Mathematical Logic

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Next

1. Propositional Logic
2. First order Logic
   - $\mathcal{L}$-formula
   - Semantics
   - Definability
   - Homomorphism
   - Proof system

Validity

Definition 5.1

A **validity** is an $\mathcal{L}$-formula φ such that for all $(\mathcal{M}, \nu)$,

$$(\mathcal{M}, \nu) \models \varphi.$$

- Validities are truths hold in every pair of $(\mathcal{M}, \nu)$. They provide no particular information about any structure. From this aspect, a single validity is not worth much of effort to study.
- However, the set of all validities is a fascinating set.

Question

Given a $\mathcal{L}$-formula, how do we tell it is a validity or not?

Things to do

We will give a pure logical description of validity:

An $\mathcal{L}$-formula is valid iff it can be proven.

To do this, we need to set up

- A set Δ of **logical axioms**.

  For connectives, quantifier, equality.
- Rule(s) of deductions.
- The notion of proofs.

Recall that a term τ is free for x_i in φ means variables in τ remain to be free in φ after substituting τ in x_i.

Definition 5.2 (Substitutable)

Suppose $\varphi \in \mathcal{L}$ and $\tau \in \mathcal{T}$.

1. Suppose that x_i is a free variable of φ.
   - τ is **substitutable for x_i** iff every variable x_j of τ is free for x_i in φ.
   - If τ is substitutable for x_i in φ, then $\varphi(x_i; \tau)$ denotes the $\mathcal{L}$-formula obtained by substituting τ for every free occurrence of x_i in φ. Similar for $\varphi(x_{i_1}, \ldots, x_{i_n}; \tau_{i_1}, \ldots, \tau_{i_n})$.
2. Suppose that c_i is a constant symbol.
   - τ is **substitutable for c_i** iff for every variable of x_j of τ, no occurrence of c_i in φ is within the scope of an occurrence of $\forall x_j$.
   - If τ is substitutable for c_i in φ, then $\varphi(c_i; \tau)$ denotes the $\mathcal{L}$-formula obtained by substituting τ for every free occurrence of c_i in φ. Similar for $\varphi(c_{i_1}, \ldots, c_{i_n}; \tau_{i_1}, \ldots, \tau_{i_n})$.

Logical axioms I

The set of logical axioms Δ is the smallest set of $\mathcal{L}$-formulas satisfying the following closure properties.

1. (Instances of Propositional Tautologies) Suppose that $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}$. Then each of the following $\mathcal{L}$-formulas is a logical axiom:

   (Group I axioms)

   - $(\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3))$
   - $\varphi_1 \to \varphi_1$
   - $\varphi_1 \to (\varphi_2 \to \varphi_1)$

Logical axioms II

(Group II axioms)

- $\varphi_1 \to (\neg\varphi_1 \to \varphi_2)$

(Group III axioms)

- $(\neg\varphi_1 \to \varphi_1) \to \varphi_1$

(Group IV axioms)

- $\neg\varphi_1 \to (\varphi_1 \to \varphi_2)$
- $\varphi_1 \to (\neg\varphi_2 \to \neg(\varphi_1 \to \varphi_2))$

Logical axioms III

2. Suppose that $\varphi \in \mathcal{L}$, $\tau \in \mathcal{T}$ and that τ is substitutable for x_i in φ. Then
$$\forall x_i \varphi \to \varphi(x_i; \tau) \in \Delta.$$
3. Suppose that $\varphi_1, \varphi_2 \in \mathcal{L}$. Then
$$\forall x_i\,(\varphi_1 \to \varphi_2) \to (\forall x_i\,\varphi_1 \to \forall x_i\,\varphi_2) \in \Delta.$$
4. Suppose $\varphi \in \mathcal{L}$ and x_i does not occur freely in φ. Then
$$\varphi \to \forall x_i\,\varphi \in \Delta.$$
[There are two cases for "does not occur freely":
   - x_i is "new";
   - x_i occurs boundedly in some $\gamma \in \Gamma$.]

Logical axioms IV

5. For every variable x_i, $x_i \hat{=} x_i \in \Delta$.
6. Suppose $\varphi_1, \varphi_2 \in \mathcal{L}$ and x_j is substitutable for x_i in φ_1 and in φ_2.

   If $\varphi_2(x_i; x_j) = \varphi_1(x_i; x_j)$, then $(x_i \hat{=} x_j) \to (\varphi_1 \to \varphi_2) \in \Delta$.
7. Suppose that $\varphi \in \Delta$. Then $\forall x_i \varphi \in \Delta$.

The notion of proof

Definition 5.3

Suppose that $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \Gamma$. Then $\Gamma \vdash \varphi$ iff there exists a finite sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $\mathcal{L}$-formulas such that

- $\varphi_1 \in \Gamma \cup \Delta$,
- $\varphi_n = \varphi$,
- for each $i \leqslant n$, either
  - $\varphi_i \in \Gamma \cup \Delta$, or
  - there exist $i_0, i_1 < i$ such that $\varphi_{i_1} \equiv (\varphi_{i_0} \to \varphi_i)$.[a]

  $\langle \varphi_1, \ldots, \varphi_n \rangle$ is called a **deduction/proof** of φ_n from Γ.

When $\Gamma \vdash \varphi$, we say that Γ **proves** φ.

[a] This rule of inference is called **Modus Ponens (MP)**.

"Consistent" and "Satisfiable"

Definition 5.4

Suppose $\Gamma \subseteq \mathcal{L}$.

- Γ is **consistent** iff for every φ,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Gamma \nvdash \neg\varphi.$$

- Γ is **satisfiable** iff there exists a structure $\mathcal{M}$ and an $\mathcal{M}$-assignment ν such that

$$(\mathcal{M}, \nu) \models \Gamma.$$

The next slide is the famous **Gödel Completeness Theorem**.

Gödel Completeness Theorem

Theorem 5.1 (Gödel Completeness, version I)

For any $\Gamma \subseteq \mathcal{L}$,

$$\Gamma \text{ is consistent} \quad \Leftrightarrow \quad \Gamma \text{ is satisfiable}$$

Another version:

Theorem 5.2 (Gödel Completeness, version II)

For any $\Gamma \subseteq \mathcal{L}$ *and any* $\varphi \in \mathcal{L}$,

$$\Gamma \models \varphi \quad \Leftrightarrow \quad \Gamma \vdash \varphi$$

Soundness

The "only if" ($\Leftarrow$) direction of the above two statements are also called **Soundness Theorem**.

Theorem 5.3 (Soundness, version II)

Suppose that $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ *and that* $\Gamma \vdash \varphi$. *Then for any* $(\mathcal{M}, \nu)$, *if* $(\mathcal{M}, \nu) \models \Gamma$,[a] *then* $(\mathcal{M}, \nu) \models \varphi$.

[a] $(\mathcal{M}, \nu) \models \Gamma$ abbreviates "$(\mathcal{M}, \nu) \models \gamma$, for every $\gamma \in \Gamma$".

Proof (Sketch).

Induction on the lengths of proofs. Inductively verify the cases $\varphi_n \in \Gamma$, Δ or obtained via MP, in particular the case in Δ.
Δ-2 as an example, to see $(\mathcal{M}, \nu) \models \forall x_i \varphi \to \varphi(x_i; \tau)$. Assume $(\mathcal{M}, \nu) \models \forall x_i \varphi$, set $\mu(x_i) = \nu(\tau)$, and copy ν at x_j $(j \neq i)$.
$(\mathcal{M}, \mu) \models \varphi(x_i; \tau)$, as $\mu \sim_{\forall x_j \varphi} \nu$, we have $(\mathcal{M}, \nu) \models \varphi(x_i; \tau)$. □

Corollary 5.4 (Soundness, version I)

Suppose $\Gamma \subseteq \mathcal{L}$. *If* Γ *is satisfiable, then* Γ *is consistent.*

Exercise 5.1

1. Prove Theorem 5.3.
2. Prove Corrolary 5.4.

We now start to prove the other direction.

Deduction

Theorem 5.5 (Deduction)

Suppose that $\Gamma \subseteq \mathcal{L}$ *and* $\varphi_1, \varphi_2 \in \mathcal{L}$. *Then*

$$\Gamma \cup \{\varphi_1\} \vdash \varphi_2 \quad \textit{iff} \quad \Gamma \vdash \varphi_1 \to \varphi_2.$$

Proof.

$\Leftarrow$: If $\Gamma \vdash \varphi_1 \to \varphi_2$ via $\bar{\theta}$, then $\Gamma \cup \{\varphi_1\} \vdash \varphi_2$ via $\bar{\theta} + \langle \varphi_1, \varphi_2 \rangle$.

$\Rightarrow$: Suppose $\Gamma \cup \{\varphi_1\} \vdash \varphi_2$ via $\bar{\theta} = \langle \theta_1, \ldots, \theta_n \rangle$. Proceed by induction on the length of $\bar{\theta}$.[a] Fix $j \leqslant n$. Assume $\Gamma \vdash \varphi_1 \to \theta_i$, $i < j$.

- $\theta_j \equiv \varphi_1$. Use $\varphi_1 \to \varphi_1 \in \Delta$-1.
- $\theta_j \in \Gamma \cup \Delta$. Use $\theta_j \to (\varphi_1 \to \theta_j) \in \Delta$-1.
- $\theta_{i_2} \equiv \theta_{i_1} \to \theta_j$, some $i_1 < i_2 < j$. Use $(\varphi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j)) \in \Delta$-1. □

[a] Exactly the same as in propositional logic

Generalization

Theorem 5.6 (Generalization)

Suppose that $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ *and* $\Gamma \vdash \varphi$. *Suppose that* x_i *is a variable and* x_i *does not occur freely in any formula in* Γ. *Then* $\Gamma \vdash \forall x_i \varphi$.

Proof.

We prove by induction on the lengths of proofs $\bar{\theta} = \langle \theta_1, \ldots, \theta_n \rangle$. Assume for each $j < n$, $\Gamma \vdash \forall x_i \theta_j$.

- $\theta_j \in \Delta$: use $\forall x_i \theta_j \in \Delta$-7.
- $\theta_j \in \Gamma$: use $\theta_j \to \forall x_i \theta_j \in \Delta$-4. $\langle \theta_j, \theta_j \to \forall x_i \theta_j, \forall x_i \theta_j \rangle$.
- MP: use $\forall x_i(\theta_{i_1} \to \theta_j) \to (\forall x_i \theta_{i_1} \to \forall x_i \theta_j) \in \Delta$-3. □

Theorem on Constants

Next theorem says that at certain settings constant symbols can be treated as (new) "free" variables.

Theorem 5.7 (Constants)

Suppose that $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ *and* $\Gamma \vdash \varphi$. *Suppose that* c_i *is a constant and* c_i *does not occur in any formula of* Γ. *Let* x_j *be a variable which is substitutable for* c_i *in* φ *and does not occur freely in* φ. *Then the following conditions hold:*

1. $\Gamma \vdash \forall x_j \varphi(c_i; x_j)$.
2. *There is a deduction* $\langle \varphi_1, \dots \varphi_n \rangle$ *for* $\Gamma \vdash \forall x_j \varphi(c_i; x_j)$ *such that*
   - c_i *does not occur in* φ_m, $m \leqslant n$.
   - *if* c_k *occurs in* φ_m, $m < n$, *then* c_k *occurs in* $\{\forall x_j \varphi(c_i; x_j)\} \cup \Gamma$.

This theorem needs two lemmas:

Lemma 5.7

Suppose that $\varphi \in \mathcal{L}$, x_i is free for x_j in φ and x_i does not occur freely in $\forall x_j \varphi$. Then

$$\vdash \forall x_j \varphi \to \forall x_i \varphi(x_j; x_i).$$

Proof.

Δ-2 gives $\vdash \forall x_j \varphi \to \varphi(x_j; x_i)$. By Deduction

$$\{\forall x_j \varphi\} \vdash \varphi(x_j; x_i).$$

x_i does not occur freely in $\Gamma = \{\forall x_j \varphi\}$. By Generalization,

$$\{\forall x_j \varphi\} \vdash \forall x_i \varphi(x_j; x_i).$$

Use Deduction again,

$$\vdash \forall x_j \varphi \to \forall x_i \varphi(x_j; x_i).$$

□

Lemma 5.7

Suppose that $\Gamma \subseteq \mathcal{L}$ *and the constant symbol* c_i *does not occur in any formula of* Γ. *Suppose that* $\langle \theta_1, \dots \theta_m \rangle$ *is a deduction from* Γ *and that the variable* x_j *does not occur in* θ_i, *for all* $i \leqslant m$. *Then*

$$\langle \theta_1(c_i; x_j), \dots, \theta_m(c_i; x_j) \rangle$$

is a deduction from Γ.

Proof.

Induction on the lengths of proofs.

- For each $\varphi \in \Gamma$, $\varphi(c_i; x_j) = \varphi$.
- By inspection, for any $\varphi \in \Delta$, if x_j does not occur in φ, then $\varphi(c_i; x_j) \in \Delta$.
- If φ is obtained by MP, use the fact that for $\varphi_1, \varphi_2 \in \mathcal{L}$,

$$(\varphi_1 \to \varphi_2)(c_i; x_j) = \varphi_1(c_i; x_j) \to \varphi_2(c_i; x_j).$$

□

Proof of Theorem 5.7

The tricky case is that x_j could appear boundedly in φ.

Let $\bar{\theta} = \langle \theta_1, \dots, \theta_n \rangle$ be a deduction of φ from Γ. Let x_k be a variable never "used" in $\bar{\theta}$ (as $\bar{\theta}$ uses finitely many symbols). Let $\Gamma_0 = \bar{\theta} \cap \Gamma$.

- By Lemma 5.7, $\Gamma_0 \vdash \varphi(c_i; x_k)$. By Generalization, $\Gamma_0 \vdash \forall x_k \varphi(c_i; x_k)$.
- By Lemma 5.7, $\vdash \forall x_k \varphi(c_i; x_k) \to (\forall x_j \varphi(c_i; x_k)(x_k; x_j))$. So $\Gamma_0 \vdash \forall x_k \varphi(c_i; x_k) \to \forall x_j \varphi(c_i; x_j)$.

Apply MP to get Clause 1 .

Clause 2 follows by induction on the lengths of the proofs.

Maximally consistent

Definition 5.5

Suppose that $\Gamma \subseteq \mathcal{L}$ is consistent. Γ is **maximally consistent** iff for any $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\Gamma \cup \{\varphi\}$ is not consistent.

Lemma 5.7

Suppose that $\Gamma \subseteq \mathcal{L}$ is maximally consistent. Then for each $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Proof.

Suppose $\neg\varphi \notin \Gamma$. By maximality, for some $\psi \in \mathcal{L}$,

$$\Gamma \cup \{\neg\varphi\} \vdash \psi \quad \text{and} \quad \Gamma \cup \{\neg\varphi\} \vdash \neg\psi$$

Applying Δ-1 and MP, we get $\Gamma \vdash \varphi$. If Γ is consistent, then so is $\Gamma \cup \{\varphi\}$. By maximality, $\varphi \in \Gamma$. □

The Henkin property

Definition 5.6

$\Gamma \subseteq \mathcal{L}$ has the **Henkin property** iff for each $\varphi \in \mathcal{L}$ and for each variable x_i, if $\exists x_i \varphi \in \Gamma$ then there exists a constant c_j such that $\varphi(x_i; c_j) \in \Gamma$.

The next theorem motivates the definition of Henkin's property.

Theorem 5.8

Suppose that $\mathcal{M} = (M, I)$ *is a structure and* ν *is a* $\mathcal{M}$*-assignment such that* $\operatorname{ran}(\nu) \subseteq \{I(c_i) \mid i \in \mathbb{N}\}$. *Let* $\Gamma = \{\varphi \mid (\mathcal{M}, \nu) \models \varphi\}$. *Then*

1. Γ *is maximally consistent.*
2. Γ *has the Henkin property iff*

$$\mathcal{M}_0 = (M_0, I_0) \preceq \mathcal{M},$$

where $M_0 = \{I(c_i) \mid i \in \mathbb{N}\}$, I_0 *is the restriction of* I *to* M_0.

Proof of theorem

1.1 Γ is consistent:

$$\Gamma \vdash \varphi \Rightarrow_s (\mathcal{M}, \nu) \models \varphi \Rightarrow (\mathcal{M}, \nu) \not\models \neg\varphi \Rightarrow_s \Gamma \nvdash \neg\varphi.$$

1.2 Γ is maximal.

$$\varphi \notin \Gamma \Rightarrow_\Gamma (\mathcal{M}, \nu) \not\models \varphi \Rightarrow (\mathcal{M}, \nu) \models \neg\varphi \Rightarrow_\Gamma \neg\varphi \in \Gamma.$$

2.1 $\Rightarrow$: Verify Tarski's Criterion. Let $D = \{a \mid \mathcal{M} \models \varphi[a, I(\bar{c})]\}$.
$D \neq \varnothing$. Want c_i s.t. $I(c_i) \in D$. Let $\psi(x_1) \equiv \varphi(x_1, \bar{c})$,

$$(\mathcal{M}, \nu) \models \exists x_1 \psi \Rightarrow \exists x_1 \psi \in \Gamma \Rightarrow_H \psi(c_i) \in \Gamma, \text{some } c_i.$$

Then $(\mathcal{M}, \nu) \models \psi(c_i)$. Hence, $I(c_i) \in D \cap M_0 \neq \varnothing$.

2.2 $\Leftarrow$: For the Henkin property, suppose $\exists x_i \varphi(x_i, \bar{x}) \in \Gamma$.
$\mathcal{M} \models \exists x_i \varphi(x_i, \bar{\nu}(\bar{x}))$. $\bar{\nu}(\bar{x}) \in M_0$ and $\mathcal{M}_0 \preceq \mathcal{M}$, so for some c_j,
$\mathcal{M}_0 \models \varphi[I_0(c_j), \bar{\nu}(\bar{x})]$. Since $I(c_j) = I_0(c_j)$, $\varphi(c_j, \bar{x}) \in \Gamma$. □

Lemma for substitution

Lemma 5.8

Suppose that $\varphi \in \mathcal{L}$ has no quantifiers and $\bar{\tau} = \langle \tau_i : i \leqslant n \rangle$, $\bar{\sigma} = \langle \sigma_i : i \leqslant n \rangle$ are two sequences of terms. Then for any sequence of variables $\bar{x} = \langle x_{m_i} : i \leqslant n \rangle$ such that none of x_{m_j} appears in any of τ_i and σ_i,

$$\{\tau_i \hat{=} \sigma_i \mid i \leqslant n\} \cup \{\varphi(\bar{x}; \bar{\tau})\} \vdash \varphi(\bar{x}; \bar{\sigma}).$$

PROOF. Prove by induction on n, the length of $\bar{x}$.

This is used in verifying that the $\mathcal{M}$-assignment ν constructed in the proof of Completeness (Theorem 5.9) is well-defined.

- Let $\bar{\tau}' = \langle \tau_i : i < n \rangle$, $\bar{\sigma}' = \langle \sigma_i : i < n \rangle$, $\bar{x}' = \langle x_{m_i} : i < n \rangle$.
- Granting the inductive hypothesis for the cases $< n$, we have
$$\{\tau_i \hat{=} \sigma_i \mid i \leqslant n\} \vdash \varphi(\bar{x}; \bar{\tau}', x_{m_n}) \to \varphi(\bar{x}; \bar{\sigma}', x_{m_n}).$$
- Since x_{m_n} does not appear in $\bar{\tau}, \bar{\sigma}$, by Generalization,
$$\{\tau_i \hat{=} \sigma_i \mid i \leqslant n\} \vdash \forall x(\varphi(\bar{x}; \bar{\tau}', x) \to \varphi(\bar{x}; \bar{\sigma}', x)).$$
then by Δ-2 (to the substitution $(x_{m_n}; \sigma_n)$) and MP,
$$\{\tau_i \hat{=} \sigma_i \mid i \leqslant n\} \vdash \varphi(\bar{x}; \bar{\tau}', \sigma_n) \to \varphi(\bar{x}; \bar{\sigma}). \qquad (\dagger)$$
- Take an "unused" x_k, by Δ-6,
$$\varnothing \vdash (x_{m_n} \hat{=} x_k) \to (\varphi(\bar{x}; \bar{\tau}', x_{m_n}) \to \varphi(\bar{x}; \bar{\tau}', x_k))$$
- Substitute $(x_{m_n}; \tau_n)$ and $(x_k; \sigma_n)$, we have
$$\varnothing \vdash (\tau_n \hat{=} \sigma_n) \to (\varphi(\bar{x}; \bar{\tau}', \tau_n) \to \varphi(\bar{x}; \bar{\tau}', \sigma_n))$$
- Combine with $(\dagger)$, then
$$\{\tau_i \hat{=} \sigma_i \mid i \leqslant n\} \vdash \varphi(\bar{x}; \bar{\tau}) \to \varphi(\bar{x}; \bar{\sigma}).$$
□

Henkin property & Satisfiability

Theorem 5.9

Suppose that $\Gamma \subseteq \mathcal{L}$ is maximally consistent and has the Henkin property. Then Γ is satisfiable.

Sketch of the proof

Define $(\mathcal{M}, \nu)$ s.t. $(\mathcal{M}, \nu) \models \Gamma$.

- Define $c_i \sim_\Gamma c_j$ iff $c_i \hat{=} c_j \in \Gamma$. $\sim_\Gamma$ is an equivalence relation.
- $M = \{[c_i]_\Gamma \mid i \in \mathbb{N}\}$.

Sketch of the proof (Cont'd)

- Define I.
  - $I(c_i) = [c_i]_\Gamma$.
  - $\overline{[c]_\Gamma} \in I(P_i)$ iff $P_i(\bar{c}) \in \Gamma$.
  - $I(F_i)(\overline{[c]_\Gamma}) = [c_k]$ iff $F_i(\bar{c}) = c_k \in \Gamma$.

  check that $I(P_i)$ and $I(F_i)$ are well-defined.

- $\nu(x_i) = I(c_j)$ if $x_i \hat{=} c_j \in \Gamma$. Check that ν is well-defined.
  - For each x_i, there is a c_j s.t. $x_i \hat{=} c_j \in \Gamma$. (uses Δ-5 and Henkin property)
  - If $x_i \hat{=} c_j \in \Gamma$ and $x_i \hat{=} c_k \in \Gamma$, then $c_j \sim_\Gamma c_k$. (uses Lemma 5.8, Δ-2,6,7).

Sketch of the proof (Cont'd)

- For any $\tau \in \mathcal{T}$, $\bar{\nu}(\tau) = [c_i]_\Gamma$ iff $\tau \hat{=} c_i \in \Gamma$.
- For every formula φ, $\varphi \in \Gamma$ iff $(\mathcal{M}, \nu) \models \varphi$.
  - Reduce to the case of sentences: replacing free variables by "new" constant symbols. (need Δ-1,2,7).
  - Induction on the length of φ. For the case of quantification, from the Henkin property, we have:

$$\begin{array}{lll} \forall x_i \psi \in \Gamma & \text{iff} & \text{for every } c_j,\ \psi(x_i; c_j) \in \Gamma \\ & \text{iff} & \text{for every } c_j,\ \mathcal{M} \models \psi(x_i; c_j) \\ & \text{iff} & \mathcal{M} \models \forall x_i \psi. \end{array}$$

Extensions of consistent sets of formulas

Theorem 5.10

Suppose that $\Gamma \subseteq \mathcal{L}$ is consistent and there are infinitely many constants c_i which do not occur in any formula of Γ. Then there is a set of formulas $\Sigma \subseteq \mathcal{L}$ such that

1. $\Gamma \subseteq \Sigma$.
2. Σ *is maximally consistent.*
3. Σ *has the Henkin property.*

Hence, Γ is satisfiable.

This theorem can be rephrased as follows:

Suppose $\Gamma \subseteq \mathcal{L}_A$ is consistent, there is extension Γ^* in a language $\mathcal{L}_{A\cup C}$, where C is a set of infinitely new constant symbols, such that 1-3 above hold for Γ^*.

Proof

Let A_Γ denote the set of all constant, function and predicate symbols used in Γ, $\{d_{i,j} \mid i, j \in \mathbb{N}\}$ enumerate the constant that do not occur in Γ. Define an increasing sequence $\langle \Sigma_i \mid i \in \mathbb{N} \rangle$ as follows: Let $\Sigma_0 = \Gamma$.

Suppose Σ_i is defined. Let $\{\varphi_{i,j} \mid j \in \mathbb{N}\}$ enumerate all the $\mathcal{L}_{A_i}$-formulas, where $A_i = A_\Gamma \cup \{d_{k,j} \mid k < i,\ j \in \mathbb{N}\}$ (so $A_0 = A_\Gamma$).

Proof (Cont'd)

We construct a sequence $\langle \Sigma_{i,j} \mid j \in \mathbb{N} \rangle$ s.t.

- $\Sigma_{i0} = \Sigma_i$.
- If $\varphi_{i,j} \notin \Sigma_{i,j}$ and $\Sigma_{i,j} \cup \{\varphi_{i,j}\}$ is consistent, then $\Sigma'_{i,j} = \Sigma_{i,j} \cup \{\varphi_{i,j}\}$; otherwise $\Sigma'_{i,j} = \Sigma_{i,j}$.

  If further $\varphi_{i,j} = \exists x_k \, \psi$, then $\Sigma_{i,j+1} = \Sigma'_{i,j} \cup \{\psi(x_k; d_{i,j})\}$; otherwise, $\Sigma_{i,j+1} = \Sigma'_{i,j}$.

Let $\Sigma_{i+1} = \bigcup_j \Sigma_{i,j}$.

At the end, let $\Sigma = \bigcup \Sigma_i$. Σ is maximally consistent and has the Henkin property.

Gödel's Completeness Theorem

Theorem 5.11 (Completeness, version I)

A set of $\mathcal{L}$ formulas is consistent iff it is satisfiable.

Theorem 5.12 (Completeness, version II)

For any $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$,

$$\Gamma \models \varphi \quad \textit{iff} \quad \Gamma \vdash \varphi.$$

Proof of Gödel's Completeness Theorem

We sketch the proof of version I.

By Soundness, if Γ is satisfiable then Γ is consistent. We show the other direction. The key is to get infinitely many unused constant symbols.

Let φ' be the formula obtained from φ by replacing c_{2i} for each c_i occurring in φ. Let $\Gamma' = \{\varphi' \mid \varphi \in \Gamma\}$. Then

- Γ is consistent $\Rightarrow$ Γ' is consistent.
  - [Need the lemma on next slide]
- Γ' is satisfiable $\Rightarrow$ Γ is satisfiable.
  - [Only need to change the interpretation.] $\square$

To complete the proof, what's left is the following lemma.

Lemma 5.12

Suppose $\rho : \{c_i \mid i \in \mathbb{N}\} \to \{c_i \mid i \in \mathbb{N}\}$ *is an injective function. Given a formula* φ *and a set of formulas* Γ*, let* φ^ρ *be the formula obtained from* φ *by replacing* $\rho(c_i)$ *for each* c_i *occurring in* φ*, and let* $\Gamma^\rho = \{\gamma^\rho \mid \gamma \in \Gamma\}$*. Then*

$$\Gamma^\rho \vdash \varphi^\rho \quad \textit{iff} \quad \Gamma \vdash \varphi.$$

Proof.

- The direction "$\Leftarrow$": translate a Γ-proof to a Γ^ρ-proof.
- The direction "$\Rightarrow$":
  - Since proofs are finite, we may assume that Γ is finite. Let $\mathcal{C}$ be the set of constant symbols appearing in $\Gamma \cup \{\varphi\}$.
  - Since $\mathcal{C}$ is finite, the complement of $\rho[\mathcal{C}]$ is infinite, there is an injective $\xi : \{c_i \mid i \in \mathbb{N}\} \to \{c_i \mid i \in \mathbb{N}\}$ such that $\xi \upharpoonright \rho[\mathcal{C}] = \mathrm{id}$.
  - Apply the direction "$\Leftarrow$" to ξ: $\Gamma^\rho \vdash \varphi^\rho \Rightarrow \Gamma^{\xi \circ \rho} \vdash \varphi^{\xi \circ \rho}$. □

Definition 5.7

Suppose that $\Gamma \subseteq \mathcal{L}_A$ and $\varphi \in \mathcal{L}_A$. Then $\Gamma \vdash_{\mathcal{L}_A} \varphi$ iff there exists a proof $\langle \varphi_1, \ldots, \varphi_n \rangle$ of φ from Γ such that for each $i \leqslant n$, φ_i is an $\mathcal{L}_A$-formula.

Theorem 5.13

Suppose that $\Gamma \subseteq \mathcal{L}_A$ *and* $\varphi \in \mathcal{L}_A$. *Then*

$$\Gamma \vdash \varphi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathcal{L}_A} \varphi$$

Theorem 5.14 (Completeness for $\mathcal{L}_A$)

Suppose that $\Gamma \subseteq \mathcal{L}_A$ *and* $\varphi \in \mathcal{L}_A$. *Then*

$$\Gamma \vdash_{\mathcal{L}_A} \varphi \qquad \textit{iff} \qquad \Gamma \models \varphi.$$

Exercise 5.2

1. Show that for every pair of $\mathcal{L}$ formulas φ and ψ, $\{\varphi, (\neg\varphi)\} \vdash \psi$.
2. Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Show that $\Gamma \vdash \varphi$.

*Craig Interpolation Theorem

Suppose $\varphi \in \mathcal{L}$ and $\Gamma \subset \mathcal{L}$. Let A_Γ denote the minimal signature for formulas in Γ, and $A_\varphi = A_{\{\varphi\}}$.

Theorem 5.15 (Craig Interpolation Theorem for $\mathcal{L}$)

Suppose that $\varphi, \psi \in \mathcal{L}$, $\Gamma \subseteq \mathcal{L}$ and that $\Gamma \vdash \varphi \to \psi$. Suppose that $A_\varphi \cap A_\psi \subseteq A_\Gamma$. Then there is a $\theta \in \mathcal{L}_{A_\Gamma}$ s.t.

1. $\Gamma \vdash (\varphi \to \theta)$,
2. $\Gamma \vdash (\theta \to \psi)$.

Proof of Craig's Interpolation

- Suppose NOT, we construct a model of $\varphi \wedge \neg\psi$.
- Let $\mathcal{L}_1 = \mathcal{L}_{A_\varphi}$, $\mathcal{L}_2 = \mathcal{L}_{A_\psi}$ and $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$.

  We shall expand $\mathcal{L}_i$'s to $\mathcal{L}_i'$'s by adding a (common) countable set C of new constant symbols, $i = 0, 1, 2$.
- Let S, T be two theories in $\mathcal{L}_1, \mathcal{L}_2$. A sentence $\sigma \in \mathcal{L}_0$ separates S and T if $S \vdash \sigma$ and $T \vdash \neg\sigma$. Say S and T are inseparable if no such σ exists.

  Observation:
  If T and U are inseparable, then both are consistent.
- Starting with $S_0 = \{\varphi\}$, $T_0 = \{\neg\psi\}$, construct two $\subseteq$-increasing sequences of finite sets of sentences S_n, T_n, $n \in \mathbb{N}$ such that the resulting theories S_ω and T_ω are two inseparable, maximal consistent theories with Henkin property in $\mathcal{L}_1'$, $\mathcal{L}_2'$ respectively.

Proof of Craig's Interpolation, cont'd

- OBSERVATION:
  1. S_ω is a maximal consistent theory in $\mathcal{L}_1'$, and T_ω is a maximal consistent theory in $\mathcal{L}_2'$.
  2. $S_\omega \cap T_\omega$ is a maximal consistent theory in $\mathcal{L}_0'$.
  3. $S_\omega \cup T_\omega$ is a consistent theory in $\mathcal{L}_1' \cup \mathcal{L}_2'$.
- Suppose $\mathcal{M} \models S_\omega$. By Henkin's property, the interpretations $\{c^{\mathcal{M}} \mid c \in C\}$, give an elementary submodel $\mathcal{M}_0 \prec \mathcal{M}$.

  Suppose $\mathcal{N} \models T_\omega$. Let $\mathcal{N}_0$ be the counterpart of $\mathcal{M}_0$.

  $\mathcal{M}_0$ and $\mathcal{N}_0$ have the same $\mathcal{L}_0$-reduct.
- Let $\mathcal{W}$ be a model of $S_\omega \cup T_\omega$ such that $\mathcal{W}|_{\mathcal{L}_1} = \mathcal{M}_0|_{\mathcal{L}_1}$ and $\mathcal{W}|_{\mathcal{L}_2} = \mathcal{N}_0|_{\mathcal{L}_2}$. Then $\mathcal{W} \models \varphi \wedge \neg\psi$. □

Implicit and explicit defintions

An important applications of Craig's interpolation is Beth's Definability Theorem, for which we need the following definitions.

Definition

Fix a language $\mathcal{L}_A$. Let P and P' be two n-place relation symbols not in A. For a set Σ of $\mathcal{L}_{A\cup\{P\}}$-sentence, let Σ' be the result of replacing P by P'. We say

1. Σ defines P implicitly if $\Sigma \cup \Sigma' \vdash \forall\bar{x}(P(\bar{x}) \leftrightarrow P'(\bar{x}))$.
2. Σ defines P explicitly if there is a formula $\varphi(\bar{x}) \in \mathcal{L}_A$ such that $\Sigma \vdash \forall\bar{x}(P(\bar{x}) \leftrightarrow \varphi(\bar{x}))$.

*Beth's Definability Theorem

Theorem (Beth)

Σ *defines* P *implicitly iff it defines* P *explicitly.*

Proof.

"$\Leftarrow$" is easy. "$\Rightarrow$". Add new constant symbols $\bar{c}$ to A, and may assume Σ, Σ' closed under conjunction.

- Let $\psi \in \Sigma$ and $\sigma \in \Sigma'$ be such that $\sigma = \psi'$ and $\psi \wedge \psi' \vdash P(\bar{c}) \rightarrow P'(\bar{c})$. Rearrang the later to $\psi \wedge P(\bar{c}) \vdash \psi' \rightarrow P'(\bar{c})$.
- Let $\sigma(\bar{c})$ be a Craig Interpolation for that: $\psi \wedge P(\bar{c}) \vdash \theta(\bar{c})$ and $\theta(\bar{c}) \vdash \psi' \rightarrow P'(\bar{c})$, these yield: $\psi \vdash P(\bar{c}) \rightarrow \theta(\bar{c})$ and $\psi \vdash \theta(\bar{c}) \rightarrow P(\bar{c})$ and hence

$$\psi \vdash P(\bar{c}) \leftrightarrow \theta(\bar{c}).$$

□