Mathematical Logic

Xianghui Shi

School of Mathematical Science
Beijing Normal University

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Gödel's Incompleteness Theorem

– An over-simplified introduction –

Hilbert's 2-nd problem, 1900

Is it possible to prove that arithmetic is consistent – free of any internal contradictions? (Need a finitistic proof)

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Gödel's Incompleteness Theorem, 1931

The answer is NO.

Set A_E of Axioms

$$\forall x(Sx \neq 0) \qquad S1$$

$$\forall x \forall y(Sx = Sy \to x = y) \qquad S2$$

$$\forall x \forall y(x < Sy \iff x \leqslant y) \qquad L1$$

$$\forall x(x \nless 0) \qquad L2$$

$$\forall x \forall y(x < y \lor x = y \lor y < x) \qquad L3$$

$$\forall x(x + 0 = x) \qquad A1$$

$$\forall x \forall y(x + Sy = S(x + y)) \qquad A2$$

$$\forall x(x \cdot 0 = 0) \qquad M1$$

$$\forall x \forall y(x \cdot Sy = x \cdot y + x) \qquad M2$$

$$\forall x(x^0 = S0) \qquad E1$$

$$\forall x \forall y(x^{Sy} = x^y \cdot x) \qquad E2$$

Representable in $\mathrm{Cn}(A_E)$

Definition 6.1

Let T be any theory in a language $\mathcal{L}$ containing $\{\hat{0}, \hat{S}\}$. Let $\varphi(\bar{x}) \in \mathcal{L}$ and a set $R \subseteq \mathbb{N}^n$. Then **φ represents R in T** iff for every $\bar{a} \in \mathbb{N}^n$,

- $\bar{a} \in R \implies T \vdash \varphi(\bar{a})$,
- $\bar{a} \notin R \implies T \vdash \neg\varphi(\bar{a})$.

A set R on $\mathbb{N}$ is **representable** in T if the above φ exists.

Thus φ represent R in $\mathrm{Th}(\mathcal{N})$ iff φ defines R in $\mathcal{N}$.

We shall be interested in objects representable in $\mathrm{Cn}(A_E)$, where $\mathrm{Cn}(T) =_{\text{def}} \{\sigma \in \mathcal{S}_{A_E} \mid T \vdash \sigma\}$.

Gödelnumbering

Assign numbers to symbols:

Parameters		Logical symbols	
0.	$\forall$	1.	(
2.	$\hat{0}$	3.	)
4.	$\hat{S}$	5.	$\neg$
6.	$\hat{<}$	7.	$\to$
8.	$\hat{+}$	9.	$=$
10.	$\hat{\times}$	11.	x_1
12.	$\hat{E}$	13.	x_2

For every x_i, $\ulcorner x_i \urcorner = 9 + 2i$. Define

$$\ulcorner s_0 \cdots s_n \urcorner = p_0^{\ulcorner s_0 \urcorner + 1} \cdots p_n^{\ulcorner s_n \urcorner + 1}$$

where p_i is the i-th prime number.

The following sets are representable in $\mathrm{Cn}(A_E)$.

- $T_1 = \{\ulcorner x_i \urcorner \mid x_i \text{ is a variable}\}$
- $T_2 = \{\ulcorner \tau \urcorner \mid \tau \text{ is a term}\}$
- $T_3 = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a well formed formula}\}$
- $T_4 = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a sentence}\}$
- $T_5 = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a logical axiom}\}$
- $T_6 = \{\langle \ulcorner \theta_0 \urcorner, \cdots, \ulcorner \theta_n \urcorner \rangle \mid \langle \theta_0, \cdots, \theta_n \rangle \text{ is a proof from } A\}$, where A is a set of formulas such that $\sharp A =_{\text{def}} \{\ulcorner \sigma \urcorner \mid \sigma \in A\}$ is representable in $\mathrm{Cn}(A_E)$.

Hierarchy of formulas

Let

$$\Sigma_k =_{\text{def}} \{\exists x_1 \forall x_2 \cdots \Box\, x_k\, \varphi \mid \varphi \text{ is quantifier free}\}$$

$$\Pi_k =_{\text{def}} \{\forall x_1 \exists x_2 \cdots \Box\, x_k\, \varphi \mid \varphi \text{ is quantifier free}\}$$

where $\Box$ is $\exists$ when k is odd, and $\forall$ when k is even.

We say a set R on $\mathbb{N}$ is Σ_k when R can be defined by a Σ_k formula. Similar for Π_k sets. If a set is both Σ_k and Π_k, we call it Δ_k. A set is **definable** iff it has a Σ_k (Π_k) definition for some k.

We shall not distinguish sets and their definitions. So $\Sigma_k, \Pi_k, \Delta_k$ often refer to the collections of Σ_k-relations, Π_k-relations, Δ_k-relations, respectively.

Definability and Representability

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Church's Thesis

A set R on $\mathbb{N}$ is decidable iff R is recursive[a] iff R is representable in $\mathrm{Cn}(A_E)$.

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So all the T's in the previous slide are recursive, Δ_1. Given a set of axioms A with $\sharp A$ recursive, $\sharp\mathrm{Cn}(A)$ is a Σ_1 subset of $\mathbb{N}$. If $\mathrm{Cn}(A)$ is complete, then $\sharp\mathrm{Cn}(A)$ is Δ_1, i.e. recursive.

A_E is Σ_1-correct

Theorem

For every Σ_1 sentence $\exists x\, \varphi$, $\mathcal{N} \models \exists x\, \varphi \implies A_E \vdash \exists x\, \varphi$.

<u>Observation</u>. For every quantifier free sentence σ,

$$\mathcal{N} \models \sigma \implies A_E \vdash \sigma.$$

[1] In other word, $\mathrm{Con}(A_E + \mathsf{GC}) \implies \mathcal{N} \models \mathsf{GC}$. However, this doesn't reduce the difficulty of the problem, since $\mathbb{N}$ is the standard part of every model of A_E.

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If $\mathcal{N} \models \exists x\,\varphi$, then for some $m \in \mathbb{N}$, $\mathcal{N} \models \varphi(\hat{S}^m\hat{0})$. $\varphi(\hat{S}^m\hat{0})$ is quantifier free, therefore $A_E \vdash \varphi(\hat{S}^m\hat{0})$, hence $A_E \vdash \exists x\,\varphi$. ☒

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Note that Goldbach's Conjecture (GC) is a Π_1 sentence. So if GC is false, then one can prove ¬GC from A_E.[1]

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Fix-point Lemma

Lemma 6.0

Given any $\varphi(x) \in \mathcal{L}_{A_E}$, *there is a* $\sigma \in \mathcal{S}_{A_E}$ *such that*

$$A_E \vdash \sigma \leftrightarrow \varphi(\ulcorner \sigma \urcorner).$$

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Let $\vartheta(x, y, z)$ represent in $\mathrm{Cn}(A_E)$ the function $f(\ulcorner\alpha\urcorner, n) = \ulcorner\alpha(n)\urcorner$.
Consider $\tau(x) =_{\text{def}} \forall z\, [\vartheta(x, x, z) \to \varphi(z)]$. Let σ be $\tau(\ulcorner\tau\urcorner)$. We show that $\mathcal{N} \models \sigma \leftrightarrow \varphi(\ulcorner\sigma\urcorner)$:
"$\to$": $\sigma \vdash \vartheta(\ulcorner\tau\urcorner, \ulcorner\tau\urcorner, \ulcorner\sigma\urcorner) \to \varphi(\ulcorner\sigma\urcorner)$. (let $z = \ulcorner\sigma\urcorner$) $A_E \vdash \vartheta(\ulcorner\tau\urcorner, \ulcorner\tau\urcorner, \ulcorner\sigma\urcorner)$, thus $A_E \cup \{\sigma\} \vdash \varphi(\ulcorner\sigma\urcorner)$.
"$\leftarrow$": By definition of ϑ, $A_E \vdash \forall z\, [\vartheta(\ulcorner\tau\urcorner, \ulcorner\tau\urcorner, z) \to z = \ulcorner\sigma\urcorner]$.
Therefore $A_E \cup \{\varphi(\ulcorner\sigma\urcorner)\} \vdash \forall z[\vartheta(\ulcorner\tau\urcorner, \ulcorner\tau\urcorner, z) \to \varphi(z)]$. ⊠

Undefinability and Incompleteness

Tarski Undefinability Theorem, 1933

The set $\sharp\mathrm{Th}(\mathcal{N})$ is not definable in $\mathcal{N}$.

As a corollary, $\sharp\mathrm{Th}(\mathcal{N})$ is not recursive.

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Towards a contradiction assume $\varphi \in \mathcal{L}_{A_E}$ defines $\sharp\mathrm{Th}(\mathcal{N})$. Apply Fix-point Lemma to $\neg\varphi$, there is a $\sigma \in \mathcal{S}_{A_E}$ s.t. $\mathcal{N} \models \sigma \leftrightarrow \neg\varphi(\ulcorner\sigma\urcorner)$.[a] Then

$$\mathcal{N} \models \sigma \quad \Leftrightarrow \quad \mathcal{N} \models \neg\varphi(\ulcorner\sigma\urcorner) \quad \Leftrightarrow \quad \mathcal{N} \not\models \sigma$$

Contradiction! ☒

[a] So σ says "φ is false of me", i.e. "I am false".

Gödel's Incompleteness Theorem, 1931

If $A \subseteq \mathrm{Th}(\mathcal{N})$ and $\sharp A$ is recursive, then $\mathrm{Cn}(A)$ is not a complete theory.

If $\mathrm{Cn}(A)$ is complete, $\mathrm{Cn}(A) = \mathrm{Th}(\mathcal{N})$. $\sharp A$ is recursive, so is $\sharp\mathrm{Cn}(A)$. Therefore $\sharp\mathrm{Th}(\mathcal{N})$ is definable in $\mathcal{N}$. Contradiction! $\boxtimes$

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Another way to put this:

$\mathrm{Th}(\mathcal{N})$ can not have a recursive set of axioms.

The next theorem says that one can not extend A_E to a complete theory by adding axioms recursively.

Strong Undecidability of $\mathrm{Cn}(A_E)$

Let T be a theory such that $T \cup A_E$ is consistent. Then $\sharp T$ is not recursive.

As a corollary, if $\sharp\Sigma$ is recursive, then $\mathrm{Cn}(\Sigma)$ is not complete.

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<u>NOTE.</u> $\sharp\mathrm{Cn}(\Sigma)$ is recursive $\implies \sharp\mathrm{Cn}(\Sigma \cup \{\sigma\})$ is recursive.

Let $T' = \mathrm{Cn}(T \cup A_E)$. If $\sharp T$ is recursive, since A_E is finite, $\sharp T'$ is recursive. Thus $\sharp T'$ is represented in $\mathrm{Cn}(A_E)$, say by φ.

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Contradiction! ☒