Deduction Exercises

1 Propositional Logic

Exercise 1. Show that

1. $\vdash \neg\neg\alpha \to \alpha$
2. $\vdash (\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
3. $\vdash \alpha \leftrightarrow \neg\neg\alpha$
4. $\vdash (\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$
5. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \beta$, then $\Gamma \vdash \beta$.

SOLUTION. 1. The following are the main steps:

- By Δ_0-6, we have $\vdash \neg\neg\alpha \to (\neg\alpha \to \alpha)$.
- By Δ_0-5, $\vdash (\neg\alpha \to \alpha) \to \alpha$.
- Applying the Deduction theorem, we get $\vdash \neg\neg\alpha \to \alpha$.

2. It is equivalent to show that

$$\{\alpha \to \neg\beta, \beta\} \vdash \neg\alpha.$$

We proceed in five steps:

- By Δ_0-3, $\{\beta\} \vdash \neg\neg\alpha \to \beta$.
- Using (1), we have $\{\alpha \to \neg\beta\} \vdash \neg\neg\alpha \to \neg\beta$.
- From Δ_0-4, we get $\vdash \beta \to (\neg\beta \to \neg\alpha)$.
- Combine the above 3 steps. We get $\{\alpha \to \neg\beta, \beta\} \vdash \neg\neg\alpha \to \neg\alpha$.
- Δ_0-5 says $\vdash (\neg\neg\alpha \to \neg\alpha) \to \neg\alpha$.

Therefore, combining the last two results, we have $\{\alpha \to \neg\beta, \beta\} \vdash \neg\alpha$.

3. We are left to show that $\vdash \alpha \to \neg\neg\alpha$.

- By Δ_0-2, $\vdash \neg\alpha \to \neg\alpha$.
- From (2), we have $\vdash (\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$.

Hence $\vdash \alpha \to \neg\neg\alpha$.

4. There are two parts:

(a) $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

(b) $\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

(a). From (2), we have $\vdash (\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$. Using (3), we have

$$\vdash (\alpha \to \beta) \to \alpha \to \neg\neg\beta.$$

Thus $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

(b). From (2) and Deduction, we have

$$\{\neg\beta \to \neg\alpha, \alpha\} \vdash \neg\neg\beta.$$

Since $\vdash \neg\neg\beta \to \beta$, we get $\{\neg\beta \to \neg\alpha, \alpha\} \vdash \beta$, and use Deduction again,

$$\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta).$$

5. (3) is the principle of **Double Negation**, (4) is **Contraposition**. By (3) and (4), the assumption can be restated as: $\Gamma \vdash \neg\beta \to \neg\alpha$ and $\Gamma \vdash \neg\alpha \to \beta$. Combining them, we have $\Gamma \vdash \neg\beta \to \beta$. By Δ_0-5, $\vdash (\neg\beta \to \beta) \to \beta$. So $\Gamma \vdash \beta$. □

2 First Order Logic

Exercise 2. Suppose x does not occur freely in α. Show that

1. $\vdash (\forall x\beta(x) \to \alpha) \leftrightarrow \exists x(\beta(x) \to \alpha)$
2. $\vdash (\alpha \to \exists x\beta(x)) \leftrightarrow \exists x(\alpha \to \beta(x))$

SOLUTION. The following tautology in Propositional logic will be frequently used in our proofs:
CLAIM. $\{\varphi, \neg\psi\} \vdash\dashv \{\neg(\varphi \to \psi)\}$.

Proof of Claim. The direction from left to right follows from Δ_0-7,

$$\vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi)).$$

For the other direction, we need the following consequences of Δ_0-6 and Δ_0-3:

$$\{\neg\varphi\} \vdash (\varphi \to \psi) \quad \text{and} \quad \{\psi\} \vdash (\varphi \to \psi)$$

Then the other direction can be obtained by applying Contraposition and Double Negation. ⊣

Now we are ready to work with the exercises.

1. We first show the direction "$\to$":

$$\vdash (\forall x\beta(x) \to \alpha) \to \neg\forall x\neg(\beta(x) \to \alpha).$$

By Contraposition, Double Negation and Deduction, it suffices to show that

$$\{\forall x\neg(\beta(x) \to \alpha)\} \vdash \neg(\forall x\beta(x) \to \alpha)$$

By the claim, it suffices to show that

- $\{\forall x\neg(\beta(x) \to \neg\alpha)\} \vdash \forall x\beta(x)$, and
- $\{\forall x\neg(\beta(x) \to \alpha)\} \vdash \neg\alpha$.

Since x is substitutable for x in $\neg(\beta(x) \to \neg\alpha)$, by Δ-2,

$$\{\forall x \neg(\beta(x) \to \alpha)\} \vdash \neg(\beta(x) \to \alpha).$$

Using the claim again, we get $\{\neg(\beta(x) \to \alpha)\} \vdash \{\beta(x), \neg\alpha\}$. Consequently, we have $\{\forall x \neg(\beta(x) \to \alpha)\} \vdash \{\beta(x), \neg\alpha\}$. As x does not occur freely in $\forall x \neg(\beta(x) \to \alpha)$, using Generalization, we obtain $\{\forall x \neg(\beta(x) \to \alpha)\} \vdash \{\forall x \beta(x), \alpha\}$. This completes the proof of the direction "$\to$".

Next we show the other direction:

$$\vdash \neg\forall x \neg(\beta(x) \to \alpha) \to (\forall x \beta(x) \to \alpha)$$

By Contraposition, Double Negation and Deduction, it suffices to show that

$$\{\neg(\forall x \beta(x) \to \alpha)\} \vdash \forall x \neg(\beta(x) \to \alpha)$$

By the claim, we have

$$\{\neg(\forall x \beta(x) \to \alpha)\} \vdash \{\forall x \beta(x),\ \neg\alpha\}.$$

Since x is substitutable for x in $\beta(x)$, from the first one we have $\{\neg(\forall x \beta(x) \to \alpha)\} \vdash \beta(x)$. Since $\{\beta(x), \neg\alpha\} \vdash \neg(\beta(x) \to \alpha)$, we have

$$\{\neg(\forall x \beta(x) \to \alpha)\} \vdash \neg(\beta(x) \to \alpha).$$

As x is no long free in the hypothesis, by Generalization, we have

$$\{\neg(\forall x \beta(x) \to \alpha)\} \vdash \forall x \neg(\beta(x) \to \alpha).$$

This problem is finished.

2. First we expand the symbol $\exists$:

$$\vdash (\alpha \to \neg\forall x \neg\beta(x)) \leftrightarrow \neg\forall x \neg(\alpha \to \beta(x))$$

We first show from left to right, i.e.

$$\{\alpha \to \neg\forall x \neg\beta(x)\} \vdash \neg\forall x \neg(\alpha \to \beta(x))$$

By Contraposition and Double Negation, we show $\{\forall x \neg(\alpha \to \beta(x))\} \vdash \neg(\alpha \to \neg\forall x \neg\beta(x))$. By the claim, we only need to show that

$$\{\forall x \neg(\alpha \to \beta(x))\} \vdash \{\alpha,\ \forall x \neg\beta(x)\}$$

Since x is substitutable for x in $\neg(\alpha \to \beta(x))$, we immediately have

$$\{\forall x \neg(\alpha \to \beta(x))\} \vdash \{\alpha,\ \neg\beta(x)\}.$$

Since x is not free in the left, the latter can be generalized. Thus we have

$$\{\forall x \neg(\alpha \to \beta(x))\} \vdash \{\alpha,\ \forall x \neg\beta(x)\}.$$

Now we look at the other direction:

$$\{\neg\forall x \neg(\alpha \to \beta(x))\} \vdash \alpha \to \neg\forall x \neg\beta(x)$$

By Contraposition and Double Negation, we consider

$$\{\neg(\alpha \to \neg\forall x \neg\beta(x))\} \vdash \forall x \neg(\alpha \to \beta(x))$$

By the claim, the hypothesis is equivalent to $\{\alpha, \forall x \neg\beta(x)\}$. Again since x is substitutable for x in $\neg\beta(x)$, we get $\{\alpha, \forall x \neg\beta(x)\} \vdash \neg\beta(x)$. Consequently,

$$\{\alpha, \forall x \neg\beta(x)\} \vdash \neg(\alpha \to \beta(x)).$$

x is not free in the hypothesis, so by Generalization,

$$\{\alpha, \forall x \neg\beta(x)\} \vdash \forall x \neg(\alpha \to \beta(x)).$$

This completes the second problem. □