^*$.

  $\mathbb{R}$ is bounded by an infinitely large hyper-real, but $\mathbb{R}$ has no least upper bound.

- If $A \subseteq \mathbb{R}$ is unbounded, then A^* contains infinite hyper-reals. (e.g. $\mathbb{N}^*, \mathbb{Q}^*$)

Definition 6.4

Define the set $\mathcal{F}$ of **finite** elements by

$$\mathcal{F} = \{x \in \mathbb{R}^* \mid |x|^* <^* y, \text{ for some } y \in \mathbb{R}^+\}.$$

Infinitesimal are elements of the following set:

$$\mathcal{I} = \{x \in \mathbb{R}^* \mid |x|^* <^* y, \text{ for all } y \in \mathbb{R}^+\}$$

We say x is **infinitely close to** to y, $x \simeq y$, iff $x -^* y \in \mathcal{I}$.

Properties

- $\mathbb{R}, \mathcal{I} \subsetneq \mathcal{F}$,
- $\mathbb{R} \cap \mathcal{I} = \{0\}$, and
- $\simeq$ is an equivalence relation.

Properties of $\mathcal{F}$

Exercise (Not assigned)

Show that

1. $\mathcal{F}$ is a subring, i.e. closed under $+^*, -^*, \times^*$.
2. $\mathcal{I}$ is an ideal, i.e. closed under $+^*, -^*$, and $\times^*$ from $\mathcal{F}$.
3. $\simeq$ is an equivalence relation on $\mathbb{R}^*$ respecting $+^*, -^*, \times^*$.

Thus $(\mathcal{F}/_{\simeq}, +^*/_{\simeq}, \times^*/_{\simeq}, \cdots) \cong (\mathbb{R}, +, \times, \cdots)$.

Standard part

- Every $x \in \mathcal{F}$ is infinitely close to a unique $r \in \mathbb{R}$, the **standard part** of x, $\mathrm{st}(x)$. Thus x has a unique decomposition: $x = \mathrm{st}(x) + i$, where i is infinitesimal.
- If $x \not\approx y$ and at least one finite, then there is a standard q strictly between x and y.

Properties of $\mathrm{st}(\cdot)$

1. $\mathrm{st} : \mathcal{F} \to \mathbb{R}$ is surjective.
2. $\mathrm{st}(x) = 0$ iff x is infinitestimal.
3. $\mathrm{st}(x +^* y) = \mathrm{st}(x) + \mathrm{st}(y)$.
4. $\mathrm{st}(x \times^* y) = \mathrm{st}(x) \times \mathrm{st}(y)$.

Thus $\mathrm{st} : \mathcal{F} \to \mathbb{R}$ is an epimorphism with kernel $\mathcal{I}$. Consequently, the quotient ring $\mathcal{F}/\mathcal{I} \cong \mathbb{R}$ (the real field).

Infinitestimal and (ε, δ)-definitions

1. **Definition.** *Let* $f : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$. *Then*

$$\lim_{x \to a} f(x) = b \quad \textit{iff} \quad x \simeq a \implies f^*(x) \simeq b.$$

2. **Standard definition** of "limit": For all standard $\varepsilon > 0$,

$$\varphi(x, \varepsilon) \equiv \exists \delta > 0\,(0 \neq |x - a| < \delta \to |f(x) - b| < \varepsilon).$$

   - 2 $\Rightarrow$ 1. If $\mathfrak{R} \models \forall x \forall \varepsilon\, \varphi(x, \varepsilon)$, then $\mathfrak{R}^* \models \forall x \forall \varepsilon\, \varphi(x, \varepsilon)$. For any $x \in \mathbb{R}^*$ such that $x \simeq a$, $|x -^* a| < \delta$, so $|f^*(x) -^* b| < \varepsilon$, for any $\varepsilon > 0$. Thus $f^*(x) \simeq b$.
   - 1 $\Rightarrow$ 2. Suppose f satisfies 1. Suppose $x \in \mathbb{R}$, $\varepsilon \in \mathbb{R}^+$. Then $\mathfrak{R}^* \models \varphi(x, \varepsilon)$, since we can take δ to be infinitesimal. By elementarity, $\mathfrak{R} \models \varphi(x, \varepsilon)$.

From the definition of “limit”, one can define “continuity”, ”derivative”, ”integral”, etc.

- f **is continuous at** a iff $x \simeq a \implies f^*(x) \simeq f(a)$.
- $f'(a) = b$ iff for every $0 \neq dx \in \mathcal{I}$, $\frac{df}{dx} \simeq b$, where $df = f^*(a + dx) - f(a)$.

Theorem

Show that in $\mathbb{R}$, $f'(a)$ exists $\Rightarrow$ f is continuous at a.

Nonstandard proof. Since

$$\frac{f^*(a + dx) - f(a)}{dx} \simeq f'(a).$$

$f'(a)$ is finite, so $f'(a) \cdot dx \in \mathcal{I}$. Thus $f^*(a + dx) \simeq f(a)$.

Let $F(x) = x^2$. Then
$F'(x) = \frac{dF}{dx} = \frac{(a+dx)^2 - a^2}{dx} = \frac{2a(dx)+(dx)^2}{dx} = 2a + dx \simeq 2a$.