Mathematical Logic

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Next

Definition 4.1 (homomorphisms between structures)

Suppose $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are two $\mathcal{L}_A$-structures. A **homomorphism** between $\mathcal{M}$ and $\mathcal{N}$ is a function $e : M \to N$ with the following properties:

- For each constant symbol $c_i \in A$,
$$e(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}.$$
- For each function symbol $F_i \in A$, if $n = \pi(F_i)$, then for each $\bar{a} = \langle a_1, \ldots, a_n \rangle \in M^n$,
$$e(F_i^{\mathcal{M}}(\bar{a})) = F_i^{\mathcal{N}}(e(\bar{a})).$$
- For each predicate symbol $P_i \in A$, if $n = \pi(P_i)$, then for each $\bar{a} = \langle a_1, \ldots, a_n \rangle \in M^n$,
$$\bar{a} \in P_i^{\mathcal{M}} \quad \text{iff} \quad e(\bar{a}) \in P_i^{\mathcal{N}}.$$

If e is 1-1, it is called an **isomorphic embedding** of $\mathcal{M}$ into $\mathcal{N}$. If, in addition, e is onto, then it is called an **isomorphism** of $\mathcal{M}$ onto $\mathcal{N}$, and write $\mathcal{M} \cong \mathcal{N}$.

Exampls

Example 4.1

Let $\mathcal{M}$ be the structure $(\mathbb{N}; +, \cdot)$. We define a function $h : \mathbb{N} \to \{0, 1\}$ by

$$h(n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Then h is a homomorphism of $\mathcal{M}$ onto $\mathcal{U} = (\{0, 1\}, J)$, where $J(+)$ and $J(\cdot)$ are given by

$J(+)$	0	1
0	0	1
1	1	0

$J(\cdot)$	0	1
0	0	0
1	0	1

Exampls

Example 4.2

Consider the two structures

$$\mathcal{M} = (\mathbb{Z}^+, <_Z) \quad \text{and} \quad \mathcal{N} = (\mathbb{N}, <_N).$$

- There is an isomorphism h from $\mathcal{M}$ onto $\mathcal{N}$: $h(n) = n - 1$.
- The identity map $\mathrm{id} : \mathbb{Z}^+ \to \mathbb{N}$ is an isomorphic embedding of $\mathcal{M}$ into $\mathcal{N}$. $\mathcal{M}$ fits coherently with the structure of $\mathcal{N}$. Due to this fact, we say that $\mathcal{M}$ is a **substructure** of $\mathcal{N}$. More generally,

Substructure

Definition 4.2

Consider two $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ such that $M \subseteq N$. We say $\mathcal{M}$ is a **substructure** of $\mathcal{N}$ (write $\mathcal{M} \subseteq \mathcal{N}$) if the following conditions are met:

- For every $c \in A$, $c^{\mathcal{M}} = c^{\mathcal{N}}$;
- For every $F \in A$, $F^{\mathcal{M}} = F^{\mathcal{N}} | M^n$, where $n = \pi(F)$;
- For every $P \in A$, $P^{\mathcal{M}} = P^{\mathcal{N}} \cap M^n$, where $n = \pi(P)$.

Also $\mathcal{N}$ is called an **extension** of $\mathcal{M}$.

Example 4.3

$(\mathbb{Q}; +_Q)$ is a substructure of $(\mathbb{C}; +_C)$.

Simple facts about substructures

- Let $\mathcal{M} = (M, I)$, $\mathcal{N} = (N, J)$ be two $\mathcal{L}_A$-structures. If $\mathcal{M} \subseteq \mathcal{N}$, M must be closed under $F^{\mathcal{N}}$ for all $F \in A$.

  This closure property holds for the 0-ary function symbols, as $c^{\mathcal{M}} \in M$, for each $c \in A$.

- Conversely, given a $\mathcal{L}_A$-structure $\mathcal{N} = (N, J)$. Suppose $\varnothing \neq M \subseteq N$ and M is closed under $F^{\mathcal{N}}$, for all $F \in A$. Then there is a unique $\mathcal{M} \subseteq \mathcal{N}$ with universe M.

  An extreme case: if $A \cap \mathcal{F} = \varnothing$ i.e. no function symbols, then *any* nonempty subset of N can be the universe of a substructure of $\mathcal{N}$.

Theorem 4.1

Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are two $\mathcal{L}_A$-structures with $M \subseteq N$. Then the following are equivalent.

1. $\mathcal{M} \subseteq \mathcal{N}$.
2. *For all atomic $\varphi \in \mathcal{L}_A$ and for all $\mathcal{M}$-assignments ν,*

$$(\mathcal{M}, \nu) \models \varphi \quad \leftrightarrow \quad (\mathcal{N}, \nu) \models \varphi.$$

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Proof.

This follows from part (1) of Homomorphism Theorem (coming up next) and the definition of substructures. □

Homomorphism Theorem

Theorem 4.2

Suppose that $e : \mathcal{M} \to \mathcal{N}$ is a homomorphism between $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$. Suppose that ν is an $\mathcal{M}$-assignment. Then

1. *$e \circ \nu$ is an $\mathcal{N}$-assignment. In fact, for any term τ,*
$$\overline{e \circ \nu}(\tau) = e(\bar{\nu}(\tau)).$$
2. *For any quantifier-free $\mathcal{L}_A$-formula φ not containing the equality symbol,*
$$(\mathcal{M}, \nu) \models \varphi \quad \leftrightarrow \quad (\mathcal{N}, e \circ \nu) \models \varphi.$$
3. *If e is injective (1-1), then in part (2), we may delete the restriction "not containing the equality symbol".*
4. *If e is surjective (onto), then in part (2), we may drop the restriction "quantifier-free".*

Proof

- Induction on the rank of terms. For instance, suppose $F \in A$ with $\pi(F) = n$ and $\bar{\tau} \in \mathcal{T}^n$.

$$\begin{aligned}
\overline{e \circ \nu}(F(\bar{\tau})) &= F^{\mathcal{N}}(\overline{e \circ \nu}(\bar{\tau})) && \text{(defn of } \bar{\nu}\text{)} \\
&= F^{\mathcal{N}}(e(\bar{\nu}(\bar{\tau}))) && \text{(by induction)} \\
&= e(F^{\mathcal{M}}(\bar{\nu}(\bar{\tau}))) && (e \text{ is a hom}) \\
&= e(\bar{\nu}(F(\bar{\tau}))) && \text{(defn of } \bar{\nu}\text{)}
\end{aligned}$$

- Induction on the rank of formulas.

a. Checking atomic formulas such as $P(\bar{\tau})$ and the inductive arguments on connective symbols are routine. This proves part (2) for quantifier-free and no-equality-symbol formulas.

b. For the case of the equality symbol,

$$\begin{aligned}(\mathcal{M},\nu) \models \tau = \sigma &\Leftrightarrow \bar{\nu}(\tau) = \bar{\nu}(\sigma) \\ &\Rightarrow e(\bar{\nu}(\tau)) = e(\bar{\nu}(\sigma)) \\ &\Leftrightarrow \overline{e \circ \nu}(\tau) = \overline{e \circ \nu}(\sigma) \\ &\Leftrightarrow (\mathcal{N}, e \circ \nu) \models \tau = \sigma\end{aligned}$$

If e is injective, then "$\Rightarrow$" can be reversed.

c. We assume e is surjective and check the inductive step on the quantifier. Let $\varphi = (\forall x \psi)$.

- $(\mathcal{M}, \nu) \models \varphi$, so for every $\mu \sim_\varphi \nu, (\mathcal{M}, \mu) \models \psi$.
- For each $\mathcal{N}$-assignment $\mu^* \sim_\varphi e \circ \nu$, since e is onto, there is an $\mathcal{M}$-assignment μ such that $\mu \sim_\varphi \nu$ and $e \circ \mu = \mu^*$, in particular, setting $\mu(x) = m$ for some $m \in M$ such that $e(m) = \mu^*(x)$.
- By inductive hypothesis, $(\mathcal{N}, e \circ \mu) \models \psi$. $\mu \sim_\varphi \nu \Rightarrow e \circ \mu \sim_\varphi e \circ \nu$. So we have $(\mathcal{N}, e \circ \nu) \models \varphi$.
- Similarly, $(\mathcal{M}, \nu) \not\models \varphi$ implies $(\mathcal{N}, e \circ \nu) \not\models \varphi$. (Why?)

□

Elementary Equivalent

Definition 4.3

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_A$-structures. Then $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** (write as $\mathcal{M} \equiv \mathcal{N}$) if and only if for each $\mathcal{L}_A$-sentences φ,

$$\mathcal{M} \models \varphi \quad \Leftrightarrow \quad \mathcal{N} \models \varphi.$$

Elementary Equivalent

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$$\mathcal{M} \models \varphi \quad \Leftrightarrow \quad \mathcal{N} \models \varphi.$$

Corollary 4.3

Isomorphic structures are elementarily equivalent:

$$\mathcal{M} \cong \mathcal{N} \quad \Rightarrow \quad \mathcal{M} \equiv \mathcal{N}.$$

Remarks

- There are elementarily equivalent structures that are not isomorphic.
- For finite $\mathcal{L}_A$-structures, $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$ are equivalent. (The case that A is finite follows from Exercise 4.1)

Example 4.4

$(\mathbb{R}; <_R)$ is elementarily equivalent to $(\mathbb{Q}; <_Q)$ (to be discussed later), but they are not isomorphic — different cardinality.

An example revisited

Example 4.5

Consider the two structures

$$\mathcal{M} = (\mathbb{Z}^+; <_Z) \quad \text{and} \quad \mathcal{N} = (\mathbb{N}, <_N).$$

$\mathcal{M} \cong \mathcal{N}$ via the map

$$h(n) = n - 1,$$

so $\mathcal{M} \equiv \mathcal{N}$. In other word, these two structures are indistinguishable by first-order sentences.

Note that $\mathrm{id} : \mathbb{Z}^+ \to \mathbb{N}$ is an isomorphic embedding. Hence for any $\mathcal{M}$-assignment ν and any quantifier-free formula φ,

$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad (\mathcal{N}, \mathrm{id} \circ \nu) \models \varphi.$$

This equivalence may fail if φ contains quantifiers. For example, let $\varphi(x) \equiv \forall y(x \neq y \to x < y)$, then

$$\mathcal{M} \models \varphi[1] \quad \text{but} \quad \mathcal{N} \not\models \varphi[1]$$

Automorphism

- An **automorphism** of the structure $\mathcal{M} = (M, I)$ is an isomorphism of $\mathcal{M}$ onto itself.
- The identity function on M is trivially an automorphism of $\mathcal{M}$.
- $\mathcal{M}$ may or may not have nontrivial automorphisms. If not, we say $\mathcal{M}$ is rigid.
- As a consequence of Homomorphism Theorem, automorphism preserve the definable relations.

Preserving the definable relations

Corollary 4.4

Let $e : \mathcal{M} \to \mathcal{M}$ be an automorphism and R be an n-ary definable relation on M. Then for any $\langle a_1, \ldots, a_n \rangle \in M$,

$$\langle a_1, \ldots, a_n \rangle \in R \quad \textit{iff} \quad \langle e(a_1), \ldots e(a_n) \rangle \in R.$$

Remark

This means that automorphisms fix definable sets. It is useful in showing that a given relation is not definable.

Non-definable sets I

- The set $\mathbb{N}$ is not definable in the structure $(\mathbb{R}; <)$.

  This is witnessed by the automorphism $e(a) = a^3$, since it maps points outside of $\mathbb{N}$ into $\mathbb{N}$.

- Take the previous example of a directed graph $\mathcal{M} = (\{a, b, c\}; \{(a, b), (a, c)\})$, where $\{(a, b), (a, c)\}$ interprets a binary symbol.

  $$b \longleftarrow a \longrightarrow c$$

  The only nontrivial automorphism of this structure is the map that fixes a and exchanges b and c, since it has to respect the directions of edges. This map does not fix $\{b\}$, so $\{b\}$ is not definable.

Non-definable sets II

- Consider the vector space

$$\mathcal{E} = (E; +, f_r)_{r \in \mathbb{R}},$$

where
  - E is the universe,
  - $+$ is the vector addition, and
  - for each $r \in \mathbb{R}$, f_r is the scalar multiplication by r.

Consider the set of unit vectors,

$$U = \{\vec{x} \mid \vec{x} \in E \text{ and } |\vec{x}| = 1\}.$$

CLAIM. U is not definable in the structure $\mathcal{E}$.
This is witnessed by the doubling map

$$e(\vec{x}) = 2\vec{x}.$$

e is an automorphism but it does not preserve U.

Exercise 4.1

Suppose that A is finite and that $\mathcal{M}$ is a finite $\mathcal{L}_A$-structure. Prove that there is an $\mathcal{L}_A$-sentence φ such that for every $\mathcal{L}_A$-structure $\mathcal{N}$, if $\mathcal{N} \models \varphi$ then $\mathcal{N} \cong \mathcal{M}$.

Exercise 4.2

Fix $A = \{P\}$, where P is a binary function symbol. For each of the following two pairs of $\mathcal{L}_A$-structures, show that they are not elementarily equivalent, by giving a sentence true in one and false in the other.

1. $(\mathbb{R}; \times)$ and $(\mathbb{R}^*; \times^*)$, where x is the usual multiplication operation on the real numbers, $\mathbb{R}^*$ is the set of the non-zero reals, and $\times^*$ is $\times$ restricted to $\mathbb{R}^*$.
2. $(\mathbb{N}; +)$ and $(\mathbb{Z}^+; +^*)$, where $+^*$ is $+$ restricted to the set $\mathbb{Z}^+$ of positive integers.
3. For each of the above structures, give a sentence true in that structure and false in the other three.

Elementary Substructure

Suppose $\mathcal{M} = (M, I)$, $\mathcal{N} = (N, J)$ and $M \subseteq N$.

$\mathcal{M}$ is a substructure of $\mathcal{N}$ iff

For all atomic $\mathcal{L}_A$-formulas φ, and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \quad \Leftrightarrow \quad (\mathcal{N}, \nu) \models \varphi.$$

This inspires the following concept

Definition 4.4 (Elementary substructure)

Let $\mathcal{M}, \mathcal{N}$ be as above. We say that $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$, and write $\mathcal{M} \preccurlyeq \mathcal{N}$, if and only if for all $\mathcal{L}_A$-formulas φ, and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \quad \Leftrightarrow \quad (\mathcal{N}, \nu) \models \varphi.$$

or equivalently, for all $\bar{a} \in \bigcup_{n \in \mathbb{N}} M^n$,

$$\mathcal{M} \models \varphi[\bar{a}] \quad \Leftrightarrow \quad \mathcal{N} \models \varphi[\bar{a}].$$

Remarks

- $\mathcal{M} \preccurlyeq \mathcal{N}$ implies $\mathcal{M} \subseteq \mathcal{N}$.
- $\mathcal{M} \preccurlyeq \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$.

As a consequence, for any finite structure, there is no other elementary substructure besides itself.

- For the converse of both claims, consider $(\mathbb{N}, <)$ and $(\mathbb{Z}^+, <)$.
- When $A = \varnothing$, for every infinite $\mathcal{L}_A$-structure, every infinite subset of its universe forms an elementary substructure.

Exercise 4.3

Let $A = \varnothing$ and $\mathcal{N}$ be the $\mathcal{L}_A$-structure whose universe is $\mathbb{N}$. Show that for every infinite $S \subseteq \mathbb{N}$, the $\mathcal{L}_A$-struture with S being its universe is an elementary substructure of $\mathcal{N}$.

Question

Question 4.5

How do we tell a substructure of $\mathcal{N}$ is an elementary substructure?

The answer is Tarski's criterion, which gives an elegant characterization in terms of definable sets. For that we give a more general notion of definability.

Definability

Definition 4.5

Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_A$-structure.

1. Suppose that $\bar{b} \in M^k$. A set $Y \subseteq M^n$ is **definable in $\mathcal{M}$ with parameter $\bar{b}$** (or **$\bar{b}$-definable in M**) iff there a $\mathcal{L}_A$-formula $\varphi(\bar{x}, \bar{y})$ such that
$$Y = Y_{\varphi, \bar{b}} =_{\text{def}} \{\bar{a} \in M^n \mid \mathcal{M} \models \varphi[\bar{a}, \bar{b}]\}$$
2. Suppose that $X \subseteq M$. A set $Y \subseteq M^n$ is **definable in $\mathcal{M}$ with parameters from X** (or **X-definable in M**) iff $Y = Y_{\varphi, \bar{b}}$ for some $\mathcal{L}_A$-formula $\varphi(\bar{x}, \bar{y})$ and some parameters $\bar{b} \in X^k$.
3. Y is **definable in $\mathcal{M}$ (without parameters)** iff it is $\varnothing$-definable in $\mathcal{M}$.

Definability Theorem

Theorem 4.6

Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_A$-structure and that $X \subseteq M$. Suppose that $Y \subseteq M^n$ is X-definable in $\mathcal{M}$ and that $e : M \to M$ is an automorphism of $\mathcal{M}$. If e fixes X, i.e. for each $b \in X$, $e(b) = b$, then:

$$Y = e[Y] = \{\langle e(a_1), \ldots, e(a_n)\rangle \mid \langle a_1, \ldots, a_n\rangle \in Y\}.$$

PROOF (SKETCH). Extend the language by adding constant symbols for each element $b \in X$, i.e. work with language $\mathcal{L}_{A*}$, where $A^* = A \cup \{c_b \mid b \in X\}$. Any $\mathcal{L}_A$-automorphism over $\mathcal{M}$ fixing X can be viewed as $\mathcal{L}_{A*}$-automorphism over $\mathcal{M}$.

Example 4.6

Suppose that $A = \varnothing$ and $\mathcal{M} = (M, \varnothing)$ is an $\mathcal{L}_A$-structure. Note that any bijection $e : M \to M$ defines an automorphism of $\mathcal{M}$. The following claims follow from the Definability Theorem. Suppose that $D \subseteq M$, then

1. D is definable in M without parameters iff $D = \varnothing$ or $D = M$.
2. D is definable in M from parameters iff D is finite or $M \backslash D$ is finite. (Exercise 4.4-3)

Theorem 4.7 (Tarski's Criterion)

Suppose $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_A$-structures, and $\mathcal{M} \subseteq \mathcal{N}$. Then the following are equivalent:

1. $\mathcal{M} \preccurlyeq \mathcal{N}$
2. *for every M-definable nonempty set $D \subseteq N$, $D \cap M \neq \varnothing$.*

2' *for every $\mathcal{L}_A$-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M^n$, if there is $a \in N$ such that $\mathcal{N} \models \varphi[a, \bar{b}]$, then there is $a' \in M$ such that $\mathcal{N} \models \varphi[a', \bar{b}]$.*

The equivalence 1 $\Leftrightarrow$ 2$'$ is also known as Tarski-Vaught Test.

Proof

We prove ❶ $\Leftrightarrow$ ❷ only.

❶ $\Rightarrow$ ❷: Let $D = \{a \mid \mathcal{N} \models \varphi[a, \bar{b}]\}$, where $\bar{b} \in M^n$ are parameters. This is implies that $\mathcal{N} \models \exists x \varphi[x, \bar{b}]$. By elementarity, $\mathcal{M} \models \exists x \varphi[x, \bar{b}]$, which means $D \cap M \neq \varnothing$.

❷ $\Rightarrow$ ❶: Induction on the ranks of formulas. Since $\mathrm{id} : \mathcal{M} \to \mathcal{N}$ is an isomorphic embedding, by the proof of Homomorphism Theorem, we only need to check the quantifier case $\varphi = \forall x \psi$.

Suppose for all $a \in M$, $\bar{b} \in M^n$,

$$\mathcal{N} \models \psi[a, \bar{b}] \quad \Leftrightarrow \quad \mathcal{M} \models \psi[a, \bar{b}]$$

Since $\mathcal{M} \subseteq \mathcal{N}$,

$$\mathcal{N} \models \forall x \psi[x, \bar{b}] \quad \Rightarrow \quad \mathcal{M} \models \forall x \psi[x, \bar{b}].$$

For the other direction, fix $\bar{b} \in M^n$, let $D = \{a \mid \mathcal{N} \models \neg\psi[a, \bar{b}]\}$. Suppose $\mathcal{N} \models \exists x \neg\psi[x, \bar{b}]$, i.e. $D \neq \varnothing$. By ❷, $D \cap M \neq \varnothing$. This means $\mathcal{M} \models \exists x \neg\psi[x, \bar{b}]$. □

Exercise 4.4

1. Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times)$. Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.
2. Let $A = \{P\}$ where P is a unary predicate symbol. Let $\mathcal{M} = (M, I)$ be the finite $\mathcal{L}_A$-structure with $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$.
   - Which subsets of M are definable in M without parameters.
   - Which subsets of M are definable in M with parameters.
3. Prove the second claim of Example 4.6.

Case study: Dense linear order

We shall study a few examples, try to classify the definable sets of these structures.

Let $A = \{P\}$, where P is a binary predicate symbol. Consider the $\mathcal{L}_A$-structure $(\mathbb{R}, <)$, where $<$ is the usual ordering of the reals.

Suppose $X \subset \mathbb{R}$ is finite. Define for each $a, b \in \mathbb{R}$, $a \sim_X b$ iff there exists an automorphism e of $(\mathbb{R}, <)$ such that e fixes X and $e(a) = b$.

An equivalence relation

The relation $\sim_X$ is an equivalence relation. Namely, for all $x, y, z \in \mathbb{R}$,

1. $x \sim_X x$;
   since the identity map id is an automorphism.
2. if $x \sim_X y$ then $y \sim_X x$;
   since the inverse of an automorphism is an automorphism.
3. if $x \sim_X y$ and $y \sim_X z$ then $x \sim_X z$.
   since the composition of automorphisms is an automorphism.

Intervals

For each $r \in \mathbb{R}$, let

$$[r]_X = \{x \in \mathbb{R} \mid x \sim_X r\}$$

be the equivalence class of r.

Definition 4.6

- **interval**: If $a \leqslant b \leqslant c$ and $a, c \in I$ then $b \in I$.
- **endpoint(s)**: $\sup(I) \cup \inf(I)$.
- Notation: (a, b), $[a, b]$.

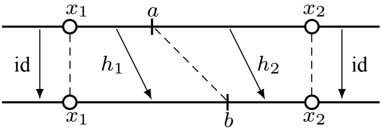
Lemma 4.7

Suppose that $X \subset \mathbb{R}$ is finite. Then for each $a \in \mathbb{R}$, $[a]_X$ is an interval. In fact,

- *if $a \in X$ then $[a]_X = \{a\}$;*
- *if $a \notin X$ then $[a]_X$ is the maximum interval $I \subset \mathbb{R}$ such that $a \in I$ and $I \cap X = \varnothing$.*

Proof of lemma 4.7

- Suppose $a \in X$. Since $a \sim_X b \Rightarrow a = b$, $[a]_X = \{a\}$.
- Suppose $X = \{x_1, \ldots, x_n\}$ and w.l.o.g. assume $a \in (x_1, x_2)$. We show that $[a]_X = (x_1, x_2)$. Suppose $b \in (x_1, x_2)$ and $b \neq a$.



Define

$$h = h_1 \cup h_2 \cup \mathrm{id}\,|_{\mathbb{R}-(x_1,x_2)}.$$

where h_1 is the linear transformation from (x_1, a) to (x_1, b),

$$h_1(t) = \frac{b - x_1}{a - x_1}(t - x_1) + x_1$$

and h_2 is the linear transformation from (a, x_2) to (b, x_2)

$$h_2(t) = x_2 - \frac{x_2 - b}{x_2 - a}(x_2 - t).$$

$a \sim_X b$ via h. So $(x_1, x_2) \subseteq [a]_X$. Since any automorphism sending a to outside (x_1, x_2) moves either x_1 or x_2, it must be that $(x_1, x_2) = [a]_X$. □

Definable subsets of $(\mathbb{R}, <)$

Theorem 4.8

Suppose that $X \subseteq \mathbb{R}$ and $A \subseteq \mathbb{R}$. Then the following are equivalent.

1. *A is X-definable in $(\mathbb{R}, <)$.*
2. *A is a finite union of intervals with endpoints in X.*

Proof.

2 $\Rightarrow$ 1 is clear. For instance,

$$(x_1, x_2) = \{a \mid (\mathbb{R}, <) \models x_1 < a \wedge a < x_2\}.$$

For 1 $\Rightarrow$ 2, suppose A is definable with parameter $\bar{p}$. The key point is that if $a \in A$ then $[a]_{\bar{p}} \subseteq A$, thus $A = \bigcup_{a \in A} [a]_X$. Each $[a]_{\bar{p}}$ is an interval, and there are only finitely many of them. □

Corollary 4.9 (Elementary substructures of $(\mathbb{R}, <)$)

Let $\mathcal{R} = (\mathbb{R}, <)$. Suppose that $M \subseteq \mathbb{R}$ and $\mathcal{M} = (M, <_M)$ is the induced substructure of $\mathcal{R}$. Then the following are equivalent

1. $\mathcal{M} \preccurlyeq \mathcal{R}$.
2. $\mathcal{M}$ *is a dense linear order without endpoints.* (DLO)

Proof.

1 $\Rightarrow$ 2: DLO $\subseteq \mathrm{Th}(\mathcal{M})$ and $\mathcal{M} \equiv \mathcal{R}$, thus $\mathcal{M} \models$ DLO.

2 $\Rightarrow$ 1: Suppose A is $\{m_1, \ldots, m_n\}$-definable in $\mathcal{R}$ and $A \neq \varnothing$. Show $A \cap M \neq \varnothing$. By Theorem 4.8, we may assume that A is an interval.

- $I = \{m_i\}$: $I \subseteq A \cap M$;
- $I = (m_i, m_{i+1})$: use "dense";
- $I = (-\infty, m_1)$ or $I = (m_n, +\infty)$: use "without endpoints". □

$(\mathbb{Q}, <)$ is a DLO, and $\mathbb{Q} \subseteq \mathbb{R}$, so $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$. Thus we have

Corollary 4.10 (Definable subsets of $(\mathbb{Q}, <)$)

Suppose that $X \subseteq \mathbb{Q}$ and that $A \subset \mathbb{Q}$. Then the following are equivalent

1. *A is X-definable in $(\mathbb{Q}, <)$.*
2. *A is a finite union of intervals with endpoints in X.*

Proof sketch (one interval as an example).

Suppose $\varphi(x; r_1, r_2)$ is a definition for $A = (r_1, r_2)$. Then

$$\forall x\, [\varphi(x, r_1, r_2) \leftrightarrow (r_1 < x \wedge x < r_2)].$$

The point is that the formula above is a first order property with parameters in $\mathbb{Q}$, therefore holds in both $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$. □

Some basics of cardinals

Definition 4.7

A set A is **countable** if $A = \varnothing$ or there is an injection $f : A \to \mathbb{N}$.

Assuming AC, this is equivalent to the following definition

Definition

A set A is **countable** if $A = \varnothing$ or there is a surjection $f : \mathbb{N} \to A$.

Intuitively, the surjection f is an enumeration of all the elements of A. We shall always assume AC. The next result follows from (countable) AC.

Theorem 4.11

Suppose that $\{A_i \mid i \in \mathbb{N}\}$ is a countable sequence of countable sets. Then $A = \bigcup_i A_i$ is a countable set.

Example 4.7

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathcal{L}_0$, $\mathcal{L}_A$ (A countable) are countable sets.
- (Cantor) $\mathcal{P}(\mathbb{N})$, $\mathbb{R}$ are uncountable.

Exercise (not assigned)

1. Show that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_A$-structure and X is a countable subset of M, then the collection of all sets $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X is countable. (Hint: Show that there are only countably many formulas and countably many finite sequences from X.)
2. Show that the set of all algebraic numbers in $\mathbb{R}$ is countable. A real r is algebraic if for some $a_0, \ldots, a_{n-1} \in \mathbb{Z}$,
$$(\mathbb{R}; +, \cdot, 0, 1) \models \textstyle\sum_{i=0}^{n} a_i r^i = 0.$$

Theorem 4.12 ((Downward) Lowenheim-Skolem)

Suppose $\mathcal{N} = (N, I)$ *is an* $\mathcal{L}_A$*-structure and* X *is a countable subset of* N*. Then there is countable* M *such that* $X \subseteq M \subseteq N$ *and* $\mathcal{M} = (M, I \upharpoonright M) \preccurlyeq \mathcal{N}$*.*[a]

[a] This is also known as **L-S-T Theorem**, where L-S-T stands for Lowenheim-Skolem-Tarski.

Proof.

The key is to construct a countable sequence $\langle M_k \mid k \in \mathbb{N} \rangle$ of countable subsets of N such that

- $M_0 = X$,
- for each $k \in \mathbb{N}$, $M_k \subseteq M_{k+1}$,
- for each $k \in \mathbb{N}$, if $A \subseteq N$ is M_k-definable in $\mathcal{N}$ and $A \neq \varnothing$, then $A \cap M_{k+1} \neq \varnothing$.

□

Arbitrary dense linear orders

Let $\mathcal{M} = (M, <)$ be an arbitrary DLO.

Question

How to characterize the definable subsets of $\mathcal{M}$?

There are examples of DLO structure $\mathcal{M}$ with the additional property that there is no nontrivial automorphism over $\mathcal{M}$. So it doesn't help to use the method of automorphism to analyze arbitrary DLOs.

However, one can appeal to the Downward L-S-T Theorem.

Countable dense linear order without endpoints

Theorem 4.13

Suppose that $\mathcal{M} = (M, <)$ is a countable dense linear order without endpoints. Then $\mathcal{M} \cong (\mathbb{Q}, <)$.

Proof.

- Enumerate $M = \{m_i\}_i$ and $\mathbb{Q} = \{Q_i\}_i$.
- (Back-and-forth argument) Suppose a finite partial isomorphism $h_n : \mathbb{Q} \to \mathcal{M}$ has been constructed up to step n.
  - Take i_n least i such that $q_i \notin \mathrm{dom}(h)$, find $m \in M$ such that $h' := h_n \cup \{(q_{i_n}, m)\}$ remains to be an isomorphism (on its domain).
  - take j_n least j such that $m_j \notin h'$, find $q \in Q$ such that $h_{n+1} := h' \cup \{(q, m_{j_n})\}$ remains to be an isomorphism.
- Let $h = \bigcup_n h_n$. Verify that h works. □

Definable subsets of arbitrary $\mathcal{M}$

Theorem 4.14

Suppose that $\mathcal{M}$ is a dense linear order without endpoints. Then

1. $\mathcal{M} \equiv (\mathbb{Q}, <)$.
2. *If $X \subseteq M$ and $A \subseteq M$ is X-definable in $\mathcal{M}$, then A is a finite union of intervals with endpoints in X.*

Proof.

Use the Downward Lowenheim Skolem. There is a countable $\mathcal{M}_0 \preccurlyeq \mathcal{M}$. $\mathcal{M}_0 \cong (\mathbb{Q}, <)$, so $\mathcal{M}_0 \equiv (\mathbb{Q}, <)$. Moreover, suppose A is definable with parameter $\bar{p} \in X^m$, choose $\mathcal{M}_0$ such that $\bar{p} \subseteq M_0$, (2) follows from that $\mathcal{M}_0 \preccurlyeq \mathcal{M}$. □

Definable subsets of $(\mathbb{R}, +, \times, <, 0, 1)$

The analysis of the definable sets in familiar mathematical structures can be quite a complicated problem, and one whose resolution involves a deep understanding of those structures.

Theorem 4.15 (Tarski-Seidenberg)

Let $\mathcal{R} = (\mathbb{R}, +, \times, <, 0, 1)$. Suppose $A \subseteq \mathbb{R}$ is definable from parameters in $\mathcal{R}$. Then A is a finite union of intervals.

Exercise 4.5

Consider the structure $\mathcal{R} = (\mathbb{R}, +, \times, <, 0, 1)$. Let $\mathcal{N}$ be the substructure of $\mathcal{R}$ given by the set of all $r \in \mathbb{R}$ such that the set $A_r = \{r\}$ is definable in $\mathcal{R}$ without parameters. Show that $\mathcal{N} \preccurlyeq \mathcal{R}$.

HINT: Use Tarski's Criterion together with the Tarski-Seidenberg Theorem.

Definable subsets of $(\mathbb{R}, +, \times, <, F, 0, 1)$

Question

What about expanded structures of the form

$$\mathcal{R}_F = (\mathbb{R}, +, \times, <, F, 0, 1),$$

where a single function $F : \mathbb{R} \to \mathbb{R}$ is added?

Theorem 4.16

Let $\mathcal{R}_F = (\mathbb{R}, +, \times, <, F, 0, 1)$.

1. *If $F(x) = \sin x$, then there is an $A \subseteq \mathbb{R}$ which is definable in $\mathcal{R}_F$ without parameters such that A is NOT a finite union of intervals.* (Let $A = \{x \in \mathbb{R} \mid \sin x = 0\}$)
2. (Wilkie, 1996) *If $F(x) = \sin \frac{1}{1+x^2}$ or $F(x) = e^x$, then every $A \subseteq \mathbb{R}$ that is definable from parameters in $\mathcal{R}_F$ is a finite union of intervals.*

Definable subsets of $(\mathbb{N}, <)$

Question

What about the definable subsets of the structure $(\mathbb{N}, <)$?

- $(\mathbb{N}, <)$ is a discrete order, and it admits no automorphism.
- First construct a structure $\mathcal{M} = (M, <)$ such that

$$(\mathbb{N}, <) \prec (M, <) \quad \text{and} \quad \mathcal{M} \neq (\mathbb{N}, <)^1.$$

- Then use automorphism of $\mathcal{M}$ is a finite union of intervals.
- Answer: every definable subsets of $\mathbb{N}$ is either finite or co-finite (i.e. its complement is finite).

[1] The existence of such $\mathcal{M}$ follows from **Compactness Theorem**, which is a corollary of the upcoming **Gödel's Completeness Theorem**.