Mathematical Logic

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Abstact logic

An **abstract logic** is a pair $\mathcal{L} = (\mathcal{S}, \models)$, where

- $\mathcal{S}$ is a set (of sentences of $\mathcal{L}$),
- $\models$ is a relation between arbitrary structure and elements of $\mathcal{S}$ (intuitively, a truth predicate),

such that $\mathcal{L}$ has the following closure properties: letting A be any vocabulary set,

$$\mathcal{S}_A = \{\varphi \mid \varphi \text{ is an } A\text{-sentence}\}$$

$$\mathrm{St}(A) = \{\mathcal{M} \mid \mathcal{M} \text{ is an } A\text{-structure}\}$$

$$\mathrm{Mod}_A^{\mathcal{L}}(\varphi) = \{\mathcal{M} \in \mathrm{St}(A) \mid \mathcal{M} \models \varphi\}$$

Abstact logic

1. *closed under negation*:
   - if $\varphi \in \mathcal{S}_A$ then $\neg\varphi \in \mathcal{S}_A$, and
   - $\mathrm{Mod}^{\mathcal{L}}_A(\varphi) = \mathrm{St}(A) \backslash \mathrm{Mod}^{\mathcal{L}}_A(\neg\varphi)$;
2. *closed under conjunction*:
   - if $\varphi, \psi \in S_A$, then $\varphi \wedge \psi \in S_A$, and
   - $\mathrm{Mod}^{\mathcal{L}}_A(\varphi \wedge \psi) = \mathrm{Mod}^{\mathcal{L}}_A(\varphi) \cap \mathrm{Mod}^{\mathcal{L}}_A(\psi)$;
3. *closed under existential quantification*: for any constant symbol $c \in A$, for any $\varphi \in \mathcal{S}_A$, there is a $\varphi' \in \mathcal{S}_A$ s.t. $\mathrm{Mod}^{\mathcal{L}}_{A\backslash\{c\}}(\varphi') = \{\mathcal{M} \mid (\mathcal{M}, c^{\mathcal{M}}) \in \mathrm{Mod}^{\mathcal{L}}_A(\varphi)$ for some $c^{\mathcal{M}} \in \mathcal{M}\}$.

4. *closed under renaming*: suppose $\pi : A \to A'$ is a renaming function, then π can be canonically extended to $\pi' : \mathrm{St}(A) \to \mathrm{St}(A')$, and for all $\varphi \in \mathcal{S}_A$ there is a $\varphi' \in \mathcal{S}_{A'}$,
$$\mathrm{Mod}^{\mathcal{L}}_{A'}(\varphi) = \{\pi'(\mathcal{M}) \mid \mathcal{M} \in \mathrm{Mod}^{\mathcal{L}}_{A}(\varphi)\}.$$
5. *closed under free expansions*: if whenever $A \subseteq A'$ and $\varphi \in \mathcal{S}_A$, there is a $\varphi' \in \mathcal{S}_{A'}$ such that $\mathrm{Mod}^{\mathcal{L}}_{A}(\varphi) = \mathrm{Mod}^{\mathcal{L}}_{A'}(\varphi')$.
6. *closed under isomorphism*: whenever $\varphi \in \mathcal{S}_A$, $\mathcal{M} \in \mathrm{Mod}^{\mathcal{L}}_{A}(\varphi)$ and $f : \mathcal{M} \cong \mathcal{N}$, then $\mathcal{N} \in \mathrm{Mod}^{\mathcal{L}}_{A}(\varphi)$.

Comparing logics

Definition

- An abstract logic $\mathcal{L} = (\mathcal{S}, \models)$ is a **sublogic** of another abstract logic $\mathcal{L}' = (\mathcal{S}', \models')$, denoted as $\mathcal{L} \leqslant \mathcal{L}'$, if for any $\varphi \in \mathcal{S}$, there is a $\varphi' \in \mathcal{S}'$ such that[a]

$$\mathrm{Mod}_A^{\mathcal{L}}(\varphi) = \mathrm{Mod}_{A'}^{\mathcal{L}'}(\varphi').$$

- Two logics $\mathcal{L}$ and $\mathcal{L}'$ are **equivalent** if $\mathcal{L} \leqslant \mathcal{L}'$ and $\mathcal{L}' \leqslant \mathcal{L}$.

[a] writing $A = A(\mathcal{S})$ and $A' = A(\mathcal{S}')$.

Intuitively, this is defining the notion of comparing expressive powers of logics.

Compactness and LST properties

Definition

- An abstract logic $\mathcal{L} = (\mathcal{S}, \models)$ satisfies the **(Countable) Compactness Property** if for any A, for any (countable) $\Sigma \subseteq \mathcal{S}_A$,

  For every finite $\Sigma_0 \subseteq \Sigma$

  $$\bigcap_{\varphi \in \Sigma_0} \mathrm{Mod}_A^{\mathcal{L}}(\varphi) \neq \varnothing \implies \bigcap_{\varphi \in \Sigma} \mathrm{Mod}_A^{\mathcal{L}}(\varphi) \neq \varnothing.$$
- An abstract logic $\mathcal{L} = (\mathcal{S}, \models)$ satisfies the **Downward Löwenheim-skolem Property** if for any countable A, every nonempty $\mathrm{Mod}_A^{\mathcal{L}}(\varphi)$, $\varphi \in \mathcal{S}$, contains a countable model.

$\mathcal{L}_{\omega,\omega}$ is the strongest

Let $\mathcal{L}_{\kappa,\lambda}$ denote the language allowing

- $<\kappa$ many conjunction/disjuctions, and
- $<\lambda$ many universal/exitential quantifications.

Then $\mathcal{L}_{\omega,\omega} =$ the first order logic.

Lindström Theorem, 1969

Let $\mathcal{L}^*$ be an abstract logic such that $\mathcal{L}_{\omega,\omega} \leqslant \mathcal{L}^*$. If $\mathcal{L}^*$ satisfies

1. (CCP) the Countable Compactness Property, and
2. (DLP) the Downward Löwenheim-Skolem Property,

then $\mathcal{L}^* \equiv \mathcal{L}_{\omega,\omega}$, i.e. $\mathcal{L}^*$ has the same expressive power as first-order logic $\mathcal{L}_{\omega,\omega}$.

Proof

- Suppose $\mathcal{L}^* = (\mathcal{S}^*, \models^*)$ satisfies CCP and DLP, and some $\varphi \in \mathcal{S}$ is not $\mathcal{L}_{\omega,\omega}$-definable, i.e. for any A, there is no $\psi \in \mathcal{L}_{\omega,\omega}$ such that $\mathrm{Mod}_A^{\mathcal{L}^*}(\varphi) = \mathrm{Mod}_A^{\mathcal{L}_{\omega,\omega}}(\psi)$.

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- We may assume that A is finite and relational.

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- We may assume that A is finite and relational.
- For each $n < \omega$, there are only finitely many (logically non-equivalent) first order A-sentences of quantifier rank $\leqslant n$, ψ_i^n, $i < k_n$.

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- We may assume that A is finite and relational.
- For each $n < \omega$, there are only finitely many (logically non-equivalent) first order A-sentences of quantifier rank $\leqslant n$, ψ_i^n, $i < k_n$.
- Call two A-structures n-equivalent if they satisfy the same ψ_i^n's.

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- We may assume that A is finite and relational.
- For each $n < \omega$, there are only finitely many (logically non-equivalent) first order A-sentences of quantifier rank $\leqslant n$, ψ_i^n, $i < k_n$.
- Call two A-structures n-equivalent if they satisfy the same ψ_i^n's.
- There are only 2^{k_n} many different n-equivalence classes and each is first order definable.

Proof, cont'd

- Since φ is not first order definable, for each n, one can find $\mathcal{M}_n$ and $\mathcal{N}_n$ such that
  "$\mathcal{M}_n \models^* \varphi$, $\mathcal{N}_n \models^* \neg\varphi$ and they are n-equivalent". (†)
- By Ehrenfeucht and Fraisse, two models $\mathcal{M}, \mathcal{N}$ are n-equivalent iff there are relations I_i, $i < n$, such that
  1. $I_0(\langle\rangle, \langle\rangle)$;
  2. $I_i(\bar{a}, \bar{b})$ implies that $\bar{a} \in M^i$ and $\bar{b} \in N^i$;
  3. If $I_i(\bar{a}, \bar{b})$, then for every $a_i \in M$ ($b_i \in N$), there is a $b_i \in N$ ($a_i \in M$) such that $I_{i+1}(\bar{a}^\frown a_i, \bar{b}^\frown b_i)$;
  4. If $I(\bar{a}, \bar{b})$, then for every atomic formula $\varphi(\bar{x})$, $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{N} \models \varphi[\bar{b}]$.

  If there are such relations I_i, $i < \omega$, then $\mathcal{M}$ and $\mathcal{N}$ are ω-equivalent.

Proof, cont'd

- By a back-and-forth argument, if $\mathcal{M}$ and $\mathcal{N}$ are countable and ω-equivalent, then $\mathcal{M} \cong \mathcal{N}$.
- Code (†) into a $\psi(n) \in \mathcal{S}^*$.
- By CCP, there is a nonstandard model of $\psi(n)$ in which n is nonstandard.
- Via the coding, one gets two other models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \models \varphi$, $\mathcal{N} \models \neg\varphi$ and they are ω-equivalent.
- By DLP, we may assume that they are countable. But then we have a contradiction! □