Mathematical Logic

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1. Propositional Logic
2. First order Logic
   - $\mathcal{L}$-formula
   - Semantics

Semantics

Definition 2.1

Suppose $A \subseteq \{c_i \mid i \in \mathbb{N}\} \cup \{F_i \mid i \in \mathbb{N}\} \cup \{P_i \mid i \in \mathbb{N}\}$. A finite sequence φ is an $\mathcal{L}_A$-formula, $\varphi \in \mathcal{L}_A$, iff

1. $\varphi \in \mathcal{L}$,
2. the constant, predicate and function symbols occurring in φ are all in A.

The set, $\mathcal{T}_A$, of $\mathcal{L}_A$-terms can be defined similarly. Both $\mathcal{T}_A$ and $\mathcal{L}_A$ can be defined inductively. This A is often called the **signature** of the language $\mathcal{L}_A$.

$\mathcal{L}_A$-structure

Definition 2.2

An $\mathcal{L}_A$-structure is a pair (M, I) with the following properties:

1. $M \neq \emptyset$,
2. I is a function with $\mathrm{dom}(I) = A$ such that for each $i \in \mathbb{N}$,
    - If $c_i \in A$, then $I(c_i) \in M$;
    - If $F_i \in A$, then $I(F_i)$ is a function
      $$I(F_i) : M^n \to M, \qquad \text{where } n = \pi(F_i).$$
    - If $P_i \in A$, then $I(P_i)$ is a relation
      $$I(P_i) \subseteq M^n, \qquad \text{where } n = \pi(P_i).$$

We shall use $c^{\mathcal{M}}, F^{\mathcal{M}}, P^{\mathcal{M}}$, instead of $I(c), I(F), I(P)$.

Examples

1. Set Theory: $A = \{\hat{\in}\}$.

   equality: Yes
   constant: None
   predicate: $\hat{\in}$ (2-ary)
   function: None

   $(V, \in)$, where V is the class of all sets, $\in$ is the membership relation, is an $\mathcal{L}_A$-structure.

   Here $\forall$ should mean "for all sets", and $\exists$ "there exists a set".

2. Number Theory: $B = \{\hat{0}, \hat{<}, \hat{S}, \hat{+}, \hat{\times}, \hat{E}\}$

equality: Yes
constant: $\hat{0}$
predicate: $\hat{<}$(2-ary)
function: $\hat{S}$ (1-ary); $\hat{+}, \hat{\times}, \hat{E}$ (2-ary)

$(\mathbb{N}, 0, <, S, +, \times, \mathrm{Exp})$, where $\mathbb{N}$ is the set of all natural numbers, and the rest use our natural interpretation, is an $\mathcal{L}_B$-structure. Here "a Exp b" is for a^b.

Here $\forall$ should mean "for all natural numbers", and $\exists$ "there exists a natural number".

3. Group Theory: $C = \{\hat{1}, \hat{+}\}$

equality: Yes
constant: $\hat{1}$
predicate: None
function: $\hat{+}$ (2-ary)

The additive group of integers $(\mathbb{Z}, 0, +)$ and the multiplicative semi-group of integers $(\mathbb{Z}, 1, \times)$ are $\mathcal{L}_C$-structures.

Here $\forall$ should mean "for all integers", and $\exists$ "there exists an integer".

Satisfaction of sentences

$\mathcal{L}_A$-structures can determine the truth values of $\mathcal{L}_A$-sentences ($\mathcal{L}_A$-formulas without free variables).

Definition 2.3

If an $\mathcal{L}_A$-sentence φ is true in an $\mathcal{L}_A$-structure $\mathcal{M}$, we say $\mathcal{M}$ is a **model** of φ, or $\mathcal{M}$ **satisfies** φ, written as $\mathcal{M} \models \varphi$.

Consider the following $\mathcal{L}_{\{\hat{\in}\}}$-sentence:

$$\varphi \equiv \exists x \forall y \neg (y \mathrel{\hat{\in}} x)$$

$(V, \in)$ and $(\mathbb{N}, <)$ are both models of φ. But φ is interpreted differently.

$(V, \in)$: There exists an empty set.

$(\mathbb{N}, <)$: There is a natural number such that no natural number is smaller.

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$(V, \in)$: There exists an empty set.

$(\mathbb{N}, <)$: There is a natural number such that no natural number is smaller.

On the other hand, $(\mathbb{N}, <)$ is not a model of the **paring** axiom:

$$\forall x \forall y \exists z \forall t (t \mathrel{\hat{\in}} z \leftrightarrow t \mathrel{\hat{=}} x \vee t \mathrel{\hat{=}} y)$$

$\mathcal{M}$-assignment

Definition 2.4

Suppose $\mathcal{M} = (M, I)$ is a $\mathcal{L}_A$-structure. A function ν is an $\mathcal{M}$-assignment if ν is a map from $\{x_i \mid i \in \mathbb{N}\}$ to M.

ν can be extended to a function $\check{\nu} : \mathcal{T} \to M$ as follows:

Base step: $\check{\nu}(\langle x_i \rangle) = \nu(x_i)$, and

$$\check{\nu}(\langle c_i \rangle) = I(c_i).$$

Recursion step: If $\tau = F_i(\tau_1 \ldots \tau_n)$, then

$$\check{\nu}(\tau) = I(F_i)(\check{\nu}(\tau_1) \ldots \check{\nu}(\tau_n)).$$

Agreement of $\mathcal{M}$-assignments

Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_A$-structure, τ a $\mathcal{L}_A$-term, φ a $\mathcal{L}_A$-formula, and ν, μ two $\mathcal{M}$-assignments.

Definition 2.5

- Let $\bar{x} = \langle x_{n_1}, \cdots, x_{n_k} \rangle$. μ, ν agree on $\bar{x}$ if $\mu(x_{n_i}) = \nu(x_{n_i})$ for $i \leqslant k$. Write as $\mu|_{\bar{x}} = \nu|_{\bar{x}}$.
- Suppose $\tau(\bar{x})$ is a term with (free) variables $\bar{x}$. ν, μ agree on the (free) variables of τ if $\mu|_{\bar{x}} = \nu|_{\bar{x}}$. Denote it as $\nu \sim_\tau \mu$.
- Suppose $\varphi(\bar{x})$ is a formula with free variables $\bar{x}$. ν, μ agree on the free variables of φ if $\mu|_{\bar{x}} = \nu|_{\bar{x}}$. Denote it as $\nu \sim_\varphi \mu$.

Extending assignment to terms and formulas

Lemma 2.0

Suppose that ν, μ agree on the variables of τ. Then $\check{\mu}(\tau) = \check{\nu}(\tau)$.

Using the unique readability of terms, prove by induction on the construction complexity of τ: for the inductive step, $\tau = F_i(\bar{\sigma})$,

$$\check{\mu}(\tau) = \check{\mu}(F_i(\bar{\sigma})) = I(F_i)(\check{\mu}(\bar{\sigma})) \xlongequal{\text{by ind.}} I(F_i)(\check{\nu}(\bar{\sigma})) = \check{\nu}(F_i(\bar{\sigma})) = \check{\nu}(\tau)$$

Therefore, we shall not distinguish ν and $\check{\nu}$.

By the unique readability of formulas, we define the **satisfaction relation** $(\mathcal{M}, \nu) \models \varphi$ by induction as follows:

Satisfaction

Definition

Atomic cases: Suppose that φ is an atomic formula. $\tau, \sigma, \tau_i \in \mathcal{T}$.

- $\varphi = P_i(\tau_1 \ldots \tau_n)$, $n = \pi(P_i)$.

$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad \langle \check{\nu}(\tau_1), \ldots, \check{\nu}(\tau_n) \rangle \in P_i^{\mathcal{M}}.$$

- $\varphi = (\sigma \hat{=} \tau)$.

$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad \check{\nu}(\sigma) = \check{\nu}(\tau).$$

Inductive cases: Suppose that φ is not an atomic formula. $\psi \in \mathcal{L}$ and $x_i \in \mathfrak{X}$.

- $\varphi = (\neg\psi)$.
$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad (\mathcal{M}, \nu) \not\models \psi.$$
- $\varphi = (\psi_1 \to \psi_2)$.
$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad \text{either } (\mathcal{M}, \nu) \not\models \psi_1 \text{ or } (\mathcal{M}, \nu) \models \psi_2.$$
- $\varphi = (\forall x_i \psi)$.
$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{M}\text{-assignments } \mu, \text{ if } \nu \sim_\varphi \mu \text{ then } (\mathcal{M}, \mu) \models \psi.$$

Satisfaction and free variables

Theorem 2.1

Suppose that $\mathcal{M} = (M, I)$ *is a* $\mathcal{L}_A$*-structure,* φ *is an* $\mathcal{L}_A$*-formula,* $\nu \sim_\varphi \mu$*. Then*

$$(\mathcal{M}, \nu) \models \varphi \quad \textit{iff} \quad (\mathcal{M}, \mu) \models \varphi$$

If φ*. is a sentence, then the satisfaction relation* $(\mathcal{M}, \mu) \models \varphi$ *is independent of* $\mathcal{M}$*-assignments (often written as* $\mathcal{M} \models \varphi$*).*

Satisfaction and free variables

Theorem 2.1

Suppose that $\mathcal{M} = (M, I)$ is a $\mathcal{L}_A$-structure, φ is an $\mathcal{L}_A$-formula, $\nu \sim_\varphi \mu$. Then

$$(\mathcal{M}, \nu) \models \varphi \quad \textit{iff} \quad (\mathcal{M}, \mu) \models \varphi$$

If φ. is a sentence, then the satisfaction relation $(\mathcal{M}, \mu) \models \varphi$ is independent of $\mathcal{M}$-assignments (often written as $\mathcal{M} \models \varphi$).

Prove by induction on the rank (construction complexity) of φ: $\mathrm{rank}(\varphi)$

For the case $\varphi = (\forall x_i \psi)$. We have $\mathrm{rank}(\psi) < \mathrm{rank}(\varphi)$.

- $(\mathcal{M}, \mu) \models \varphi$, by definition, for all $\mathcal{M}$-assignments γ, $\gamma \sim_\varphi \mu$ implies $(\mathcal{M}, \gamma) \models \psi$.
- Since $\nu \sim_\varphi \mu$, we have $(\mathcal{M}, \nu) \models \psi$. Take any $\rho \sim_\psi \nu$, we have $\rho \sim_\varphi \nu \sim_\varphi \mu$, by the ind. hypo., $(\mathcal{M}, \rho) \models \psi$.
- By definition again, $(\mathcal{M}, \nu) \models \varphi$.

Notational conventions

Theorem 2.1 allows the following notational conventions.

Definition 2.6

Let $\mathcal{M} = (M, I)$ is an $\mathcal{L}_A$-structure and $a_1 \dots a_n \in M$, let ν be any $\mathcal{M}$-assignment. Then

1. Write $\bar{\nu}(\tau)$ as $\tau[a_1 \dots a_n]$, for any ν such that $\nu(x_i) = a_i$, $i \leqslant n$, where $\tau = \tau(x_1 \dots x_n) \in \mathcal{T}_A$.
2. Write $\mathcal{M} \models \varphi[a_1 \dots a_n]$, if $(\mathcal{M}, \nu) \models \varphi$, for any ν such that $\nu(x_i) = a_i$, $i \leqslant n$, where $\varphi = \varphi(x_1 \dots x_n) \in \mathcal{L}_A$.

- Another way to define $(\mathcal{M}, \nu) \models \forall x_i \varphi$

  $(\mathcal{M}, \nu) \models \forall x_i \varphi$ iff for every $d \in M$, $(\mathcal{M}, \nu_d^{x_i}) \models \varphi$, where

$$\nu_d^{x_i}(x_j) = \begin{cases} \nu(x_j), & j \neq i; \\ d, & j = i. \end{cases}$$

- $(\mathcal{M}, \nu) \models \varphi \wedge \psi$ iff $(\mathcal{M}, \nu) \models \varphi$ and $(\mathcal{M}, \nu) \models \psi$.

  Similar for $\vee$ and $\leftrightarrow$.
- $(\mathcal{M}, \nu) \models \exists x_i \varphi$ iff $(\mathcal{M}, \nu_d^{x_i}) \models \varphi$, for some $d \in M$.

Substitution

Definition 2.7

1. $\tau(x_1 \dots x_n; \tau_1 \dots \tau_n)$
2. $\varphi(x_1 \dots x_n; \tau_1 \dots \tau_n)$
3. term τ is **free for x_i in** φ:

   (free) variables in τ remain to be free in φ after substitution.

Lemma 2.1

1. $\tau(x_1 \dots x_n; \tau_1 \dots \tau_n)$ *is a term,*
2. $\varphi(x_1 \dots x_n; \tau_1 \dots \tau_n)$ *is a formula.*

Substitution

Definition 2.7

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2. $\varphi(x_1 \dots x_n; \tau_1 \dots \tau_n)$
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   (free) variables in τ remain to be free in φ after substitution.

Lemma 2.1

1. $\tau(x_1 \dots x_n; \tau_1 \dots \tau_n)$ *is a term,*
2. $\varphi(x_1 \dots x_n; \tau_1 \dots \tau_n)$ *is a formula.*

Prove by induction: (1) on terms, (2) on formulas.

Substitution Theorem

Theorem 2.2

Let $\mathcal{M} = (M, I)$ *be an* $\mathcal{L}_A$*-structure,* $\tau_1 \dots \tau_n \in \mathcal{T}_A$ *and* ν *be an* $\mathcal{M}$*-assignment such that* $\check{\nu}(\tau_i) = b_i$, $i \leqslant n$. *Then*

1. *For* $\tau(x_1 \dots x_n) \in \mathcal{T}_A$,
$$\check{\nu}(\tau(x_1 \dots x_n; \tau_1 \dots \tau_n)) = \tau[b_1 \dots b_n].$$
2. *For* $\varphi(x_1 \dots x_n) \in \mathcal{L}_A$*, assuming* $\bar{\tau}$ *are free for* $\bar{x}$ *in* φ,
$$(\mathcal{M}, \nu) \models \varphi(x_1 \dots x_n; \tau_1 \dots \tau_n) \Leftrightarrow \mathcal{M} \models \varphi[b_1 \dots b_n].$$

For short, we shall use $\tau(\bar{x}; \bar{\tau})$, $\varphi(\bar{x}; \bar{\tau})$, $\tau[\bar{b}]$ and $\varphi[\bar{b}]$.
For any φ, write $\varphi^*(\bar{t}) := \varphi(\bar{x}; \bar{\tau}(\bar{t}))$. $(\mathcal{M}, \nu) \models \varphi(\bar{x}; \bar{\tau})$ means $(\mathcal{M}, \nu) \models \varphi^*(\bar{t})$. We show that

$$(\mathcal{M}, \nu) \models \varphi^*(\bar{t}) \quad \text{iff} \quad \mathcal{M} \models \varphi[\bar{b}] \text{ i.e. } \varphi[\nu(\bar{\tau})].$$

Proof of substitution

Induction on the ranks of τ and φ respectively. As an example, we prove for the case: $\varphi = (\psi_1 \to \psi_2)$.

$$\begin{aligned} & (\mathcal{M}, \nu) \models (\psi_1 \to \psi_2)(\bar{x}; \bar{\tau}) \\ \text{iff } & (\mathcal{M}, \nu) \not\models \psi_1(\bar{x}; \bar{\tau}) \text{ or } (\mathcal{M}, \nu) \models \psi_2(\bar{x}; \bar{\tau}) && \text{(defn of } \models\text{)} \\ \text{iff } & \mathcal{M} \models \neg\psi_1[\bar{b}] \text{ or } \mathcal{M} \models \psi_2[\bar{b}] && \text{(ind. hyp.)} \\ \text{iff } & \mathcal{M} \models (\psi_1 \to \psi_2)[\bar{b}] \end{aligned}$$

The non-trivial case is $\varphi = (\forall x_0 \psi)$. Note that

$$\begin{aligned} \varphi^*(\bar{t}) &= (\forall x_0 \psi)(\bar{x}; \bar{\tau}(\bar{t})) \\ &= \forall x_0 \psi(\bar{x}, x_0; \bar{\tau}(\bar{t}), x_0) \\ &= \forall x_0 \psi^*(\bar{t}, x_0) \end{aligned}$$

Here $\bar{t}$ denotes the free variables involved in substitution ($\bar{\tau}$ for $\bar{x}$).

1. Start with $(\mathcal{M}, \nu) \models \varphi^*(\bar{t})$. By the definition of $\models$, for all $\mathcal{M}$-assignments μ, if $\mu \sim_{\bar{t}} \nu$, then $(\mathcal{M}, \mu) \models \psi^*(\bar{t}, x_0)$.
2. By the ind. hyp., for each such μ, $\mathcal{M} \models \psi[\mu(\bar{\tau}), \mu(x_0)]$.
3. As $\mu(x_0)$ is arbitrary, for any $\mathcal{M}$-assignments ρ such that $\rho(\bar{x}) = \bar{b}$, then $(\mathcal{M}, \rho) \models \psi(\bar{x}, x_0)$. Thus $\mathcal{M} \models \varphi[\rho(\bar{x})]$ i.e. $\varphi[\bar{b}]$.

Conversely,

1. from $\mathcal{M} \models \varphi[\bar{b}]$, we have that for all $\mathcal{M}$-assignment ρ such that $\rho(\bar{x}) = \bar{b}$, $(\mathcal{M}, \rho) \models \varphi(\bar{x})$; hence $(\mathcal{M}, \rho) \models \psi(\bar{x}, x_0)$
2. Suppose μ is such that $\mu \sim_{\bar{t}} \nu$. Then $\mu(\bar{\tau}) = \nu(\bar{\tau}) = \bar{b}$.
3. Set ρ_μ such that $\rho_\mu(\bar{x}) = \mu(\bar{\tau})$ and $\rho_\mu(x_0) = \mu(x_0)$. So $\mathcal{M} \models \psi[\rho_\mu(\bar{x}), \rho_\mu(x_0)]$.
4. By the ind. hyp., $(\mathcal{M}, \rho_\mu) \models \psi^*(\bar{t}, x_0)$.
5. As $\mu(x_0)$ and hence $\rho_\mu(x_0)$ are arbitrary, we have $(\mathcal{M}, \rho_\mu) \models \varphi^*(\bar{t})$.

Theorem 2.3

Suppose that

1. $A_1, \dots, A_n$ *are propositional symbols,* $(\bar{A})$
2. θ *is a propositional tautology,*
3. $\varphi_1, \dots, \varphi_n$ *are* $\mathcal{L}_A$ *formulas,* $(\bar{\varphi})$
4. $\psi \equiv \theta(\bar{A}; \bar{\varphi})$ *is the result of substituting for each* i, φ_i *for each occurrence of* A_i *in* θ.

Then $(\mathcal{M}, \nu) \models \psi$ *holds for every* $(\mathcal{M}, \nu)$.

PROOF. Omitted. The key point is that

$$(\mathcal{M}, \nu) \models \psi \quad \text{iff} \quad \theta(\bar{A}; \overline{(\mathcal{M}, \nu) \models \varphi_i})^1 \text{ is true.}$$

[1] This is a truth assignment for $\{A_1, \cdots, A_n\}$.

Satisfaction and the interpretation of constants

Theorem 2.4

Let $\mathcal{M} = (M, I)$ be an $\mathcal{L}_A$-structure, ν be an $\mathcal{M}$-assignment.

Suppose φ is a $\mathcal{L}_A$-formula, $\bar{c} = \langle c_{n_i} : 1 \leqslant i \leqslant k \rangle$ enumerates the constants symbols occuring in φ, and all variables occuring in φ are contained in $\bar{x} = \langle x_i : 1 \leqslant i \leqslant m \rangle$.

Let $\hat{\nu}$ be the $\mathcal{M}$-assignment where

$$\hat{\nu}(x_j) = \begin{cases} \nu(x_j), & \text{if } x_j \neq x_{m+i} \text{ for any } i = 1, \cdots, k \\ I(c_{n_i}), & \text{if } x_j = x_{m+i} \text{ for some } i = 1, \cdots, k \end{cases}$$

Let $\hat{\varphi}$ be the formula obtained by substituting x_{m+i} for all occurrences of c_{n_i} in φ, for all $i = 1, \cdots, k$. Then

$$(\mathcal{M}, \nu) \models \varphi \quad \text{iff} \quad (\mathcal{M}, \hat{\nu}) \models \hat{\varphi}.$$

Exercise 2.1

1. Let $\mathcal{L}_A$ be a language with one binary relation symbol. Give an example of a sentence $\varphi \in \mathcal{L}_A$ and $\mathcal{L}_A$-structures M_1 and M_2 such that $M_1 \models \varphi$ and $M_2 \not\models \varphi$.
2. Do there exist an $\mathcal{L}_A$-structure M, an M-assignment ν, and an $\mathcal{L}_A$-formula φ such that $(M, \nu) \models \varphi$ and $(M, \nu) \models (\neg\varphi)$? Do there exist such M and ν such that $(M, \nu) \not\models \varphi$ and $(M, \nu) \not\models (\neg\varphi)$?
3. Prove Theorem 2.3.