Mathematical Logic

Xianghui Shi

School of Mathematical Science
Beijing Normal University

Spring, 2025

Course Information

Reference

- T.A. Slaman and W. H. Woodin. *Mathematical Logic (The Berkeley undergraduate course)*. Preprint. December 2019.
- H. B. Enderton. *A mathematical introduction to logic*.
  - 人民邮电出版社. 2006. 【另有中文译本，但不推荐】
  - Academic Press, Burlington, MA, second edition, 2001.

Course Webpage

- `http://math0.bnu.edu.cn/~shi/teaching/spring2025/logic/`

Introduction

Traditional logic

- "**Logic**" originates from ancient greek $\lambda o\gamma\iota\kappa\eta$ (logike).[1]
- the study of the principles of reasoning; a part of philosophy; **philosophical logic**.
- Theories of logic were developed in many cultures in history, including China, India, Greece and the Islamic world.
- Modern treatment descends from the Greek tradition, particularly Plato-Aristotelian logic, which was further developed by Islamic logicians and then medieval European logicians.
- mostly studied with rhetoric (修辞学), the syllogism (三段论) and philosophy.
- Informal logic examines arguments expressed in *natural language* while formal logic uses *formal language*.

[1] Cf. `https://en.wikipedia.org/wiki/Logic`

Mathematical Logic

Mathematical logic began to diverge as a distinct field in the mid-19th century, motivated by the interest to the **foundations of mathematics**.[2]

- a subfield of logic and mathematics.
- also called 'logistic', 'symbolic logic', the 'algebra of logic', and 'formal logic'.
- has numerous applications in mathematics, computer science, linguistics and philosophy.
- Themes of research in mathematical logic:
    - the expressive or deductive power of formal systems of logic,
    - uses of logic to establish foundations of mathematics.

[2] Cf. `https://en.wikipedia.org/wiki/Mathematical_logic`

History

- Antient Greece (around 400 BC):
  - Aristotle's theory of syllogisms
  - Euclid's axioms for planar geometry
- 1700s:
  Leibniz and Lambert treated logic in an algebraic way
- Developments of mathematical logic in the 19th century:
  - G. Boole and A. De Morgan presented systematic mathematical treatments of propositional logic in the mid 19th.
  - G. Frege and B. Russell developed logic with quantifiers (predicative logic, aka first-order logic) in the late 19th.

- axiomatic frameworks for fundamental areas of mathematics such as **geometry**, **arithmetic** and **analysis** were developed in late 19th century.
  - Arithmetic: Peano (1888)
  - Geometry: Lobachevsky (1826); Hilbert (1899)
  - Analysis: Weierstrass, Bolzano, Cauchy, Dedekind

  Also Georg Cantor developed the fundamental concepts of infinite **set theory**.

- early 20th century:
  - paradoxes in informal set theory: Russell paradox – the 3rd crisis of mathematics.
  - Hilbert's program:
    - prove the consistency of foundational theories.
  - Hilbert's 23 problems:[3]
    - 1 (Gödel 1940, Cohen 1963),
    - 2 (Gödel 1931, Gentzen 1936),
    - 10 (Robinson-Davis-Putnam-Matiyasevich 1970).
  - Axiomatic set theory: Zermelo-Fraenkel set theory (ZF)
  - K. Gödel, G. Gentzen etc: provided partial resolution to Hilbert's program, and clarified the issues involved in proving consistency.

[3] Cf. https://en.wikipedia.org/wiki/Hilbert%27s_problems

Logic problems in Hilbert's 23

1. The continuum hypothesis.
   - Proven to be impossible to prove or disprove within Zermelo–Fraenkel set theory with or without the axiom of choice. (Gödel 1940, Cohen 1963)
2. Prove that the axioms of arithmetic are consistent.
   - Gödel's second incompleteness theorem (1931), shows that no proof of its consistency can be carried out within arithmetic itself.
   - Gentzen (1936) proved that the consistency of arithmetic follows from the well-foundedness of the ordinal ε_0.

10. Find an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers.
    - Partial progress (logic part) was made by Julia Robinson, Martin Davis and Hilary Putnam.
    - Matiyasevich (1970) proved that there is no such algorithm.

17. Express a nonnegative rational function as quotient of sums of squares.
    - YES by E. Artin (1927).
    - A. Robinson (1979) gave a much simpler proof with model theoretic method.

Great Logicians

David Hilbert

1862 - 1943

Ernst F. F. Zermelo

1871 - 1953

Bertrand A.W. Russell

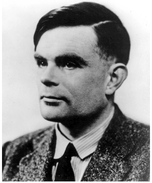
1872 - 1970

Alfred Tarski

1902 - 1983

Kurt Gödel

1906 - 1978



Alan Turing

1912 - 1954

Subfields

Contemporary mathematical logic is roughly divided into four areas:

- **set theory**: sets
- **model theory**: models of various theories
- **recursion theory**: the properties of (in)computable functions and the Turing degrees
- **proof theory**: formal proofs in various logical deduction systems

Different focus, but no sharp border line. These areas share basic results on logic, particularly first-order logic, and definability.

Plan for this course

This course introduces the so-called classical logic (which includes two basic formal/symbolic logic systems) and sets up the basics for the more advanced branches of mathematics.

1. Propositional/Sentential logic: $\mathcal{L}_0$
    - formulas and proof systems for $\mathcal{L}_0$
2. First-order/Predicative logic: $\mathcal{L}$
    - formulas and proof systems for $\mathcal{L}$
    - Completeness and compactness theorems for $\mathcal{L}$
    - a bit more on model theory.