Mathematical Logic

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Part II. First Order Logic

Next

1. Propositional Logic

2. First order Logic
   - $\mathcal{L}$-formula

Predicate and Quantifier

Example 1.1

A proposition P:

$$2 \text{ is a prime.}$$

A predicate $P(x)$:

$$x \text{ is a prime.}$$

P is true. $P(x)$ has no truth value unless

1. x is replaced by a real object, or
2. $P(x)$ is prefixed by quantifiers:

$$\forall x\, P(x), \qquad \exists x\, P(x).$$

First-Order language

- Equality symbol: $\hat{=}$
- Logical symbols: $($ $)$ $\neg$ $\rightarrow$ $\forall$
- Non-logical symbols:
  - variables: $\mathfrak{X} = \{x_i \mid i \in \mathbb{N}\}$
  - constants: $\mathfrak{C} = \{c_i \mid i \in \mathbb{N}\}$
  - functions: $\mathfrak{F} = \{F_i \mid i \in \mathbb{N}\}$
  - predicates: $\mathfrak{P} = \{P_i \mid i \in \mathbb{N}\}$

  In addition, for each $k \in \mathbb{Z}^+$, there are infinitely many F_i's and P_i's with k arguments. Use $\pi : \{F_i \mid i \in \mathbb{N}\} \cup \{P_i \mid i \in \mathbb{N}\} \rightarrow \mathbb{Z}^+$ to denote the arity function.

Let S denote the set of all above symbols.

Terms

Building blocks of formulas for the first-order language are terms.

Definition 1.1

The set of terms, $\mathcal{T}$, is defined as the smallest set of sequences T satisfying the following properties.

1. For each $i \in \mathbb{N}$, $x_i \in T$ and $c_i \in T$.
2. If $F_i \in \mathfrak{F}$, $n = \pi(F_i)$ and $\tau_1, \ldots, \tau_n \in T$, then
$$F_i(\tau_1 \ldots \tau_n) \in T.^{a}$$

[a] $F_i(\tau_1 \cdots \tau_n)$ is the concatenation $\underbrace{F_i}\underbrace{(}\underbrace{\tau_1 \cdots \tau_n}\underbrace{)}$.

Readability of terms

The definition of $\mathcal{T}$ is external, the following is an internal version.

Theorem 1.1

For each $\tau \in \mathcal{T}$, exactly one of the following conditions applies:

1. *There is an $i \in \mathbb{N}$ s.t. $\tau = x_i$ or c_i.*
2. *There is an $i \in \mathbb{N}$ s.t. $\pi(F_i) = n$ and there are $\tau_1, \ldots, \tau_n \in \mathcal{T}$ s.t. $\tau = F_i(\tau_1 \ldots \tau_n)$.*

Moreover, the elements of $\mathcal{T}$ are uniquely readable.

Unique Readability of terms

Theorem 1.2

For each $\tau \in \mathcal{T}$, exactly one of the following conditions applies:

1. *There is an $i \in \mathbb{N}$ s.t. $\tau = x_i$ or c_i.*
2. *There is an $i \in \mathbb{N}$ s.t. $\pi(F_i) = n$ and there are $\tau_1, \ldots, \tau_n \in \mathcal{T}$ s.t. $\tau = F_i(\tau_1 \ldots \tau_n)$.*

Further, in (2), F_i and $\tau_1, \ldots, \tau_n$ are unique.

Lemma 1

If $\tau \in \mathcal{T}$, then no proper initial segment of τ is in $\mathcal{T}$.

As for $\mathcal{L}_0$, prove by induction on $|\tau|$, the length of τ. The key is the first symbol of the sequence for a term for each case.

formulas

Definition 1.2

The set of formulas, $\mathcal{L}$, is the smallest set L of finite sequences of symbols in S such that:

- (Atomic formulas)
  1. If $P_i \in \mathfrak{P}$, $n = \pi(P_i)$ and $\tau_1, \ldots, \tau_n \in \mathcal{T}$, then $P_i(\tau_1 \ldots \tau_n) \in L$.
  2. If $\tau_1, \tau_2 \in \mathcal{T}$, then $(\tau_1 \hat{=} \tau_2) \in L$.
- (Compound formulas)
  1. If $\varphi \in L$, then $(\neg\varphi) \in L$.
  2. If $\varphi_1, \varphi_2 \in L$, then $(\varphi_1 \to \varphi_2) \in L$.
  3. If $\varphi \in L$ and $x_i \in \mathfrak{X}$, then $(\forall x_i \varphi) \in L$.

Readability of formulas

Theorem 1.3 (Readability)

Suppose that $\varphi \in \mathcal{L}$. Then exactly one of the following conditions applies:

1. *There is an $i \in \mathbb{N}$ and $\tau_, \ldots, \tau_n \in \mathcal{T}$ s.t. $\varphi = P_i(\tau_1 \ldots \tau_n)$, where $n = \pi(P_i)$.*
2. *There are $\tau_1, \tau_2 \in \mathcal{T}$ s.t. $\varphi = (\tau_1 \hat{=} \tau_2)$.*
3. *There is a $\psi \in \mathcal{L}$ s.t. $\varphi = (\neg\psi)$.*
4. *There are $\psi_1, \psi_2 \in \mathcal{L}$ s.t. $\varphi = (\psi_1 \to \psi_2)$.*
5. *There is a $\psi \in \mathcal{L}$ and $x_i \in \mathfrak{X}$ s.t. $\varphi = (\forall x_i \psi)$.*

Readability of formulas

Theorem 1.3 (Readability)

Suppose that $\varphi \in \mathcal{L}$. Then exactly one of the following conditions applies:

1. *There is an $i \in \mathbb{N}$ and $\tau_, \ldots, \tau_n \in \mathcal{T}$ s.t. $\varphi = P_i(\tau_1 \ldots \tau_n)$, where $n = \pi(P_i)$.* P_i
2. *There are $\tau_1, \tau_2 \in \mathcal{T}$ s.t. $\varphi = (\tau_1 \hat{=} \tau_2)$.* $(x_i, (c_i$ or $(F_i$
3. *There is a $\psi \in \mathcal{L}$ s.t. $\varphi = (\neg\psi)$.* $(\neg$
4. *There are $\psi_1, \psi_2 \in \mathcal{L}$ s.t. $\varphi = (\psi_1 \to \psi_2)$.* $(P_i$ or $(($
5. *There is a $\psi \in \mathcal{L}$ and $x_i \in \mathfrak{X}$ s.t. $\varphi = (\forall x_i \psi)$.* $(\forall$

Unique Readability of formulas

Theorem 1.4 (Unique Readability)

Further, in each of the above cases, the terms and/or subformulas mentioned in that case are unique.

Again, the following lemma is needed

Lemma 2

If $\varphi \in \mathcal{L}$, then no proper initial segment of φ is in $\mathcal{L}$.

Once again, the key is the leading two symbols of a sequence for a formula in $\mathcal{L}$.

Construction tree for an $\mathcal{L}$-formula

$$(\forall x_1((\neg(\forall x_3 E(x_3, F(x_1, x_3)))) \to (\forall x_2(E(F(x_3, x_2), x_1) \to (\forall x_4(E(x_4, x_2) \to E(x_4, x_1)))))))$$

Construction tree for an $\mathcal{L}$-formula

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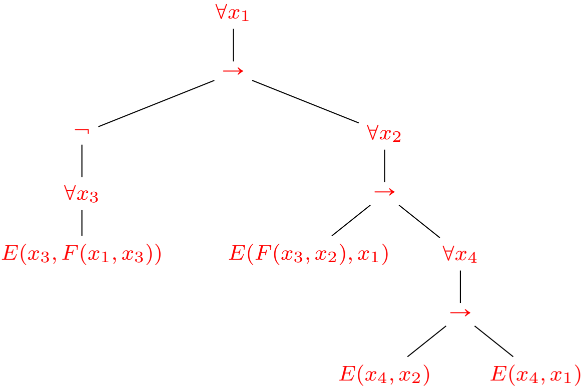

Construction sequence for an $\mathcal{L}$-formula

Definition 1.3

Suppose $\vec{\varphi} = \langle \psi_1, \cdots, \psi_n \rangle$ is a finite sequence of finite sequences. We say $\vec{\varphi}$ is a formula-witness if for all $i \leqslant n$, one of the following holds:

1. ψ_i is an atomic formula,
2. For some $j < i$, $\psi_i = (\neg \psi_j)$,
3. For some $j_1, j_2 < i$, $\psi_i = (\psi_{j_1} \to \psi_{j_2})$,
4. For some $j < i$ and some $k \in \mathbb{N}$, $\psi_i = (\forall x_k \psi_j)$.

Construction sequence for an $\mathcal{L}$-formula

Lemma 3

Suppose that $\vec{\varphi} = \langle \psi_1, \cdots, \psi_n \rangle$ is a formula-witness (for ψ_n). Then for all $i \leqslant n$, $\psi_i \in \mathcal{L}$.

Lemma 4

Suppose φ is a finite sequence. Then the following are equivalent.

1. *φ is a formula,*
2. *There is a formula witness for φ.*

Lemma 5

Suppose $\varphi, \psi \in \mathcal{L}$. Then the following are equivalent.

1. *ψ is a subformula of φ,*
2. *Suppose $\langle \psi_1, \cdots, \psi_n \rangle$ is a formula-witness for φ, then $\psi = \psi_k$ for some $k \leqslant n$.*

Free and bounded variables

Each occurrence of $\forall x_i$ in a formula σ defines a unique subformula of φ.

Lemma 6

Suppose $\varphi \in \mathcal{L}$, $x_i \in \mathfrak{X}$, s, t are finite sequences such that $\varphi = s + \langle(\forall x_i\rangle + t$. Then there is a unique formula $\psi \in \mathcal{L}$ such that $s + \psi$ is an initial segment of φ.

We call the occurrence of this ψ in φ the scope of x_i in φ.

Proof of Lemma 2

- The uniqueness follows from Lemma 2.
- We show the existence. Prove by induction on $|\varphi|$.
- The case $|\varphi| = 1$ is vacuous. Assume the conclusion holds for all formulas of length $< |\varphi|$.
- Three cases:
  - φ is atomic: no occurrence of $\langle(\forall x_i\rangle$.
  - $\varphi = (\neg\varphi_1)$: reduce φ to φ_1.
  - $\varphi = (\varphi_1 \to \varphi_2)$: reduce φ to φ_1 or φ_2.
  - $\varphi = (\forall x_j \varphi_1)$: either reduce φ to φ_1, or $s = \langle\rangle$ (in which case $\psi = \varphi$).

A recursive definition of scope

Define the scope function $\text{scope}(n, \forall x_i, \varphi)$ $(n \geqslant 1)$, the scope of the n-th occurrence of $\forall x_i$ in φ, by recursion on φ:

- If φ is an atomic formula, then $\text{scope}(n, \forall x_i, \varphi) = \langle\rangle$.
- If $\varphi = (\neg\psi)$, then $\text{scope}(n, \forall x_i, \varphi) = \text{scope}(n, \forall x_i, \psi)$.

- If $\varphi = (\psi_1 \rightarrow \psi_2)$, then
  - If the n-th occurrence of $\forall x_i$ is in ψ_1,
  $$\text{scope}(n, \forall x_i, \varphi) = \text{scope}(n, \forall x_i, \psi_1).$$
  - If the n-th occurrence of $\forall x_i$ is in ψ_2,
  $$\text{scope}(n, \forall x_i, \varphi) = \text{scope}(n - k, \forall x_i, \psi_2),$$
  where k is the number of occurrences of $\forall x_i$ in ψ_1.
  - Otherwise, $\text{scope}(n, \forall x_i, \varphi) = \langle\rangle$.

- If $\varphi = (\forall x_j \psi)$, then

$$\text{scope}(n, \forall x_i, \varphi) = \begin{cases} \varphi, & \text{if } n = 1 \text{ and } i = j; \\ \text{scope}(n-1, \forall x_i, \psi), & \text{if } n > 1 \text{ and } i = j; \\ \text{scope}(n, \forall x_i, \psi), & \text{if } i \neq j. \end{cases}$$

Definition 1.4

1. An occurrence of x_i in φ is free if x_i does not occur within the scope of any occurrence of $\forall x_i$ in φ. Otherwise, the occurrence is called bounded.
2. x_i is a free variable if there is a free occurrence of x_i in φ.
3. x_i is bounded variable if it occurs in φ but no free occurrence.

Definition 1.5

A formula φ is a sentence iff it has no free variables.

On notation

- $\vee$, $\wedge$, $\leftrightarrow$ and $\exists$ are used as abbreviations.
- Parentheses may be dropped whenever it causes no confusion, for instance, the outmost parentheses can always be dropped.
- When parentheses are dropped, connectives and quantifiers subject to the following priorities:

$$(\neg, \forall x_i, \exists x_j) \qquad (\vee, \wedge) \qquad (\rightarrow, \leftrightarrow).$$

On notation

- Symbols of the same priority group are ordered according to their occurrence, the one appears at the right has higher priority. For example,

$$\exists x_1 \neg P(x_1, x_2) \to \forall x_2 Q(x_1, x_2)$$

abbreviates

$$(((\neg(\forall x_1(\neg(\neg P(x_1, x_2)))))) \to (\forall x_2 Q(x_1, x_2)))$$

Exercise 1.1

1. Prove the Unique Readability Theorem for $\mathcal{L}$ (Theorem 1.4).
2. Consider the set of sequences defined as in Definition 1.2 except that the last clause is changed to read,

   $$\text{"If } \varphi \in L \text{ and } x_i \in \mathfrak{X}, \text{ then } \forall x_i \varphi \in L\text{"}$$

   in which the parentheses are omitted. Is this set uniquely readable?
3. Consider the set of sequences defined as in Definition 1.2 except that the last clause is changed to read,

   $$\text{"If } \varphi_1, \varphi_2 \in L, \text{ then } \varphi_1 \to \varphi_2 \in L\text{"}$$

   in which the parentheses are omitted. Is this set uniquely readable?