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Set Theory

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1. Axioms of Set Theory

Axioms of Zermelo-Fraenkel

1.1. ***Axiom of Extensionality.*** *If X and Y have the same elements, then $X = Y$.*

1.2. ***Axiom of Pairing.*** *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. ***Axiom Schema of Separation.*** *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. ***Axiom of Union.*** *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. ***Axiom of Power Set.*** *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. ***Axiom of Infinity.*** *There exists an infinite set.*

1.7. ***Axiom Schema of Replacement.*** *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. ***Axiom of Regularity.*** *Every nonempty set has an $\in$-minimal element.*

1.9. ***Axiom of Choice.*** *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

1.10. ***Axiom Schema of Comprehension (false).*** *If P is a property, then there exists a set $Y = \{x : P(x)\}$.*

This principle, however, is false:

1.11. Russell's Paradox. Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the Schema of Comprehension and keep its weak version, the *Schema of Separation*:

If P is a property, then for any X there exists a set $Y = \{x \in X : P(x)\}$.

Once we give up the full Comprehension Schema, Russell's Paradox is no longer a threat; moreover, it provides this useful information: The set of all sets does not exist. (Otherwise, apply the Separation Schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full Comprehension by Separation presents us with a new problem. The Separation Axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

Language of Set Theory, Formulas

The Axiom Schema of Separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation*.

The *formulas* of set theory are built up from the *atomic formulas*

$$x \in y, \qquad x = y$$

by means of *connectives*

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \neg\varphi, \qquad \varphi \to \psi, \qquad \varphi \leftrightarrow \psi$$

(conjunction, disjunction, negation, implication, equivalence), and *quantifiers*

$$\forall x\, \varphi, \qquad \exists x\, \varphi.$$

In practice, we shall use in formulas other symbols, namely defined predicates, operations, and constants, and even use formulas informally; but it will be tacitly understood that each such formula can be written in a form that only involves $\in$ and $=$ as nonlogical symbols.

Concerning formulas with free variables, we adopt the notational convention that all free variables of a formula

$$\varphi(u_1, \ldots, u_n)$$

are among $u_1, \ldots, u_n$ (possibly some u_i are not free, or even do not occur, in φ). A formula without free variables is called a *sentence*.

Classes

Although we work in ZFC which, unlike alternative axiomatic set theories, has only one type of object, namely sets, we introduce the informal notion of a *class*. We do this for practical reasons: It is easier to manipulate classes than formulas.

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}$$

a *class*. Members of the class C are all those sets x that satisfy $\varphi(x, p_1, \ldots, p_n)$:

$$x \in C \quad \text{if and only if} \quad \varphi(x, p_1, \ldots, p_n).$$

We say that C is *definable from* $p_1, \ldots, p_n$; if $\varphi(x)$ has no parameters p_i then the class C is *definable*.

Two classes are considered equal if they have the same elements: If

$$C = \{x : \varphi(x, p_1, \ldots, p_n)\}, \qquad D = \{x : \psi(x, q_1, \ldots, q_m)\},$$

then $C = D$ if and only if for all x

$$\varphi(x, p_1, \ldots, p_n) \leftrightarrow \psi(x, q_1, \ldots, q_m).$$

The *universal class*, or *universe*, is the class of all sets:

$$V = \{x : x = x\}.$$

We define *inclusion* of classes (C is a *subclass* of D)

$$C \subset D \text{ if and only if for all } x,\ x \in C \text{ implies } x \in D,$$

and the following operations on classes:

$$\begin{aligned} C \cap D &= \{x : x \in C \text{ and } x \in D\},\\ C \cup D &= \{x : x \in C \text{ or } x \in D\},\\ C - D &= \{x : x \in C \text{ and } x \notin D\},\\ \bigcup C &= \{x : x \in S \text{ for some } S \in C\} = \bigcup\{S : S \in C\}. \end{aligned}$$

Every set can be considered a class. If S is a set, consider the formula $x \in S$ and the class

$$\{x : x \in S\}.$$

That the set S is uniquely determined by its elements follows from the Axiom of Extensionality.

A class that is not a set is a *proper class.*

Extensionality

If X and Y have the same elements, then $X = Y$:

$$\forall u\,(u \in X \leftrightarrow u \in Y) \to X = Y.$$

The converse, namely, if $X = Y$ then $u \in X \leftrightarrow u \in Y$, is an axiom of predicate calculus. Thus we have

$$X = Y \quad \text{if and only if} \quad \forall u\,(u \in X \leftrightarrow u \in Y).$$

The axiom expresses the basic idea of a set: A set is determined by its elements.

Pairing

For any a and b there exists a set $\{a, b\}$ that contains exactly a and b:

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \leftrightarrow x = a \vee x = b).$$

By Extensionality, the set c is unique, and we can define the *pair*

$$\{a, b\} = \text{the unique } c \text{ such that } \forall x\,(x \in c \leftrightarrow x = a \vee x = b).$$

The *singleton* $\{a\}$ is the set

$$\{a\} = \{a, a\}.$$

Since $\{a, b\} = \{b, a\}$, we further define an *ordered pair*

$$(a, b)$$

so as to satisfy the following condition:

(1.1) $$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d.$$

For the formal definition of an ordered pair, we take

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

We leave the verification of (1.1) to the reader (Exercise 1.1).

We further define ordered triples, quadruples, etc., as follows:

$$\begin{aligned} (a, b, c) &= ((a, b), c),\\ (a, b, c, d) &= ((a, b, c), d),\\ &\ \vdots\\ (a_1, \ldots, a_{n+1}) &= ((a_1, \ldots, a_n), a_{n+1}). \end{aligned}$$

It follows that two ordered n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equal if and only if $a_1 = b_1,\ \ldots,\ a_n = b_n$.

Separation Schema

Let $\varphi(u, p)$ be a formula. For any X and p, there exists a set $Y = \{u \in X : \varphi(u, p)\}$:

(1.2) $$\forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

For each formula $\varphi(u, p)$, the formula (1.2) is an Axiom (of Separation). The set Y in (1.2) is unique by Extensionality.

Note that a more general version of Separation Axioms can be proved using ordered n-tuples: Let $\psi(u, p_1, \ldots, p_n)$ be a formula. Then

(1.3) $$\forall X\,\forall p_1 \ldots \forall p_n\,\exists Y\,\forall u\,(u \in Y \leftrightarrow u \in X \wedge \psi(u, p_1, \ldots, p_n)).$$

Simply let $\varphi(u, p)$ be the formula

$$\exists p_1, \ldots \exists p_n \, (p = (p_1, \ldots, p_n) \text{ and } \psi(u, p_1, \ldots, p_n))$$

and then, given X and $p_1, \ldots, p_n$, let

$$Y = \{u \in X : \varphi(u, (p_1, \ldots, p_n))\}.$$

We can give the Separation Axioms the following form: Consider the class $C = \{u : \varphi(u, p_1, \ldots, p_n)\}$; then by (1.3)

$$\forall X \, \exists Y \, (C \cap X = Y).$$

Thus the intersection of a class C with any set is a set; or, we can say even more informally

a subclass of a set is a set.

One consequence of the Separation Axioms is that the intersection and the difference of two sets is a set, and so we can define the operations

$$X \cap Y = \{u \in X : u \in Y\} \quad \text{and} \quad X - Y = \{u \in X : u \notin Y\}.$$

Similarly, it follows that the empty class

$$\emptyset = \{u : u \neq u\}$$

is a set—the *empty set*; this, of course, only under the assumption that at least one set X exists (because $\emptyset \subset X$):

$$\exists X \, (X = X). \tag{1.4}$$

We have not included (1.4) among the axioms, because it follows from the Axiom of Infinity.

Two sets X, Y are called *disjoint* if $X \cap Y = \emptyset$.

If C is a nonempty class of sets, we let

$$\bigcap C = \bigcap \{X : X \in C\} = \{u : u \in X \text{ for every } X \in C\}.$$

Note that $\bigcap C$ is a set (it is a subset of any $X \in C$). Also, $X \cap Y = \bigcap \{X, Y\}$.

Another consequence of the Separation Axioms is that the universal class V is a proper class; otherwise,

$$S = \{x \in V : x \notin x\}$$

would be a set.

Union

For any X there exists a set $Y = \bigcup X$:

$$\forall X \, \exists Y \, \forall u \, (u \in Y \leftrightarrow \exists z \, (z \in X \wedge u \in z)). \tag{1.5}$$

Let us introduce the abbreviations

$$(\exists z \in X) \, \varphi \quad \text{for} \quad \exists z \, (z \in X \wedge \varphi),$$

and

$$(\forall z \in X) \, \varphi \quad \text{for} \quad \forall z \, (z \in X \rightarrow \varphi).$$

By (1.5), for every X there is a unique set

$$Y = \{u : (\exists z \in X) \, u \in z\} = \bigcup \{z : z \in X\} = \bigcup X,$$

the *union* of X.

Now we can define

$$X \cup Y = \bigcup \{X, Y\}, \qquad X \cup Y \cup Z = (X \cup Y) \cup Z, \qquad \text{etc.},$$

and also

$$\{a, b, c\} = \{a, b\} \cup \{c\},$$

and in general

$$\{a_1, \ldots, a_n\} = \{a_1\} \cup \ldots \cup \{a_n\}.$$

We also let

$$X \bigtriangleup Y = (X - Y) \cup (Y - X),$$

the *symmetric difference* of X and Y.

Power Set

For any X there exists a set $Y = P(X)$:

$$\forall X \, \exists Y \, \forall u \, (u \in Y \leftrightarrow u \subset X).$$

A set U is a *subset* of X, $U \subset X$, if

$$\forall z \, (z \in U \rightarrow z \in X).$$

If $U \subset X$ and $U \neq X$, then U is a *proper subset* of X.

The set of all subsets of X,

$$P(X) = \{u : u \subset X\},$$

is called the *power set* of X.

Using the Power Set Axiom we can define other basic notions of set theory.

The *product* of X and Y is the set of all pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}. \tag{1.6}$$

The notation $\{(x, y) : \dots\}$ in (1.6) is justified because

$$\{(x, y) : \varphi(x, y)\} = \{u : \exists x\, \exists y\, (u = (x, y) \text{ and } \varphi(x, y))\}.$$

The product $X \times Y$ is a set because

$$X \times Y \subset PP(X \cup Y).$$

Further, we define

$$X \times Y \times Z = (X \times Y) \times Z,$$

and in general

$$X_1 \times \dots \times X_{n+1} = (X_1 \times \dots \times X_n) \times X_{n+1}.$$

Thus

$$X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) : x_1 \in X_1 \wedge \dots \wedge x_n \in X_n\}.$$

We also let

$$X^n = \underbrace{X \times \dots \times X}_{n \text{ times}}.$$

An *n-ary relation* R is a set of n-tuples. R is a relation *on* X if $R \subset X^n$. It is customary to write $R(x_1, \dots, x_n)$ instead of

$$(x_1, \dots, x_n) \in R,$$

and in case that R is binary, then we also use

$$x\ R\ y$$

for $(x, y) \in R$.

If R is a binary relation, then the *domain* of R is the set

$$\operatorname{dom}(R) = \{u : \exists v\, (u, v) \in R\},$$

and the *range* of R is the set

$$\operatorname{ran}(R) = \{v : \exists u\, (u, v) \in R\}.$$

Note that $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are sets because

$$\operatorname{dom}(R) \subset \bigcup\bigcup R, \qquad \operatorname{ran}(R) \subset \bigcup\bigcup R.$$

The *field* of a relation R is the set $\operatorname{field}(R) = \operatorname{dom}(R) \cup \operatorname{ran}(R)$.

In general, we call a class R an n-ary relation if all its elements are n-tuples; in other words, if

$$R \subset V^n = \text{the class of all } n\text{-tuples},$$

where C^n (and $C \times D$) is defined in the obvious way.

A binary relation f is a *function* if $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$. The unique y such that $(x, y) \in f$ is the *value* of f at x; we use the standard notation

$$y = f(x)$$

or its variations $f : x \mapsto y$, $y = f_x$, etc. for $(x, y) \in f$.

f is a function *on* X if $X = \operatorname{dom}(f)$. If $\operatorname{dom}(f) = X^n$, then f is an *n-ary function* on X.

f is a function *from* X to Y,

$$f : X \to Y,$$

if $\operatorname{dom}(f) = X$ and $\operatorname{ran}(f) \subset Y$. The set of all functions from X to Y is denoted by Y^X. Note that Y^X is a set:

$$Y^X \subset P(X \times Y).$$

If $Y = \operatorname{ran}(f)$, then f is a function *onto* Y. A function f is *one-to-one* if

$$f(x) = f(y) \quad \text{implies} \quad x = y.$$

An *n-ary operation on* X is a function $f : X^n \to X$.

The *restriction* of a function f to a set X (usually a subset of $\operatorname{dom}(f)$) is the function

$$f{\restriction}X = \{(x, y) \in f : x \in X\}.$$

A function g is an *extension* of a function f if $g \supset f$, i.e., $\operatorname{dom}(f) \subset \operatorname{dom}(g)$ and $g(x) = f(x)$ for all $x \in \operatorname{dom}(f)$.

If f and g are functions such that $\operatorname{ran}(g) \subset \operatorname{dom}(f)$, then the *composition* of f and g is the function $f \circ g$ with domain $\operatorname{dom}(f \circ g) = \operatorname{dom}(g)$ such that $(f \circ g)(x) = f(g(x))$ for all $x \in \operatorname{dom}(g)$.

We denote the *image* of X by f either $f``X$ or $f(X)$:

$$f``X = f(X) = \{y : (\exists x \in X)\, y = f(x)\},$$

and the *inverse image* by

$$f_{-1}(X) = \{x : f(x) \in X\}.$$

If f is one-to-one, then f^{-1} denotes the *inverse* of f:

$$f^{-1}(x) = y \quad \text{if and only if} \quad x = f(y).$$

The previous definitions can also be applied to classes instead of sets. A class F is a function if it is a relation such that $(x, y) \in F$ and $(x, z) \in F$

implies $y = z$. For example, $F``C$ or $F(C)$ denotes the image of the class C by the function F.

It should be noted that a function is often called a *mapping* or a *correspondence* (and similarly, a set is called a *family* or a *collection*).

An *equivalence relation* on a set X is a binary relation $\equiv$ which is *reflexive*, *symmetric*, and *transitive*: For all $x, y, z \in X$,

$$x \equiv x,$$
$$x \equiv y \text{ implies } y \equiv x,$$
$$\text{if } x \equiv y \text{ and } y \equiv z \text{ then } x \equiv z.$$

A family of sets is *disjoint* if any two of its members are disjoint. A *partition* of a set X is a disjoint family P of nonempty sets such that

$$X = \bigcup\{Y : Y \in P\}.$$

Let $\equiv$ be an equivalence relation on X. For every $x \in X$, let

$$[x] = \{y \in X : y \equiv x\}$$

(the *equivalence class* of x). The set

$$X/{\equiv} = \{[x] : x \in X\}$$

is a partition of X (the *quotient* of X by $\equiv$). Conversely, each partition P of X defines an equivalence relation on X:

$$x \equiv y \quad \text{if and only if} \quad (\exists Y \in P)(x \in Y \text{ and } y \in Y).$$

If an equivalence relation is a class, then its equivalence classes may be proper classes. In Chapter 6 we shall introduce a trick that enables us to handle equivalence classes as if they were sets.

Infinity

There exists an infinite set.

To give a precise formulation of the Axiom of Infinity, we have to define first the notion of finiteness. The most obvious definition of finiteness uses the notion of a natural number, which is as yet undefined. We shall define natural numbers (as finite ordinals) in Chapter 2 and give only a quick treatment of natural numbers and finiteness in the exercises below.

In principle, it is possible to give a definition of finiteness that does not mention numbers, but such definitions necessarily look artificial.

We therefore formulate the Axiom of Infinity differently:

$$\exists S\,(\emptyset \in S \wedge (\forall x \in S) x \cup \{x\} \in S).$$

We call a set S with the above property *inductive*. Thus we have:

Axiom of Infinity. *There exists an inductive set.*

The axiom provides for the existence of infinite sets. In Chapter 2 we show that an inductive set is infinite (and that an inductive set exists if there exists an infinite set).

We shall introduce natural numbers and finite sets in Chapter 2, as a part of the introduction of ordinal numbers. In Exercises 1.3–1.9 we show an alternative approach.

Replacement Schema

If a class F is a function, then for every set X, $F(X)$ is a set.

For each formula $\varphi(x, y, p)$, the formula (1.7) is an Axiom (of Replacement):

$$(1.7)\quad \forall x\,\forall y\,\forall z\,(\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z) \rightarrow \forall X\,\exists Y\,\forall y\,(y \in Y \leftrightarrow (\exists x \in X)\,\varphi(x, y, p)).$$

As in the case of Separation Axioms, we can prove the version of Replacement Axioms with several parameters: Replace p by $p_1, \ldots, p_n$.

If $F = \{(x, y) : \varphi(x, y, p)\}$, then the premise of (1.7) says that F is a function, and we get the formulation above. We can also formulate the axioms in the following ways:

If a class F is a function and $\operatorname{dom}(F)$ *is a set, then* $\operatorname{ran}(F)$ *is a set.*
If a class F is a function, then $\forall X\,\exists f\,(F{\restriction}X = f)$.

The remaining two axioms, Choice and Regularity, will by introduced in Chapters 5 and 6.

Exercises

1.1. Verify (1.1).

1.2. There is no set X such that $P(X) \subset X$.

Let

$$\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}.$$

$\boldsymbol{N}$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\}, \qquad 3 = \{0, 1, 2\}, \qquad \ldots.$$

If $n \in \boldsymbol{N}$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (on $\boldsymbol{N}$) by $n < m$ if and only if $n \in m$.

A set T is *transitive* if $x \in T$ implies $x \subset T$.

1.3. If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive, and for each n, $n = \{m \in \boldsymbol{N} : m < n\}$.

1.4. If X is inductive, then the set $\{x \in X : x$ is transitive$\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

1.5. If X is inductive, then the set $\{x \in X : x$ is transitive and $x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for each $n \in \boldsymbol{N}$.

1.6. If X is inductive, then $\{x \in X : x$ is transitive and every nonempty $z \subset x$ has an $\in$-minimal element$\}$ is inductive (t is $\in$-*minimal* in z if there is no $s \in z$ such that $s \in t$).

1.7. Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.
[Pick $n \in X$ and look at $X \cap n$.]

1.8. If X is inductive then so is $\{x \in X : x = \emptyset$ or $x = y \cup \{y\}$ for some $y\}$. Hence each $n \neq 0$ is $m+1$ for some m.

1.9 (Induction). Let A be a subset of $\boldsymbol{N}$ such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

A set X *has n elements* (where $n \in \boldsymbol{N}$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in \boldsymbol{N}$, and *infinite* if it is not finite.

A set S is *T-finite* if every nonempty $X \subset P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$ and $u \neq v$. S is T-*infinite* if it is not T-finite. (T is for Tarski.)

1.10. Each $n \in \boldsymbol{N}$ is T-finite.

1.11. $\boldsymbol{N}$ is T-infinite; the set $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

1.12. Every finite set is T-finite.

1.13. Every infinite set is T-infinite.
[If S is infinite, consider $X = \{u \subset S : u$ is finite$\}$.]

1.14. The Separation Axioms follow from the Replacement Schema.
[Given φ, let $F = \{(x,x) : \varphi(x)\}$. Then $\{x \in X : \varphi(x)\} = F(X)$, for every X.]

1.15. Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8) $\forall X\, \exists Y \bigcup X \subset Y$, i.e., $\forall X\, \exists Y\, (\forall x \in X)(\forall u \in x)\, u \in Y$,

(1.9) $\forall X\, \exists Y\, P(X) \subset Y$, i.e., $\forall X\, \exists Y\, \forall u\, (u \subset X \rightarrow u \in Y)$,

(1.10) If a class F is a function, then $\forall X\, \exists Y\, F(X) \subset Y$.

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).

Historical Notes

Set theory was invented by Georg Cantor. The first attempt to consider infinite sets is attributed to Bolzano (who introduced the term *Menge*). It was however Cantor who realized the significance of one-to-one functions between sets and introduced the notion of cardinality of a set. Cantor originated the theory of cardinal and ordinal numbers as well as the investigations of the topology of the real line. Much of the development in the first four chapters follows Cantor's work. The main reference to Cantor's work is his collected works, Cantor [1932]. Another source of references to the early research in set theory is Hausdorff's book [1914].

Cantor started his investigations in [1874], where he proved that the set of all real numbers is uncountable, while the set of all algebraic reals is countable. In [1878] he gave the first formulation of the celebrated Continuum Hypothesis.

The axioms for set theory (except Replacement and Regularity) are due to Zermelo [1908]. The Replacement Schema is due to Fraenkel [1922a] and Skolem (see [1970], pp. 137–152).

Exercises 1.12 and 1.13: Tarski [1925a].

2. Ordinal Numbers

In this chapter we introduce ordinal numbers and prove the Transfinite Recursion Theorem.

Linear and Partial Ordering

Definition 2.1. A binary relation $<$ on a set P is a *partial ordering* of P if:

(i) $p \not< p$ for any $p \in P$;
(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$ is called a *partially ordered set*. A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

If $<$ is a partial (linear) ordering, then the relation $\leq$ (where $p \leq q$ if either $p < q$ or $p = q$) is also called a partial (linear) ordering (and $<$ is sometimes called a *strict* ordering).

Definition 2.2. If $(P, <)$ is a partially ordered set, X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X$ and $(\forall x \in X)\, a \not< x$;
a is a *minimal* element of X if $a \in X$ and $(\forall x \in X)\, x \not< a$;
a is the *greatest* element of X if $a \in X$ and $(\forall x \in X)\, x \leq a$;
a is the *least* element of X if $a \in X$ and $(\forall x \in X)\, a \leq x$;
a is an *upper bound* of X if $(\forall x \in X)\, x \leq a$;
a is a *lower bound* of X if $(\forall x \in X)\, a \leq x$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted $\sup X$ $(\inf X)$. Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is *order-preserving* if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called *increasing*.

A one-to-one function of P onto Q is an *isomorphism* of P and Q if both f and f^{-1} are order-preserving; $(P, <)$ is then *isomorphic* to $(Q, <)$. An isomorphism of P onto itself is an *automorphism* of $(P, <)$.

Well-Ordering

Definition 2.3. A linear ordering $<$ of a set P is a *well-ordering* if every nonempty subset of P has a least element.

The concept of well-ordering is of fundamental importance. It is shown below that well-ordered sets can be compared by their lengths; ordinal numbers will be introduced as order-types of well-ordered sets.

Lemma 2.4. *If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction. □

Corollary 2.5. *The only automorphism of a well-ordered set is the identity.*

Proof. By Lemma 2.4, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x. □

Corollary 2.6. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* □

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an *initial segment* of W (given by u).

Lemma 2.7. *No well-ordered set is isomorphic to an initial segment of itself.*

Proof. If $\operatorname{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 2.4. □

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;*
(iii) *W_2 is isomorphic to an initial segment of W_1.*

Proof. For $u \in W_i$, $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}.$$

Using Lemma 2.7, it is easy to see that f is a one-to-one function. If h is an isomorphism between $W_1(x)$ and $W_2(y)$, and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case (i) holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. Thus if $\mathrm{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$, we have $\mathrm{ran}(f) = W_2(y_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where $x_0 =$ the least element of $W_1 - \mathrm{dom}(f)$. Thus case (ii) holds.

Similarly, if $\mathrm{dom}(f) \neq W_1$, then case (iii) holds.

In view of Lemma 2.7, the three cases are mutually exclusive. □

If W_1 and W_2 are isomorphic, we say that they have the same *order-type*. Informally, an ordinal number is the order-type of a well-ordered set.

We shall now give a formal definition of ordinal numbers.

Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta, \qquad \text{and} \quad \alpha = \{\beta : \beta < \alpha\}.$$

Definition 2.9. A set T is *transitive* if every element of T is a subset of T. (Equivalently, $\bigcup T \subset T$, or $T \subset P(T)$.)

Definition 2.10. A set is an *ordinal number* (an *ordinal*) if it is transitive and well-ordered by $\in$.

We shall denote ordinals by lowercase Greek letters α, β, γ, ... The class of all ordinals is denoted by *Ord*.

We define

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta.$$

Lemma 2.11.

(i) $0 = \emptyset$ *is an ordinal.*
(ii) *If* α *is an ordinal and* $\beta \in \alpha$, *then* β *is an ordinal.*
(iii) *If* $\alpha \neq \beta$ *are ordinals and* $\alpha \subset \beta$, *then* $\alpha \in \beta$.
(iv) *If* α, β *are ordinals, then either* $\alpha \subset \beta$ *or* $\beta \subset \alpha$.

Proof. (i), (ii) by definition.

(iii) If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.

(iv) Clearly, $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$, and $\gamma \in \beta$, by (iii). Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a *strict* ordering of α). □

Using Lemma 2.11 one gets the following facts about ordinal numbers (the proofs are routine):

(2.1) $<$ is a linear ordering of the class *Ord*.

(2.2) For each α, $\alpha = \{\beta : \beta < \alpha\}$.

(2.3) If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$.

(2.4) If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.

(2.5) For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). In view of (2.4), the class *Ord* is a proper class; otherwise, consider $\sup \mathit{Ord} + 1$.

We can now prove that the above definition of ordinals provides us with order-types of well-ordered sets.

Theorem 2.12. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. The uniqueness follows from Lemma 2.7. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exist). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ. □

If $\alpha = \beta + 1$, then α is a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a *limit ordinal*. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

The existence of limit ordinals other than 0 follows from the Axiom of Infinity; see Exercise 2.3.

Definition 2.13 (Natural Numbers). We denote the least nonzero limit ordinal ω (or $\boldsymbol{N}$). The ordinals less than ω (elements of $\boldsymbol{N}$) are called *finite ordinals*, or *natural numbers*. Specifically,

$$0 = \emptyset, \qquad 1 = 0 + 1, \qquad 2 = 1 + 1, \qquad 3 = 2 + 1, \qquad \text{etc.}$$

A set X is *finite* if there is a one-to-one mapping of X onto some $n \in \boldsymbol{N}$. X is *infinite* if it is not finite.

We use letters n, m, l, k, j, i (most of the time) to denote natural numbers.

Induction and Recursion

Theorem 2.14 (Transfinite Induction). *Let C be a class of ordinals and assume that*:

(i) $0 \in C$;
(ii) *if* $\alpha \in C$, *then* $\alpha + 1 \in C$;
(iii) *if* α *is a nonzero limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$, *then* $\alpha \in C$.

Then C is the class of all ordinals.

Proof. Otherwise, let α be the least ordinal $\alpha \notin C$ and apply (i), (ii), or (iii). □

A function whose domain is the set $\boldsymbol{N}$ is called an (*infinite*) *sequence* (A *sequence in* X is a function $f : \boldsymbol{N} \to X$.) The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

or variants thereof. A *finite sequence* is a function s such $\operatorname{dom}(s) = \{i : i < n\}$ for some $n \in \boldsymbol{N}$; then s is a *sequence of length* n.

A *transfinite sequence* is a function whose domain is an ordinal:

$$\langle a_\xi : \xi < \alpha \rangle.$$

It is also called an *α-sequence* or a *sequence of length* α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an *enumeration* of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}.$$

Sometimes we shall call a "sequence"

$$\langle a_\alpha : \alpha \in Ord \rangle$$

a function (a proper class) on *Ord*.

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequences), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

such that

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$.

We shall give a general version of this theorem, so that we can also construct sequences $\langle a_\alpha : \alpha \in Ord \rangle$.

Theorem 2.15 (Transfinite Recursion). *Let G be a function (on V), then* (2.6) *below defines a unique function F on Ord such that*

$$F(\alpha) = G(F{\restriction}\alpha)$$

for each α.

In other words, if we let $a_\alpha = F(\alpha)$, then for each α,

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle).$$

(Note that we tacitly use Replacement: $F{\restriction}\alpha$ is a set for each α.)

Corollary 2.16. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ such that* $\operatorname{ran}(G) \subset X$ *there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta \rangle$ in X such that $a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$ for every $\alpha < \theta$.* □

Proof. Let

(2.6) $F(\alpha) = x \leftrightarrow$ there is a sequence $\langle a_\xi : \xi < \alpha \rangle$ such that:
(i) $(\forall \xi < \alpha)\, a_\xi = G(\langle a_\eta : \eta < \xi \rangle)$;
(ii) $x = G(\langle a_\xi : \xi < \alpha \rangle)$.

For every α, if there is an α-sequence that satisfies (i), then such a sequence is unique: If $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ are two α-sequences satisfying (i), one shows $a_\xi = b_\xi$ by induction on ξ. Thus $F(\alpha)$ is determined uniquely by (ii), and therefore F is a function. It follows, again by induction, that for each α there is an α-sequence that satisfies (i) (at limit steps, we use Replacement to get the α-sequence as the union of all the ξ-sequences, $\xi < \alpha$). Thus F is defined for all $\alpha \in Ord$. It obviously satisfies

$$F(\alpha) = G(F{\restriction}\alpha).$$

If F' is any function on *Ord* that satisfies

$$F'(\alpha) = G(F'{\restriction}\alpha)$$

then it follows by induction that $F'(\alpha) = F(\alpha)$ for all α. □

Definition 2.17. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a *nondecreasing* sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \le \gamma_\eta$). We define the *limit* of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}.$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is *normal* if it is increasing and *continuous*, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$.

Ordinal Arithmetic

We shall now define addition, multiplication and exponentiation of ordinal numbers, using Transfinite Recursion.

Definition 2.18 (Addition). For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$.

Definition 2.19 (Multiplication). For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

Definition 2.20 (Exponentiation). For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β,
(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.

As defined, the operations $\alpha + \beta$, $\alpha \cdot \beta$ and α^β are normal functions in the second variable β. Their properties can be proved by transfinite induction. For instance, $+$ and $\cdot$ are associative:

Lemma 2.21. *For all ordinals α, β and γ,*

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
(ii) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Proof. By induction on γ. □

Neither $+$ nor $\cdot$ are commutative:

$$1 + \omega = \omega \neq \omega + 1, \qquad 2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega.$$

Ordinal sums and products can be also defined geometrically, as can sums and products of arbitrary linear orders:

Definition 2.22. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The *sum* of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ if and only if

(i) $x, y \in A$ and $x <_A y$, or
(ii) $x, y \in B$ and $x <_B y$, or
(iii) $x \in A$ and $y \in B$.

Definition 2.23. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The *product* of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1, b_1) < (a_2, b_2) \text{ if and only if either } b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2).$$

Lemma 2.24. *For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are, respectively, isomorphic to the sum and to the product of α and β.*

Proof. By induction on β. □

Ordinal sums and products have some properties of ordinary addition and multiplication of integers. For instance:

Lemma 2.25.

(i) *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.*
(ii) *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta = \beta$.*
(iii) *If $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.*
(iv) *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.*
(v) *If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$.*

Proof. (i), (iii) and (v) are proved by induction on γ.
(ii) Let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$; δ is unique by (i).
(iv) Let β be the greatest ordinal such that $\alpha \cdot \beta \leq \gamma$. □

For more, see Exercises 2.10 and 2.11.

Theorem 2.26 (Cantor's Normal Form Theorem). *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. By induction on α. For $\alpha = 1$ we have $1 = \omega^0 \cdot 1$; for arbitrary $\alpha > 0$ let β be the greatest ordinal such that $\omega^\beta \leq \alpha$. By Lemma 2.25(iv) there exists a unique δ and a unique $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$; this δ must necessarily be finite. The uniqueness of the normal form is proved by induction. □

In the normal form it is possible to have $\alpha = \omega^\alpha$; see Exercise 2.12. The least ordinal with this property is called ε_0.

Well-Founded Relations

Now we shall define an important generalization of well-ordered sets.

A binary relation E on a set P is *well-founded* if every nonempty $X \subset P$ has an E-*minimal* element, that is $a \in X$ such that there is no $x \in X$ with $x \mathrel{E} a$.

Clearly, a well-ordering of P is a well-founded relation.

Given a well-founded relation E on a set P, we can define the *height* of E, and assign to each $x \in P$ an ordinal number, the *rank* of x in E.

Theorem 2.27. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals such that for all $x \in P$,*

$$\rho(x) = \sup\{\rho(y) + 1 : y \mathrel{E} x\}. \tag{2.7}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the *height* of E.

Proof. We shall define a function ρ satisfying (2.7) and then prove its uniqueness. By induction, let

$$P_0 = \emptyset, \qquad P_{\alpha+1} = \{x \in P : \forall y\,(y \mathrel{E} x \to y \in P_\alpha)\},$$
$$P_\alpha = \bigcup_{\xi<\alpha} P_\xi \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let θ be the least ordinal such that $P_{\theta+1} = P_\theta$ (such θ exists by Replacement). First, it should be easy to see that $P_\alpha \subset P_{\alpha+1}$ for each α (by induction). Thus $P_0 \subset P_1 \subset \ldots \subset P_\theta$. We claim that $P_\theta = P$. Otherwise, let a be an E-minimal element of $P - P_\theta$. It follows that each $x \mathrel{E} a$ is in P_θ, and so $a \in P_{\theta+1}$, a contradiction. Now we define $\rho(x)$ as the least α such that $x \in P_{\alpha+1}$. It is obvious that if $x \mathrel{E} y$, then $\rho(x) < \rho(y)$, and (2.7) is easily verified. The ordinal θ is the height of E.

The uniqueness of ρ is established as follows: Let ρ' be another function satisfying (2.7) and consider an E-minimal element of the set $\{x \in P : \rho(x) \neq \rho'(x)\}$. □

Exercises

2.1. The relation "$(P,<)$ is isomorphic to $(Q,<)$" is an equivalence relation (on the class of all partially ordered sets).

2.2. α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

2.3. If a set X is inductive, then $X \cap Ord$ is inductive. The set $\boldsymbol{N} = \bigcap\{X : X$ is inductive$\}$ is the least limit ordinal $\neq 0$.

2.4. (Without the Axiom of Infinity). Let $\omega =$ least limit $\alpha \neq 0$ if it exists, $\omega = Ord$ otherwise. Prove that the following statements are equivalent:

(i) There exists an inductive set.
(ii) There exists an infinite set.
(iii) ω is a set.

[For (ii) → (iii), apply Replacement to the set of all finite subsets of X.]

2.5. If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in W such that $a_0 > a_1 > a_2 > \ldots$.

2.6. There are arbitrarily large limit ordinals; i.e., $\forall\alpha\,\exists\beta > \alpha\,(\beta$ is a limit).
[Consider $\lim_{n\to\omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

2.7. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord\rangle$ has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.
[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n\to\omega} \alpha_n$.]

2.8. For all α, β and γ,

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,
(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$,
(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

2.9. (i) Show that $(\omega + 1) \cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$.
(ii) Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.

2.10. If $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$,

2.11. Find α, β, γ such that

(i) $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$,
(ii) $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$,
(iii) $\alpha < \beta$ and $\alpha^\gamma = \beta^\gamma$.

2.12. Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.

A limit ordinal $\gamma > 0$ is called *indecomposable* if there exist no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$.

2.13. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α.

2.14. If E is a well-founded relation on P, then there is no sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in P such that $a_1 \mathrel{E} a_0$, $a_2 \mathrel{E} a_1$, $a_3 \mathrel{E} a_2$, ...

2.15 (Well-Founded Recursion). Let E be a well-founded relation on a set P, and let G be a function. Then there exists a function F such that for all $x \in P$, $F(x) = G(x, F{\restriction}\{y \in P : y \mathrel{E} x\})$.

Historical Notes

The theory of well-ordered sets was developed by Cantor, who also introduced transfinite induction. The idea of identifying an ordinal number with the set of smaller ordinals is due to Zermelo and von Neumann.

3. Cardinal Numbers

Cardinality

Two sets X, Y have the same *cardinality* (cardinal number, cardinal),

$$|X| = |Y|, \tag{3.1}$$

if there exists a one-to-one mapping of X onto Y.

The relation (3.1) is an equivalence relation. We assume that we can assign to each set X its *cardinal number* $|X|$ so that two sets are assigned the same cardinal just in case they satisfy condition (3.1). Cardinal numbers can be defined either using the Axiom of Regularity (via equivalence classes of (3.1)), or using the Axiom of Choice. In this chapter we define cardinal numbers of well-orderable sets; as it follows from the Axiom of Choice that every set can be well-ordered, this defines cardinals in ZFC.

We recall that a set X is *finite* if $|X| = |n|$ for some $n \in \boldsymbol{N}$; then X is said to *have n elements*. Clearly, $|n| = |m|$ if and only if $n = m$, and so we define *finite cardinals* as natural numbers, i.e., $|n| = n$ for all $n \in \boldsymbol{N}$.

The ordering of cardinal numbers is defined as follows:

$$|X| \le |Y| \tag{3.2}$$

if there exists a one-to-one mapping of X into Y. We also define the strict ordering $|X| < |Y|$ to mean that $|X| \le |Y|$ while $|X| \ne |Y|$. The relation $\le$ in (3.2) is clearly transitive. Theorem 3.2 below shows that it is indeed a partial ordering, and it follows from the Axiom of Choice that the ordering is linear—any two sets are comparable in this ordering.

The concept of cardinality is central to the study of infinite sets. The following theorem tells us that this concept is not trivial:

Theorem 3.1 (Cantor). *For every set X, $|X| < |P(X)|$.*

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a function of X onto $P(X)$. Hence $|P(X)| \ne |X|$.

The function $f(x) = \{x\}$ is a one-to-one function of X into $P(X)$ and so $|X| \le |P(X)|$. It follows that $|X| < |P(X)|$. □

In view of the following theorem, $<$ is a partial ordering of cardinal numbers.

Theorem 3.2 (Cantor-Bernstein). *If $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.*

Proof. If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $|A| = |B|$.

We define (by induction) for all $n \in \boldsymbol{N}$:

$$A_0 = A, \qquad A_{n+1} = f(A_n),$$
$$B_0 = B, \qquad B_{n+1} = f(B_n).$$

Let g be the function on A defined as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

Then g is a one-to-one mapping of A onto B, as the reader will easily verify. Thus $|A| = |B|$. □

The arithmetic operations on cardinals are defined as follows:

$$\begin{aligned} \kappa + \lambda &= |A \cup B| && \text{where } |A| = \kappa,\ |B| = \lambda, \text{ and } A, B \text{ are disjoint,} \\ \kappa \cdot \lambda &= |A \times B| && \text{where } |A| = \kappa,\ |B| = \lambda, \\ \kappa^\lambda &= |A^B| && \text{where } |A| = \kappa,\ |B| = \lambda. \end{aligned} \tag{3.3}$$

Naturally, the definitions in (3.3) are meaningful only if they are independent of the choice of A and B. Thus one has to check that, e.g., if $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$.

Lemma 3.3. *If $|A| = \kappa$, then $|P(A)| = 2^\kappa$.*

Proof. For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \in A - X. \end{cases}$$

The mapping $f : X \to \chi_X$ is a one-to-one correspondence between $P(A)$ and $\{0,1\}^A$. □

Thus Cantor's Theorem 3.1 can be formulated as follows:

$$\kappa < 2^\kappa \quad \text{for every cardinal } \kappa.$$

A few simple facts about cardinal arithmetic:

(3.4) $+$ and $\cdot$ are associative, commutative and distributive.

(3.5) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(3.6) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(3.7) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.

(3.8) If $\kappa \le \lambda$, then $\kappa^\mu \le \lambda^\mu$.

(3.9) If $0 < \lambda \le \mu$, then $\kappa^\lambda \le \kappa^\mu$.

(3.10) $\kappa^0 = 1$; $1^\kappa = 1$; $0^\kappa = 0$ if $\kappa > 0$.

To prove (3.4)–(3.10), one has only to find the appropriate one-to-one functions.

Alephs

An ordinal α is called a *cardinal number* (a cardinal) if $|\alpha| \ne |\beta|$ for all $\beta < \alpha$. We shall use κ, λ, μ, ... to denote cardinal numbers.

If W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal such that } |W| = |\alpha|.$$

Clearly, $|W|$ is a cardinal number.

Every natural number is a cardinal (a *finite cardinal*); and if S is a finite set, then $|S| = n$ for some n.

The ordinal ω is the least infinite cardinal. Note that all infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*.

Lemma 3.4.

(i) *For every α there is a cardinal number greater than α.*
(ii) *If X is a set of cardinals, then $\sup X$ is a cardinal.*

For every α, let α^+ be the least cardinal number greater than α, the *cardinal successor* of α.

Proof. (i) For any set X, let

(3.11) $h(X) =$ the least α such that there is no one-to-one function of α into X.

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < |h(\alpha)|$ by (3.11). That proves (i).

(ii) Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be such that $\beta < \kappa \le \alpha$. Then $|\kappa| = |\{f(\xi) : \xi < \kappa\}| \le \beta$, a contradiction. Thus α is a cardinal. □

Using Lemma 3.4, we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type:

$$\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+,$$
$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Infinite sets that are not countable are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

To prove Theorem 3.5 we use a pairing function for ordinal numbers:

The Canonical Well-Ordering of $\alpha \times \alpha$

We define a well-ordering of the class *Ord* × *Ord* of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class *Ord*, and we have a one-to-one function Γ of Ord^2 onto *Ord*. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.

We define:

(3.12)
$$\begin{aligned}(\alpha,\beta) < (\gamma,\delta) \leftrightarrow\ & \text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\},\\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma,\\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\},\ \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

The relation $<$ defined in (3.12) is a linear ordering of the class *Ord* × *Ord*. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\},$$

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(3.13)\qquad (\alpha,\beta) < (\gamma,\delta) \quad\text{if and only if}\quad \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta).$$

Note that $\Gamma(\omega\times\omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha\times\alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is also continuous, and so $\Gamma(\alpha\times\alpha) = \alpha$ for arbitrarily large α.

Proof of Theorem 3.5. Consider the canonical one-to-one mapping Γ of $Ord\times Ord$ onto Ord. We shall show that $\Gamma(\omega_\alpha\times\omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\Gamma(\omega_\alpha\times\omega_\alpha) \neq \omega_\alpha$. Let $\beta,\gamma < \omega_\alpha$ be such that $\Gamma(\beta,\gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ such that $\delta > \beta$ and $\delta > \gamma$. Since $\delta\times\delta$ is an initial segment of $Ord\times Ord$ in the canonical well-ordering and contains (β,γ), we have $\Gamma(\delta\times\delta) \supset \omega_\alpha$, and so $|\delta\times\delta| \geq \aleph_\alpha$. However, $|\delta\times\delta| = |\delta|\cdot|\delta|$, and by the minimality of α, $|\delta|\cdot|\delta| = |\delta| < \aleph_\alpha$. A contradiction. □

As a corollary we have

$$(3.14)\qquad \aleph_\alpha + \aleph_\beta = \aleph_\alpha\cdot\aleph_\beta = \max\{\aleph_\alpha,\aleph_\beta\}.$$

Exponentiation of cardinals will be dealt with in Chapter 5. Without the Axiom of Choice, one cannot prove that $2^{\aleph_\alpha}$ is an aleph (or that $P(\omega_\alpha)$ can be well-ordered), and there is very little one can prove about $2^{\aleph_\alpha}$ or $\aleph_\alpha^{\aleph_\beta}$.

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle\alpha_\xi : \xi < \beta\rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi\to\beta}\alpha_\xi = \alpha$. Similarly, $A\subset\alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

cf α = the least limit ordinal β such that there is an increasing β-sequence $\langle\alpha_\xi : \xi<\beta\rangle$ with $\lim_{\xi\to\beta}\alpha_\xi = \alpha$.

Obviously, $\operatorname{cf}\alpha$ is a limit ordinal, and $\operatorname{cf}\alpha \leq \alpha$. Examples: $\operatorname{cf}(\omega+\omega) = \operatorname{cf}\aleph_\omega = \omega$.

Lemma 3.6. $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$.

Proof. If $\langle\alpha_\xi : \xi<\beta\rangle$ is cofinal in α and $\langle\xi(\nu) : \nu<\gamma\rangle$ is cofinal in β, then $\langle\alpha_{\xi(\nu)} : \nu<\gamma\rangle$ is cofinal in α. □

Two useful facts about cofinality:

Lemma 3.7. *Let $\alpha > 0$ be a limit ordinal.*

(i) *If $A\subset\alpha$ and $\sup A = \alpha$, then the order-type of A is at least* $\operatorname{cf}\alpha$.

(ii) *If $\beta_0\leq\beta_1\leq\ldots\leq\beta_\xi\leq\ldots$, $\xi<\gamma$, is a nondecreasing γ-sequence of ordinals in α and $\lim_{\xi\to\gamma}\beta_\xi = \alpha$, then $\operatorname{cf}\gamma = \operatorname{cf}\alpha$.*

Proof. (i) The order-type of A is the length of the increasing enumeration of A which is an increasing sequence with limit α.

(ii) If $\gamma = \lim_{\nu\to\operatorname{cf}\gamma}\xi(\nu)$, then $\alpha = \lim_{\nu\to\operatorname{cf}\gamma}\beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle\beta_{\xi(\nu)} : \nu<\operatorname{cf}\gamma\rangle$ has an increasing subsequence of length $\leq\operatorname{cf}\gamma$, with the same limit. Thus $\operatorname{cf}\alpha\leq\operatorname{cf}\gamma$.

To show that $\operatorname{cf}\gamma\leq\operatorname{cf}\alpha$, let $\alpha = \lim_{\nu\to\operatorname{cf}\alpha}\alpha_\nu$. For each $\nu<\operatorname{cf}\alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota)$, $\iota<\nu$, such that $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu\to\operatorname{cf}\alpha}\beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu\to\operatorname{cf}\alpha}\xi(\nu) = \gamma$, and so $\operatorname{cf}\gamma\leq\operatorname{cf}\alpha$. □

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf}\omega_\alpha = \omega_\alpha$. It is *singular* if $\operatorname{cf}\omega_\alpha < \omega_\alpha$.

Lemma 3.8. *For every limit ordinal α, $\operatorname{cf}\alpha$ is a regular cardinal.*

Proof. It is easy to see that if α is not a cardinal, then using a mapping of $|\alpha|$ onto α, one can construct a cofinal sequence in α of length $\leq|\alpha|$, and therefore $\operatorname{cf}\alpha<\alpha$.

Since $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$, it follows that $\operatorname{cf}\alpha$ is a cardinal and is regular. □

Let κ be a limit ordinal. A subset $X\subset\kappa$ is *bounded* if $\sup X<\kappa$, and *unbounded* if $\sup X = \kappa$.

Lemma 3.9. *Let κ be an aleph.*

(i) *If $X\subset\kappa$ and $|X|<\operatorname{cf}\kappa$ then X is bounded.*
(ii) *If $\lambda<\operatorname{cf}\kappa$ and $f:\lambda\to\kappa$ then the range of f is bounded.*

It follows from (i) that every unbounded subset of a regular cardinal has cardinality κ.

Proof. (i) Lemma 3.7(i).

(ii) If $X = \operatorname{ran}(f)$ then $|X|\leq\lambda$, and use (i). □

There are arbitrarily large singular cardinals. For each α, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality ω.

Using the Axiom of Choice, we shall show in Chapter 5 that every $\aleph_{\alpha+1}$ is regular. (The Axiom of Choice is necessary.)

Lemma 3.10. *An infinite cardinal κ is singular if and only if there exists a cardinal $\lambda<\kappa$ and a family $\{S_\xi : \xi<\lambda\}$ of subsets of κ such that $|S_\xi|<\kappa$ for each $\xi<\lambda$, and $\kappa = \bigcup_{\xi<\lambda}S_\xi$. The least cardinal λ that satisfies the condition is* $\operatorname{cf}\kappa$.

Proof. If κ is singular, then there is an increasing sequence $\langle\alpha_\xi : \xi<\operatorname{cf}\kappa\rangle$ with $\lim_\xi\alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi<\lambda$.

If the condition holds, let $\lambda<\kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi<\lambda\}$ such that $\kappa = \bigcup_{\xi<\lambda}S_\xi$ and $|S_\xi|<\kappa$ for each $\xi<\lambda$. For every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda\rangle$ is nondecreasing, and by the minimality of λ, $\beta_\xi < \kappa$ for all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf} \kappa \le \lambda$.

Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: If $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where ξ is the least ξ such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$. □

One cannot prove without the Axiom of Choice that ω_1 is not a countable union of countable sets. Compare this with Exercise 3.13

The only cardinal inequality we have proved so far is Cantor's Theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$, and in particular $\kappa < \kappa^\kappa$ (for $\kappa \ne 1$). The following theorem gives a better inequality. This and other cardinal inequalities will also follow from König's Theorem 5.10, to be proved in Chapter 5.

Theorem 3.11. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.*

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find $f : \operatorname{cf}\kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi\to\operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf}\kappa$, let

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \ne f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi.$$

Such γ exists since $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \le |\alpha_\xi| < \kappa$. Obviously, $f \ne f_\alpha$ for all $\alpha < \kappa$. □

Consequently, $\kappa^\lambda > \kappa$ whenever $\lambda \ge \operatorname{cf}\kappa$.

An uncountable cardinal κ is *weakly inaccessible* if it is a limit cardinal and is regular. There will be more about inaccessible cardinals later, but let me mention at this point that existence of (weakly) inaccessible cardinals is not provable in ZFC.

To get an idea of the size of an inaccessible cardinal, note that if $\aleph_\alpha > \aleph_0$ is limit and regular, then $\aleph_\alpha = \operatorname{cf}\aleph_\alpha = \operatorname{cf}\alpha \le \alpha$, and so $\aleph_\alpha = \alpha$.

Since the sequence of alephs is a normal sequence, it has arbitrarily large fixed points; the problem is whether some of them are regular cardinals. For instance, the least fixed point $\aleph_\alpha = \alpha$ has cofinality ω:

$$\kappa = \lim\langle \omega, \omega_\omega, \omega_{\omega_\omega}, \ldots\rangle = \lim_{n\to\omega} \kappa_n$$
$$\text{where } \kappa_0 = \omega,\ \kappa_{n+1} = \omega_{\kappa_n}.$$

Exercises

3.1. (i) A subset of a finite set is finite.
(ii) The union of a finite set of finite sets is finite.
(iii) The power set of a finite set is finite.
(iv) The image of a finite set (under a mapping) is finite.

3.2. (i) A subset of a countable set is at most countable.
(ii) The union of a finite set of countable sets is countable.
(iii) The image of a countable set (under a mapping) is at most countable.

3.3. $\boldsymbol{N} \times \boldsymbol{N}$ is countable.
[$f(m, n) = 2^m(2n + 1) - 1$.]

3.4. (i) The set of all finite sequences in $\boldsymbol{N}$ is countable.
(ii) The set of all finite subsets of a countable set is countable.

3.5. Show that $\Gamma(\alpha \times \alpha) \le \omega^\alpha$.

3.6. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \ge |B| > 0$, then B is a projection of A. Conversely, using the Axiom of Choice, one shows that if B is a projection of A, then $|A| \ge |B|$. This, however, cannot be proved without the Axiom of Choice.

3.7. If B is a projection of ω_α, then $|B| \le \aleph_\alpha$.

3.8. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$.
[The set is a projection of the set of finite sequences.]

3.9. If B is a projection of A, then $|P(B)| \le |P(A)|$.
[Consider $g(X) = f_{-1}(X)$, where f maps A onto B.]

3.10. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.
[Use $|\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ and project $P(\omega_\alpha \times \omega_\alpha)$: If $R \subset \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R)$ be its order-type.]

3.11. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.
[Use Exercises 3.10 and 3.9.]

3.12. If $\aleph_\alpha$ is an uncountable limit cardinal, then $\operatorname{cf}\omega_\alpha = \operatorname{cf}\alpha$; ω_α is the limit of a cofinal sequence $\langle \omega_\xi : \xi < \operatorname{cf}\alpha\rangle$ of cardinals.

3.13 (ZF). Show that ω_2 is not a countable union of countable sets.
[Assume that $\omega_2 = \bigcup_{n<\omega} S_n$ with S_n countable and let α_n be the order-type of S_n. Then $\alpha = \sup_n \alpha_n \le \omega_1$ and there is a mapping of $\omega \times \alpha$ onto ω_2.]

A set S is *Dedekind-finite* (D-finite) if there is no one-to-one mapping of S onto a proper subset of S. Every finite set is D-finite. Using the Axiom of Choice, one proves that every infinite set is D-infinite, and so D-finiteness is the same as finiteness. Without the Axiom of Choice, however, one cannot prove that every D-finite set is finite.

The set $\boldsymbol{N}$ of all natural numbers is D-infinite and hence every S such that $|S| \ge \aleph_0$, is D-infinite.

3.14. S is D-infinite if and only if S has a countable subset.

[If S is D-infinite, let $f : S \to X \subset S$ be one-to-one. Let $x_0 \in S - X$ and $x_{n+1} = f(x_n)$. Then $S \supset \{x_n : n < \omega\}$.]

3.15. (i) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.
(ii) The set of all finite one-to-one sequences in a D-finite set is D-finite.
(iii) The union of a disjoint D-finite family of D-finite sets is D-finite.

On the other hand, one cannot prove without the Axiom of Choice that a projection, power set, or the set of all finite subsets of a D-finite set is D-finite, or that the union of a D-finite family of D-finite sets is D-finite.

3.16. If A is an infinite set, then $PP(A)$ is D-infinite.

[Consider the set $\{\{X \subset A : |X| = n\} : n < \omega\}$.]

Historical Notes

Cardinal numbers and alephs were introduced by Cantor. The proof of the Cantor-Bernstein Theorem is Bernstein's; see Borel [1898], p. 103. (There is an earlier proof by Dedekind.) The first proof of $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ appeared in Hessenberg [1906], p. 593. Regularity of cardinals was investigated by Hausdorff, who also raised the question of existence of regular limit cardinals. D-finiteness was formulated by Dedekind.

4. Real Numbers

The set of all real numbers $\boldsymbol{R}$ (the *real line* or the *continuum*) is the unique ordered field in which every nonempty bounded set has a least upper bound. The proof of the following theorem marks the beginning of Cantor's theory of sets.

Theorem 4.1 (Cantor). *The set of all real numbers is uncountable.*

Proof. Let us assume that the set $\boldsymbol{R}$ of all reals is countable, and let c_0, $c_1, \ldots, c_n, \ldots, n \in \boldsymbol{N}$, be an enumeration of $\boldsymbol{R}$. We shall find a real number different from each c_n.

Let $a_0 = c_0$ and $b_0 = c_{k_0}$ where k_0 is the least k such that $a_0 < c_k$. For each n, let $a_{n+1} = c_{i_n}$ where i_n is the least i such that $a_n < c_i < b_n$, and $b_{n+1} = c_{k_n}$ where k_n is the least k such that $a_{n+1} < c_k < b_n$. If we let $a = \sup\{a_n : n \in \boldsymbol{N}\}$, then $a \neq c_k$ for all k. □

The Cardinality of the Continuum

Let $\mathfrak{c}$ denote the cardinality of $\boldsymbol{R}$. As the set $\boldsymbol{Q}$ of all rational numbers is dense in $\boldsymbol{R}$, every real number r is equal to $\sup\{q \in \boldsymbol{Q} : q < r\}$ and because $\boldsymbol{Q}$ is countable, it follows that $\mathfrak{c} \leq |P(\boldsymbol{Q})| = 2^{\aleph_0}$.

Let $\boldsymbol{C}$ (the *Cantor set*) be the set of all reals of the form $\sum_{n=1}^{\infty} a_n/3^n$, where each $a_n = 0$ or 2. $\boldsymbol{C}$ is obtained by removing from the closed interval $[0, 1]$, the open intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, etc. (the middle-third intervals). $\boldsymbol{C}$ is in a one-to-one correspondence with the set of all ω-sequences of 0's and 2's and so $|\boldsymbol{C}| = 2^{\aleph_0}$.

Therefore $\mathfrak{c} \geq 2^{\aleph_0}$, and so by the Cantor-Bernstein Theorem we have

$$\mathfrak{c} = 2^{\aleph_0}. \tag{4.1}$$

By Cantor's Theorem 4.1 (or by Theorem 3.1), $\mathfrak{c} > \aleph_0$. Cantor conjectured that every set of reals is either at most countable or has cardinality of the continuum. In ZFC, every infinite cardinal is an aleph, and so $2^{\aleph_0} \geq \aleph_1$. Cantor's conjecture then becomes the statement

$$2^{\aleph_0} = \aleph_1$$

known as the *Continuum Hypothesis* (CH).

Among sets of cardinality $\mathfrak{c}$ are the set of all sequences of natural numbers, the set of all sequences of real numbers, the set of all complex numbers. This is because $\aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

Cantor's proof of Theorem 4.1 yielded more than uncountability of $\boldsymbol{R}$; it showed that the set of all transcendental numbers has cardinality $\mathfrak{c}$ (cf. Exercise 4.5).

The Ordering of $\boldsymbol{R}$

A linear ordering $(P, <)$ is *complete* if every nonempty bounded subset of P has a least upper bound. We stated above that $\boldsymbol{R}$ is the unique complete ordered field. We shall generally disregard the field properties of $\boldsymbol{R}$ and will concern ourselves more with the order properties.

One consequence of being a complete ordered field is that $\boldsymbol{R}$ contains the set $\boldsymbol{Q}$ of all rational numbers as a dense subset. The set $\boldsymbol{Q}$ is countable and its ordering is dense.

Definition 4.2. A linear ordering $(P, <)$ is *dense* if for all $a < b$ there exists a c such that $a < c < b$.

A set $D \subset P$ is a *dense subset* if for all $a < b$ in P there exists a $d \in D$ such that $a < d < b$.

The following theorem proves the uniqueness of the ordered set $(\boldsymbol{R}, <)$. We say that an ordered set is *unbounded* if it has neither a least nor a greatest element.

Theorem 4.3 (Cantor).

(i) *Any two countable unbounded dense linearly ordered sets are isomorphic.*
(ii) $(\boldsymbol{R}, <)$ *is the unique complete linear ordering that has a countable dense subset isomorphic to* $(\boldsymbol{Q}, <)$.

Proof. (i) Let $P_1 = \{a_n : n \in \boldsymbol{N}\}$ and let $P_2 = \{b_n : n \in \boldsymbol{N}\}$ be two such linearly ordered sets. We construct an isomorphism $f : P_1 \to P_2$ in the following way: We first define $f(a_0)$, then $f^{-1}(b_0)$, then $f(a_1)$, then $f^{-1}(b_1)$, etc., so as to keep f order-preserving. For example, to define $f(a_n)$, if it is not yet defined, we let $f(a_n) = b_k$ where k is the least index such that f remains order-preserving (such a k always exists because f has been defined for only finitely many $a \in P_1$, and because P_2 is dense and unbounded).

(ii) To prove the uniqueness of $\boldsymbol{R}$, let C and C' be two complete dense unbounded linearly ordered sets, let P and P' be dense in C and C', respectively, and let f be an isomorphism of P onto P'. Then f can be extended (uniquely) to an isomorphism f^* of C and C': For $x \in C$, let $f^*(x) = \sup\{f(p) : p \in P \text{ and } p \leq x\}$. □

The existence of $(\boldsymbol{R}, <)$ is proved by means of *Dedekind cuts* in $(\boldsymbol{Q}, <)$. The following theorem is a general version of this construction:

Theorem 4.4. *Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ such that*:

(i) *$P \subset C$, and $<$ and $\prec$ agree on P;*
(ii) *P is dense in C.*

Proof. A *Dedekind cut* in P is a pair (A, B) of disjoint nonempty subsets of P such that

(i) $A \cup B = P$;
(ii) $a < b$ for any $a \in A$ and $b \in B$;
(iii) A does not have a greatest element.

Let C be the set of all Dedekind cuts in P and let $(A_1, B_1) \preccurlyeq (A_2, B_2)$ if $A_1 \subset A_2$ (and $B_1 \supset B_2$). The set C is complete: If $\{(A_i, B_i) : i \in I\}$ is a nonempty bounded subset of C, then $(\bigcup_{i \in I} A_i, \bigcap_{i \in I} B_i)$ is its supremum. For $p \in P$, let

$$A_p = \{x \in P : x < p\}, \qquad B_p = \{x \in P : x \geq p\}.$$

Then $P' = \{(A_p, B_p) : p \in P\}$ is isomorphic to P and is dense in C. □

Suslin's Problem

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete and contains a countable dense subset.

Since $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$, every nonempty open interval of $\boldsymbol{R}$ contains a rational number. Hence if S is a disjoint collection of open intervals, S is at most countable. (Let $\langle r_n : n \in \boldsymbol{N} \rangle$ be an enumeration of the rationals. To each $J \in S$ assign $r_n \in J$ with the least possible index n.)

Let P be a dense linearly ordered set. If every disjoint collection of open intervals in P is at most countable, then we say that P satisfies the *countable chain condition.*

Suslin's Problem. *Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?*

This question cannot be decided in ZFC; we shall return to the problem in Chapter 9.

The Topology of the Real Line

The real line is a metric space with the metric $d(a,b) = |a-b|$. Its metric topology coincides with the order topology of $(\boldsymbol{R}, <)$. Since $\boldsymbol{Q}$ is a dense set in $\boldsymbol{R}$ and since every Cauchy sequence of real numbers converges, $\boldsymbol{R}$ is a separable complete metric space. (A metric space is *separable* if it has a countable dense set; it is *complete* if every Cauchy sequence converges.)

Open sets are unions of open intervals, and in fact, every open set is the union of open intervals with rational endpoints. This implies that the number of all open sets in $\boldsymbol{R}$ is the continuum and so is the number of all closed sets in $\boldsymbol{R}$ (Exercise 4.6).

Every open interval has cardinality $\mathfrak{c}$, therefore every nonempty open set has cardinality $\mathfrak{c}$. We show below that every uncountable closed set has cardinality $\mathfrak{c}$. Proving this was Cantor's first step in the search for the proof of the Continuum Hypothesis. In Chapter 11 we show that CH holds for Borel and analytic sets as well.

A nonempty closed set is *perfect* if it has no isolated points. Theorems 4.5 and 4.6 below show that every uncountable closed set contains a perfect set.

Theorem 4.5. *Every perfect set has cardinality* $\mathfrak{c}$.

Proof. Given a perfect set P, we want to find a one-to-one function F from $\{0,1\}^\omega$ into P. Let S be the set of all finite sequences of 0's and 1's. By induction on the length of $s \in S$ one can find closed intervals I_s such that for each n and all $s \in S$ of length n,

(i) $I_s \cap P$ is perfect,
(ii) the diameter of I_s is $\le 1/n$,
(iii) $I_{s^\frown 0} \subset I_s$, $I_{s^\frown 1} \subset I_s$ and $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$.

For each $f \in \{0,1\}^\omega$, the set $P \cap \bigcap_{n=0}^\infty I_{f\restriction n}$ has exactly one element, and we let $F(f)$ to be this element of P. □

The same proof gives a more general result: Every perfect set in a separable complete metric space contains a closed copy of the Cantor set (Exercise 4.19).

Theorem 4.6 (Cantor-Bendixson). *If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is at most countable.*

Corollary 4.7. *If F is a closed set, then either $|F| \le \aleph_0$ or $|F| = 2^{\aleph_0}$.* □

Proof. For every $A \subset \boldsymbol{R}$, let

$$A' = \text{the set of all limit points of } A$$

It is easy to see that A' is closed, and if A is closed then $A' \subset A$. Thus we let

$$F_0 = F, \qquad F_{\alpha+1} = F'_\alpha,$$
$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma \quad \text{if } \alpha > 0 \text{ is a limit ordinal.}$$

Since $F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots$, there exists an ordinal θ such that $F_\alpha = F_\theta$ for all $\alpha \ge \theta$. (In fact, the least θ with this property must be countable, by the argument below.) We let $P = F_\theta$.

If P is nonempty, then $P' = P$ and so it is perfect. Thus the proof is completed by showing that $F - P$ is at most countable.

Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. We have $F - P = \bigcup_{\alpha<\theta}(F_\alpha - F'_\alpha)$; hence if $a \in F - P$, then there is a unique α such that a is an isolated point of F_α. We let $k(a)$ be the least k such that a is the only point of F_α in the interval J_k. Note that if $\alpha \le \beta$, $b \ne a$ and $b \in F_\beta - F'_\beta$, then $b \notin J_{k(a)}$, and hence $k(b) \ne k(a)$. Thus the correspondence $a \mapsto k(a)$ is one-to-one, and it follows that $F - P$ is at most countable. □

A set of reals is called *nowhere dense* if its closure has empty interior. The following theorem shows that $\boldsymbol{R}$ is not the union of countably many nowhere dense sets ($\boldsymbol{R}$ is not of *the first category*).

Theorem 4.8 (The Baire Category Theorem). *If $D_0, D_1, \ldots, D_n, \ldots$, $n \in \boldsymbol{N}$, are dense open sets of reals, then the intersection $D = \bigcap_{n=0}^\infty D_n$ is dense in $\boldsymbol{R}$.*

Proof. We show that D intersects every nonempty open interval I. First note that for each n, $D_0 \cap \ldots \cap D_n$ is dense and open. Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. Let $I_0 = I$, and let, for each n, $I_{n+1} = J_k = (q_k, r_k)$, where k is the least k such that the closed interval $[q_k, r_k]$ is included in $I_n \cap D_n$. Then $a \in D \cap I$, where $a = \lim_{k\to\infty} q_k$. □

Borel Sets

Definition 4.9. An *algebra of sets* is a collection $\mathcal{S}$ of subsets of a given set S such that

(4.2) (i) $S \in \mathcal{S}$,
(ii) if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$,
(iii) if $X \in \mathcal{S}$ then $S - X \in \mathcal{S}$.

(Note that $\mathcal{S}$ is also closed under intersections.)

A *σ-algebra* is additionally closed under countable unions (and intersections):

(iv) If $X_n \in \mathcal{S}$ for all n, then $\bigcup_{n=0}^\infty X_n \in \mathcal{S}$.

For any collection $\mathcal{X}$ of subsets of S there is a smallest algebra (σ-algebra) $\mathcal{S}$ such that $\mathcal{S} \supset \mathcal{X}$; namely the intersection of all algebras (σ-algebras) $\mathcal{S}$ of subsets of S for which $\mathcal{X} \subset \mathcal{S}$.

Definition 4.10. A set of reals B is *Borel* if it belongs to the smallest σ-algebra $\mathcal{B}$ of sets of reals that contains all open sets.

In Chapter 11 we investigate Borel sets in more detail. In particular, we shall classify Borel sets by defining a hierarchy of ω_1 levels. For that we need however a weak version of the Axiom of Choice that is not provable in ZF alone. At this point we mention the lowest level of the hierarchy (beyond open sets and closed sets): The intersections of countably many open sets are called *G_δ sets*, and the unions of countably many closed sets are called *F_σ sets*.

Lebesgue Measure

We assume that the reader is familiar with the basic theory of Lebesgue measure. As we shall return to the subject in Chapter 11 we do not define the concept of measure at this point. We also caution the reader that some of the basic theorems on Lebesgue measure require the Countable Axiom of Choice (to be discussed in Chapter 5).

Lebesgue measurable sets form a σ-algebra and contain all open intervals (the measure of an interval is its length). Thus all Borel sets are Lebesgue measurable.

The Baire Space

The *Baire space* is the space $\mathcal{N} = \omega^\omega$ of all infinite sequences of natural numbers, $\langle a_n : n \in \boldsymbol{N}\rangle$, with the following topology: For every finite sequence $s = \langle a_k : k < n\rangle$, let

(4.3) $$O(s) = \{f \in \mathcal{N} : s \subset f\} = \{\langle c_k : k \in \boldsymbol{N}\rangle : (\forall k < n)\, c_k = a_k\}.$$

The sets (4.3) form a basis for the topology of $\mathcal{N}$. Note that each $O(s)$ is also closed.

The Baire space is separable and is metrizable: consider the metric $d(f, g) = 1/2^{n+1}$ where n is the least number such that $f(n) \neq g(n)$. The countable set of all eventually constant sequences is dense in $\mathcal{N}$. This separable metric space is complete, as every Cauchy sequence converges.

Every infinite sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ of positive integers defines a *continued fraction* $1/(a_0 + 1/(a_1 + 1/(a_2 + \ldots)))$, an irrational number between 0 and 1. Conversely, every irrational number in the interval $(0, 1)$ can be so represented, and the one-to-one correspondence is a homeomorphism. It follows that the Baire space is homeomorphic to the space of all irrational numbers.

For various reasons, modern descriptive set theory uses the Baire space rather than the real line. Often the functions in ω^ω are called reals.

Clearly, the space $\mathcal{N}$ satisfies the Baire Category Theorem; the proof is similar to the proof of Theorem 4.8 above. The Cantor-Bendixson Theorem holds as well. For completeness we give a description of perfect sets in $\mathcal{N}$.

Let *Seq* denote the set of all finite sequences of natural numbers. A (*sequential*) *tree* is a set $T \subset Seq$ that satisfies

(4.4) $$\text{if } t \in T \text{ and } s = t{\restriction}n \text{ for some } n, \text{ then } s \in T.$$

If $T \subset Seq$ is a tree, let $[T]$ be the set of all *infinite paths* through T:

(4.5) $$[T] = \{f \in \mathcal{N} : f{\restriction}n \in T \text{ for all } n \in \boldsymbol{N}\}.$$

The set $[T]$ is a closed set in the Baire space: Let $f \in \mathcal{N}$ be such that $f \notin [T]$. Then there is $n \in \boldsymbol{N}$ such that $f{\restriction}n = s$ is not in T. In other words, the open set $O(s) = \{g \in \mathcal{N} : g \supset s\}$, a neighborhood of f, is disjoint from $[T]$. Hence $[T]$ is closed.

Conversely, if F is a closed set in $\mathcal{N}$, then the set

(4.6) $$T_F = \{s \in Seq : s \subset f \text{ for some } f \in F\}$$

is a tree, and it is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f{\restriction}n \in T$ for all $n \in \boldsymbol{N}$, then for each n there is some $g \in F$ such that $g{\restriction}n = f{\restriction}n$; and since F is closed, it follows that $f \in F$.

If f is an isolated point of a closed set F in $\mathcal{N}$, then there is $n \in \mathcal{N}$ such that there is no $g \in F$, $g \neq f$, such that $g{\restriction}n = f{\restriction}n$. Thus the following definition:

A nonempty sequential tree T is *perfect* if for every $t \in T$ there exist $s_1 \supset t$ and $s_2 \supset t$, both in T, that are *incomparable*, i.e., neither $s_1 \supset s_2$ nor $s_2 \supset s_1$.

Lemma 4.11. *A closed set $F \subset \mathcal{N}$ is perfect if and only if the tree T_F is a perfect tree.* □

The Cantor-Bendixson analysis for closed sets in the Baire space is carried out as follows: For each tree $T \subset Seq$, we let

(4.7) $$T' = \{t \in T : \text{there exist incomparable } s_1 \supset t \text{ and } s_2 \supset t \text{ in } T\}.$$

(Thus T is perfect if and only if $\emptyset \neq T = T'$.)

The set $[T]-[T']$ is at most countable: For each $f \in [T]$ such that $f \notin [T']$, let $s_f = f{\restriction}n$ where n is the least number such that $f{\restriction}n \notin T'$. If $f, g \in [T] - [T']$, then $s_f \neq s_g$, by (4.7). Hence the mapping $f \mapsto s_f$ is one-to-one and $[T] - [T']$ is at most countable.

Now we let

(4.8) $$T_0 = T, \qquad T_{\alpha+1} = T'_\alpha,$$
$$T_\alpha = \bigcap_{\beta<\alpha} T_\beta \quad \text{if } \alpha > 0 \text{ is a limit ordinal.}$$

Since $T_0 \supset T_1 \supset \ldots \supset T_\alpha \supset \ldots$, and T_0 is at most countable, there is an ordinal $\theta < \omega_1$ such that $T_{\theta+1} = T_\theta$. If $T_\theta \neq \emptyset$, then it is perfect.

Now it is easy to see that $[\bigcap_{\beta<\alpha} T_\beta] = \bigcap_{\beta<\alpha}[T_\beta]$, and so

$$[T] - [T_\theta] = \bigcup_{\alpha<\theta} ([T_\alpha] - [T'_\alpha]); \tag{4.9}$$

hence (4.9) is at most countable. Thus if $[T]$ is an uncountable closed set in $\mathcal{N}$, the sets $[T_\theta]$ and $[T] - [T_\theta]$ constitute the decomposition of $[T]$ into a perfect and an at most countable set.

In modern descriptive set theory one often speaks about the *Lebesgue measure* on $\mathcal{N}$. This measure is the extension of the product measure m on Borel sets in the Baire space induced by the probability measure on $\boldsymbol{N}$ that gives the singleton $\{n\}$ measure $1/2^{n+1}$. Thus for every sequence $s \in Seq$ of length $n \geq 1$ we have

$$m(O(s)) = \prod_{k=0}^{n-1} 1/2^{s(k)+1}. \tag{4.10}$$

Polish Spaces

Definition 4.12. A *Polish space* is a topological space that is homeomorphic to a separable complete metric space.

Examples of Polish spaces include $\boldsymbol{R}$, $\mathcal{N}$, the Cantor space, the unit interval $[0,1]$, the unit circle T, the Hilbert cube $[0,1]^\omega$, etc.

Every Polish space is a continuous image of the Baire space. In Chapter 11 we prove a somewhat more general statement.

Exercises

4.1. The set of all continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^\mathfrak{c}$).

[A continuous function on $\boldsymbol{R}$ is determined by its values at rational points.]

4.2. There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

[For every sequence $a = \langle a_n : n \in \boldsymbol{N}\rangle$ of natural numbers consider the order-type

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \ldots$$

where ξ is the order-type of the integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.]

A real number is *algebraic* if it is a root of a polynomial whose coefficients are integers. Otherwise, it is *transcendental.*

4.3. The set of all algebraic reals is countable.

4.4. If S is a countable set of reals, then $|\boldsymbol{R} - S| = \mathfrak{c}$.

[Use $\boldsymbol{R} \times \boldsymbol{R}$ rather than $\boldsymbol{R}$ (because $|\boldsymbol{R} \times \boldsymbol{R}| = 2^{\aleph_0}$).]

4.5. (i) The set of all irrational numbers has cardinality $\mathfrak{c}$.

(ii) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

4.6. The set of all open sets of reals has cardinality $\mathfrak{c}$.

4.7. The Cantor set is perfect.

4.8. If P is a perfect set and (a,b) is an open interval such that $P \cap (a,b) \neq \emptyset$, then $|P \cap (a,b)| = \mathfrak{c}$.

4.9. If $P_2 \not\subset P_1$ are perfect sets, then $|P_2 - P_1| = \mathfrak{c}$.

[Use Exercise 4.8.]

If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

4.10. If P is perfect then $P^* = P$.

[Use Exercise 4.8.]

4.11. If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.

[$P = P^* \subset F^*$.]

4.12. If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$; thus $F^* = P$.

[Every $a \in F^*$ is a limit point of P since $|F - P| \leq \aleph_0$.]

4.13. If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.

[Use Exercise 4.9.]

4.14. $\boldsymbol{Q}$ is not the intersection of a countable collection of open sets.

[Use the Baire Category Theorem.]

4.15. If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

4.16. Let $f : \boldsymbol{R} \to \boldsymbol{R}$. Show that the set of all x at which f is continuous is a G_δ set.

4.17. (i) $\mathcal{N} \times \mathcal{N}$ is homeomorphic to $\mathcal{N}$.

(ii) $\mathcal{N}^\omega$ is homeomorphic to $\mathcal{N}$.

4.18. The tree T_F in (4.6) has no maximal node, i.e., $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential trees without maximal nodes.

4.19. Every perfect Polish space has a closed subset homeomorphic to the Cantor space.

4.20. Every Polish space is homeomorphic to a G_δ subspace of the Hilbert cube.

[Let $\{x_n : n \in \boldsymbol{N}\}$ be a dense set, and define $f(x) = \langle d(x, x_n) : n \in \boldsymbol{N}\rangle$.]

Historical Notes

Theorems 4.1, 4.3 and 4.5 are due to Cantor. The construction of real numbers by completion of the rationals is due to Dedekind [1872].

Suslin's Problem: Suslin [1920].

Theorem 4.6: Cantor, Bendixson [1883].

Theorem 4.8: Baire [1899].

Exercise 4.5: Cantor.

5. The Axiom of Choice and Cardinal Arithmetic

The Axiom of Choice

Axiom of Choice **(AC).** *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

for every $X \in S$.

The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

The Axiom of Choice differs from other axioms of ZF by postulating the existence of a set (i.e., a choice function) without defining it (unlike, for instance, the Axiom of Pairing or the Axiom of Power Set). Thus it is often interesting to know whether a mathematical statement can be proved without using the Axiom of Choice. It turns out that the Axiom of Choice is independent of the other axioms of set theory and that many mathematical theorems are unprovable in ZF without AC.

In some trivial cases, the existence of a choice function can be proved outright in ZF:

(i) when every $X \in S$ is a singleton $X = \{x\}$;
(ii) when S is finite; the existence of a choice function for S is proved by induction on the size of S;
(iii) when every $X \in S$ is a finite set of real numbers; let $f(X) =$ the least element of X.

On the other hand, one cannot prove existence of a choice function (in ZF) just from the assumption that the sets in S are finite; even when every $X \in S$ has just two elements (e.g., sets of reals), we cannot necessarily prove that S has a choice function.

Using the Axiom of Choice, one proves that every set can be well-ordered, and therefore every infinite set has cardinality equal to some $\aleph_\alpha$. In particular,

any two sets have comparable cardinals, and the ordering

$$|X| \le |Y|$$

is a well-ordering of the class of all cardinals.

Theorem 5.1 (Zermelo's Well-Ordering Theorem). *Every set can be well-ordered.*

Proof. Let A be a set. To well-order A, it suffices to construct a transfinite one-to-one sequence $\langle a_\alpha : \alpha < \theta \rangle$ that enumerates A. That we can do by induction, using a choice function f for the family S of all nonempty subsets of A. We let for every α

$$a_\alpha = f(A - \{a_\xi : \xi < \alpha\})$$

if $A - \{a_\xi : \xi < \alpha\}$ is nonempty. Let θ be the least ordinal such that $A = \{a_\xi : \xi < \theta\}$. Clearly, $\langle a_\alpha : \alpha < \theta \rangle$ enumerates A. □

In fact, Zermelo's Theorem 5.1 is equivalent to the Axiom of Choice: If every set can be well-ordered, then every family S of nonempty sets has a choice function. To see this, well-order $\bigcup S$ and let $f(X)$ be the least element of X for every $X \in S$.

Of particular importance is the fact that the set of all real numbers can be well-ordered. It follows that $2^{\aleph_0}$ is an aleph and so $2^{\aleph_0} \ge \aleph_1$.

The existence of a well-ordering of $\boldsymbol{R}$ yields some interesting counterexamples. Well known is Vitali's construction of a nonmeasurable set (Exercise 10.1); another example is an uncountable set of reals without a perfect subset (Exercise 5.1).

If every set can be well-ordered, then every infinite set has a countable subset: Well-order the set and take the first ω elements. Thus every infinite set is Dedekind-infinite, and so finiteness and Dedekind finiteness coincide.

Dealing with cardinalities of sets is much easier when we have the Axiom of Choice. In the first place, any two sets have comparable cardinals. Another consequence is:

(5.2) *if f maps A onto B then $|B| \le |A|$.*

To show (5.2), we have to find a one-to-one function from B to A. This is done by choosing one element from $f_{-1}(\{b\})$ for each $b \in B$.

Another consequence of the Axiom of Choice is:

(5.3) *The union of a countable family of countable sets is countable.*

(By the way, this often used fact cannot be proved in ZF alone.) To prove (5.3) let A_n be a countable set for each $n \in \boldsymbol{N}$. For each n, let us *choose* an enumeration $\langle a_{n,k} : k \in \boldsymbol{N}\rangle$ of A_n. That gives us a projection of $\boldsymbol{N} \times \boldsymbol{N}$ onto $\bigcup_{n=0}^{\infty} A_n$:

$$(n, k) \mapsto a_{n,k}.$$

Thus $\bigcup_{n=0}^{\infty} A_n$ is countable.

In a similar fashion, one can prove a more general statement.

Lemma 5.2. $|\bigcup S| \le |S| \cdot \sup\{|X| : X \in S\}$.

Proof. Let $\kappa = |S|$ and $\lambda = \sup\{|X| : X \in S\}$. We have $S = \{X_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$, we choose an enumeration $X_\alpha = \{a_{\alpha,\beta} : \beta < \lambda_\alpha\}$, where $\lambda_\alpha \le \lambda$. Again we have a projection

$$(\alpha, \beta) \mapsto a_{\alpha,\beta}$$

of $\kappa \times \lambda$ onto $\bigcup S$, and so $|\bigcup S| \le \kappa \cdot \lambda$. □

In particular, the union of $\aleph_\alpha$ sets, each of cardinality $\aleph_\alpha$, has cardinality $\aleph_\alpha$.

Corollary 5.3. *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

Proof. This is because otherwise $\omega_{\alpha+1}$ would be the union of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$. □

Using the Axiom of Choice in Mathematics

In algebra and point set topology, one often uses the following version of the Axiom of Choice. We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \le c$ for every $x \in X$.

We say that a nonempty $C \subset P$ is a *chain* in P if C is linearly ordered by $<$.

Theorem 5.4 (Zorn's Lemma). *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Proof. We construct (using a choice function for nonempty subsets of P), a chain in P that leads to a maximal element of P. We let, by induction,

$$a_\alpha = \text{an element of } P \text{ such that } a_\alpha > a_\xi \text{ for every } \xi < \alpha \text{ if there is one.}$$

Clearly, if $\alpha > 0$ is a limit ordinal, then $C_\alpha = \{a_\xi : \xi < \alpha\}$ is a chain in P and a_α exists by the assumption. Eventually, there is θ such that there is no $a_{\theta+1} \in P$, $a_{\theta+1} > a_\theta$. Thus a_θ is a maximal element of P. □

Like Zermelo's Theorem 5.1, Zorn's Lemma 5.4 is equivalent to the Axiom of Choice (in ZF); see Exercise 5.5.

There are numerous examples of proofs using Zorn's Lemma. To mention only of few:

Every vector space has a basis.
Every field has a unique algebraic closure.
The Hahn-Banach Extension Theorem.
Tikhonov's Product Theorem for compact spaces.

The Countable Axiom of Choice

Many important consequences of the Axiom of Choice, particularly many concerning the real numbers, can be proved from a weaker version of the Axiom of Choice.

The Countable Axiom of Choice. *Every countable family of nonempty sets has a choice function.*

For instance, the countable AC implies that the union of countably many countable sets is countable. In particular, the real line is not a countable union of countable sets. Similarly, it follows that $\aleph_1$ is a regular cardinal. On the other hand, the countable AC does not imply that the set of all reals can be well-ordered.

Several basic theorems about Borel sets and Lebesgue measure use the countable AC; for instance, one needs it to show that the union of countably many F_σ sets is F_σ. In modern descriptive set theory one often works without the Axiom of Choice and uses the countable AC instead. In some instances, descriptive set theorists use a somewhat stronger principle (that follows from AC):

***The Principle of Dependent Choices* (DC).** *If E is a binary relation on a nonempty set A, and if for every $a \in A$ there exists $b \in A$ such that $b \mathrel{E} a$, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that*

$$a_{n+1} \mathrel{E} a_n \quad \text{for all } n \in \boldsymbol{N}. \tag{5.4}$$

The Principle of Dependent Choices is stronger than the Countable Axiom of Choice; see Exercise 5.7.

As an application of DC we have the following characterization of well-founded relations and well-orderings:

Lemma 5.5.

(i) *A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descending sequence*

$$a_0 > a_1 > \ldots > a_n > \ldots$$

in A.

(ii) *A relation E on P is well-founded if and only if there is no infinite sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in P such that*

$$a_{n+1} \mathrel{E} a_n \quad \text{for all } n \in \boldsymbol{N}. \tag{5.5}$$

Proof. Note that (i) is a special case of (ii) since a well-ordering is a well-founded linear ordering.

If $a_0, a_1, \ldots, a_n, \ldots$ is a sequence that satisfies (5.5), then the set $\{a_n : n \in \boldsymbol{N}\}$ has no E-minimal element and hence E is not well-founded.

Conversely, if E is not well-founded, then there is a nonempty set $A \subset P$ with no E-minimal element. Using the Principle of Dependent Choices we construct a sequence $a_0, a_1, \ldots, a_n, \ldots$ that satisfies (5.5). □

Cardinal Arithmetic

In the presence of the Axiom of Choice, every set can be well-ordered and so every infinite set has the cardinality of some $\aleph_\alpha$. Thus addition and multiplication of infinite cardinal numbers is simple: If κ and λ are infinite cardinals then

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

The exponentiation of cardinals is more interesting. The rest of Chapter 5 is devoted to the operations 2^κ and κ^λ, for infinite cardinals κ and λ.

Lemma 5.6. *If $2 \le \kappa \le \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$.*

Proof.

$$2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda. \tag{5.6}$$ □

If κ and λ are infinite cardinals and $\lambda < \kappa$ then the evaluation of κ^λ is more complicated. First, if $2^\lambda \ge \kappa$ then we have $\kappa^\lambda = 2^\lambda$ (because $\kappa^\lambda \le (2^\lambda)^\lambda = 2^\lambda$), but if $2^\lambda < \kappa$ then (because $\kappa^\lambda \le \kappa^\kappa = 2^\kappa$) we can only conclude

$$\kappa \le \kappa^\lambda \le 2^\kappa. \tag{5.7}$$

Not much more can be claimed at this point, except that by Theorem 3.11 in Chapter 3 ($\kappa^{\operatorname{cf}\kappa} > \kappa$) we have

$$\kappa < \kappa^\lambda \quad \text{if } \lambda \ge \operatorname{cf}\kappa. \tag{5.8}$$

If λ is a cardinal and $|A| \ge \lambda$, let

$$[A]^\lambda = \{X \subset A : |X| = \lambda\}. \tag{5.9}$$

Lemma 5.7. *If $|A| = \kappa \ge \lambda$, then the set $[A]^\lambda$ has cardinality κ^λ.*

Proof. On the one hand, every $f : \lambda \to A$ is a subset of $\lambda \times A$, and $|f| = \lambda$. Thus $\kappa^\lambda \le |[\lambda \times A]|^\lambda = |[A]^\lambda|$. On the other hand, we construct a one-to-one function $F : [A]^\lambda \to A^\lambda$ as follows: If $X \subset A$ and $|X| = \lambda$, let $F(X)$ be some function f on λ whose range is X. Clearly, F is one-to-one. □

If λ is a limit cardinal, let

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda\}. \tag{5.10}$$

For the sake of completeness, we also define $\kappa^{<\lambda^+} = \kappa^\lambda$ for infinite successor cardinals λ^+.

If κ is an infinite cardinal and $|A| \ge \kappa$, let

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subset A : |X| < \kappa\}. \tag{5.11}$$

It follows from Lemma 5.7 and Lemma 5.8 below that the cardinality of $P_\kappa(A)$ is $|A|^{<\kappa}$.

Infinite Sums and Products

Let $\{\kappa_i : i \in I\}$ be an indexed set of cardinal numbers. We define

$$\sum_{i\in I} \kappa_i = \Bigl|\bigcup_{i\in I} X_i\Bigr|, \tag{5.12}$$

where $\{X_i : i \in I\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in I$.

This definition does not depend on the choice of $\{X_i\}_i$; this follows from the Axiom of Choice (see Exercise 5.9).

Note that if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \kappa.$$

In general, we have the following

Lemma 5.8. *If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i. \tag{5.13}$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. On the one hand, since $\kappa_i \le \kappa$ for all i, we have $\sum_{i<\lambda} \kappa \le \lambda \cdot \kappa$. On the other hand, since $\kappa_i \ge 1$ for all i, we have $\lambda = \sum_{i<\lambda} 1 \le \sigma$, and since $\sigma \ge \kappa_i$ for all i, we have $\sigma \ge \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\sigma \ge \lambda \cdot \kappa$. □

In particular, if $\lambda \le \sup_{i<\lambda} \kappa_i$, we have

$$\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i.$$

Thus we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \sum_{i<\lambda} \kappa_i$$

where $\lambda < \kappa$ and for each i, $\kappa_i < \kappa$.

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the *product* is defined as follows:

$$\prod_{i\in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}. \tag{5.14}$$

Note that if some X_i is empty, then the product is empty. If all the X_i are nonempty, then AC implies that the product is nonempty.

If $\{\kappa_i : i \in I\}$ is a family of cardinal numbers, we define

$$\prod_{i\in I} \kappa_i = \Bigl|\prod_{i\in I} X_i\Bigr|, \tag{5.15}$$

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i| = \kappa_i$ for each $i \in I$. (We abuse the notation by using $\prod$ both for the product of sets and for the product of cardinals.)

Again, it follows from AC that the definition does not depend on the choice of the sets X_i (Exercise 5.10).

If $\kappa_i = \kappa$ for each $i \in I$, and $|I| = \lambda$, then $\prod_{i\in I} \kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance, $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$, or $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$. Or if I is a disjoint union $I = \bigcup_{j\in J} A_j$, then

$$\prod_{i\in I} \kappa_i = \prod_{j\in J}\Bigl(\prod_{i\in A_j} \kappa_i\Bigr). \tag{5.16}$$

If $\kappa_i \ge 2$ for each $i \in I$, then

$$\sum_{i\in I} \kappa_i \le \prod_{i\in I} \kappa_i. \tag{5.17}$$

(The assumption $\kappa_i \ge 2$ is necessary: $1+1 > 1\cdot 1$.) If I is finite, then (5.17) is certainly true; thus assume that I is infinite. Since $\prod_{i\in I} \kappa_i \ge \prod_{i\in I} 2 = 2^{|I|} > |I|$, it suffices to show that $\sum_i \kappa_i \le |I| \cdot \prod_i \kappa_i$. If $\{X_i : i \in I\}$ is a disjoint family, we assign to each $x \in \bigcup_i X_i$ a pair (i, f) such that $x \in X_i$, $f \in \prod_i X_i$ and $f(i) = x$. Thus we have (5.17).

Infinite product of cardinals can be evaluated using the following lemma:

Lemma 5.9. *If λ is an infinite cardinal and $\langle \kappa_i : i < \lambda\rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

Proof. Let $\kappa = \sup_i \kappa_i$. Since $\kappa_i \le \kappa$ for each $i < \lambda$, we have

$$\prod_{i<\lambda} \kappa_i \le \prod_{i<\lambda} \kappa = \kappa^\lambda.$$

To prove that $\kappa^\lambda \le \prod_{i<\lambda} \kappa_i$, we consider a partition of λ into λ disjoint sets A_j, each of cardinality λ:

$$\lambda = \bigcup_{j<\lambda} A_j. \tag{5.18}$$

(To get a partition (5.18), we can, e.g., use the canonical pairing function $\Gamma : \lambda \times \lambda \to \lambda$ and let $A_j = \Gamma(\lambda \times \{j\})$.) Since a product of nonzero cardinals is greater than or equal to each factor, we have $\prod_{i\in A_j} \kappa_i \ge \sup_{i\in A_j} \kappa_i = \kappa$, for each $j < \lambda$. Thus, by (5.16),

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda}\Bigl(\prod_{i\in A_j} \kappa_i\Bigr) \ge \prod_{j<\lambda} \kappa = \kappa^\lambda.$$ □

The strict inequalities in cardinal arithmetic that we proved in Chapter 3 can be obtained as special cases of the following general theorem.

Theorem 5.10 (König). *If $\kappa_i < \lambda_i$ for every $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. We shall show that $\sum_i \kappa_i \not\ge \prod_i \lambda_i$. Let T_i, $i \in I$, be such that $|T_i| = \lambda_i$ for each $i \in I$. It suffices to show that if Z_i, $i \in I$, are subsets of $T = \prod_{i\in I} T_i$, and $|Z_i| \le \kappa_i$ for each $i \in I$, then $\bigcup_{i\in I} Z_i \ne T$.

For every $i \in I$, let S_i be the projection of Z_i into the ith coordinate:

$$S_i = \{f(i) : f \in Z_i\}.$$

Since $|Z_i| < |T_i|$, we have $S_i \subset T_i$ and $S_i \ne T_i$. Now let $f \in T$ be a function such that $f(i) \notin S_i$ for every $i \in I$. Obviously, f does not belong to any Z_i, $i \in I$, and so $\bigcup_{i\in I} Z_i \ne T$. □

Corollary 5.11. *$\kappa < 2^\kappa$ for every κ.*

Proof. $\underbrace{1+1+\ldots}_{\kappa \text{ times}} < \underbrace{2\cdot 2\cdot \ldots}_{\kappa \text{ times}}$. □

Corollary 5.12. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Proof. It suffices to show that if $\kappa_i < 2^{\aleph_\alpha}$ for $i < \omega_\alpha$, then $\sum_{i<\omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$. Let $\lambda_i = 2^{\aleph_\alpha}$.

$$\sum_{i<\omega_\alpha} \kappa_i < \prod_{i<\omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}. \qquad \square$$

Corollary 5.13. $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$.

Proof. We show that if $\kappa_i < \aleph_\alpha^{\aleph_\beta}$ for $i < \omega_\beta$, then $\sum_{i<\omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$. Let $\lambda_i = \aleph_\alpha^{\aleph_\beta}$.

$$\sum_{i<\omega_\beta} \kappa_i < \prod_{i<\omega_\beta} \lambda_i = (\aleph_\alpha^{\aleph_\beta})^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}. \qquad \square$$

Corollary 5.14. *$\kappa^{\operatorname{cf}\kappa} > \kappa$ for every infinite cardinal κ.*

Proof. Let $\kappa_i < \kappa$, $i < \operatorname{cf}\kappa$, be such that $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$. Then

$$\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i < \prod_{i<\operatorname{cf}\kappa} \kappa = \kappa^{\operatorname{cf}\kappa}. \qquad \square$$

The Continuum Function

Cantor's Theorem 3.1 states that $2^{\aleph_\alpha} > \aleph_\alpha$, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$, for all α. The *Generalized Continuum Hypothesis* (GCH) is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all α. GCH is independent of the axioms of ZFC. Under the assumption of GCH, cardinal exponentiation is evaluated as follows:

Theorem 5.15. *If* GCH *holds and κ and λ are infinite cardinals then*:

(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.*
(ii) *If $\operatorname{cf}\kappa \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.*
(iii) *If $\lambda < \operatorname{cf}\kappa$, then $\kappa^\lambda = \kappa$.*

Proof. (i) Lemma 5.6.
(ii) This follows from (5.7) and (5.8).
(iii) By Lemma 3.9(ii), the set κ^λ is the union of the sets α^λ, $\alpha < \kappa$, and $|\alpha^\lambda| \leq 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$. $\square$

The *beth function* is defined by induction:

$$\beth_0 = \aleph_0, \qquad \beth_{\alpha+1} = 2^{\beth_\alpha},$$
$$\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Thus GCH is equivalent to the statement $\beth_\alpha = \aleph_\alpha$ for all α.

We shall now investigate the general behavior of the *continuum function* 2^κ, without assuming GCH.

Theorem 5.16.

(i) *If $\kappa < \lambda$ then $2^\kappa \leq 2^\lambda$.*
(ii) $\operatorname{cf} 2^\kappa > \kappa$.
(iii) *If κ is a limit cardinal then $2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa}$.*

Proof. (ii) By Corollary 5.12,
(iii) Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i 2^{<\kappa} = (2^{<\kappa})^{\operatorname{cf}\kappa} \leq (2^\kappa)^{\operatorname{cf}\kappa} \leq 2^\kappa. \qquad \square$$

For regular cardinals, the only conditions Theorem 5.16 places on the continuum function are $2^\kappa > \kappa$ and $2^\kappa \leq 2^\lambda$ if $\kappa < \lambda$. We shall see that these are the only restrictions on 2^κ for regular κ that are provable in ZFC.

Corollary 5.17. *If κ is a singular cardinal and if the continuum function is eventually constant below κ, with value λ, then $2^\kappa = \lambda$.*

Proof. If κ is a singular cardinal that satisfies the assumption of the theorem, then there is μ such that $\operatorname{cf}\kappa \leq \mu < \kappa$ and that $2^{<\kappa} = \lambda = 2^\mu$. Thus

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = (2^\mu)^{\operatorname{cf}\kappa} = 2^\mu. \qquad \square$$

The *gimel function* is the function

$$\gimel(\kappa) = \kappa^{\operatorname{cf}\kappa}. \tag{5.19}$$

If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then the cardinal $\lambda = 2^{<\kappa}$ is a limit of a nondecreasing sequence

$$\lambda = 2^{<\kappa} = \lim_{\alpha\to\kappa} 2^{|\alpha|}$$

of length κ. By Lemma 3.7(ii), we have

$$\operatorname{cf}\lambda = \operatorname{cf}\kappa.$$

Using Theorem 5.16(iii), we get

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda}. \tag{5.20}$$

If κ is a regular cardinal, then $\kappa = \operatorname{cf}\kappa$; and since $2^\kappa = \kappa^\kappa$, we have

$$2^\kappa = \kappa^{\operatorname{cf}\kappa}. \tag{5.21}$$

Thus (5.20) and (5.21) show that the continuum function can be defined in terms of the gimel function:

Corollary 5.18.

(i) *If κ is a successor cardinal, then $2^\kappa = \gimel(\kappa)$.*
(ii) *If κ is a limit cardinal and if the continuum function below κ is eventually constant, then $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$.*
(iii) *If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then $2^\kappa = \gimel(2^{<\kappa})$.* $\square$

Cardinal Exponentiation

We shall now investigate the function κ^λ for infinite cardinal numbers κ and λ.

We start with the following observation: If κ is a regular cardinal and $\lambda < \kappa$, then every function $f : \lambda \to \kappa$ is bounded (i.e., $\sup\{f(\xi) : \xi < \lambda\} < \kappa$). Thus

$$\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda.$$

and so

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda.$$

In particular, if κ is a successor cardinal, we obtain the *Hausdorff formula*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}. \tag{5.22}$$

(Note that (5.22) holds for all α and β.)

In general, we can compute κ^λ using the following lemma. If κ is a limit cardinal, we use the notation $\lim_{\alpha\to\kappa} \alpha^\lambda$ to abbreviate $\sup\{\mu^\lambda : \mu$ is a cardinal and $\mu < \kappa\}$.

Lemma 5.19. *If κ is a limit cardinal, and $\lambda \geq \operatorname{cf}\kappa$, then*

$$\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa}.$$

Proof. Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have $\kappa^\lambda \leq (\prod_{i<\operatorname{cf}\kappa} \kappa_i)^\lambda = \prod_i \kappa_i^\lambda \leq \prod_i (\lim_{\alpha\to\kappa} \alpha^\lambda) = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} \leq (\kappa^\lambda)^{\operatorname{cf}\kappa} = \kappa^\lambda$. □

Theorem 5.20. *Let λ be an infinite cardinal. Then for all infinite cardinals κ, the value of κ^λ is computed as follows, by induction on κ:*

(i) *If $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > \lambda$ and if $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then:*
  (a) *if $\operatorname{cf}\kappa > \lambda$ then $\kappa^\lambda = \kappa$,*
  (b) *if $\operatorname{cf}\kappa \leq \lambda$ then $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$.*

Proof. (i) Lemma 5.6

(ii) $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$.

(iii) If κ is a successor cardinal, we use the Hausdorff formula. If κ is a limit cardinal, we have $\lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa > \lambda$ then every $f : \lambda \to \kappa$ is bounded and we have $\kappa^\lambda = \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa \leq \lambda$ then by Lemma 5.19, $\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} = \kappa^{\operatorname{cf}\kappa}$. □

Theorem 5.20 shows that all cardinal exponentiation can be defined in terms of the gimel function:

Corollary 5.21. *For every κ and λ, the value of κ^λ is either 2^λ, or κ, or $\gimel(\mu)$ for some μ such that $\operatorname{cf}\mu \leq \lambda < \mu$.*

Proof. If $\kappa^\lambda > 2^\lambda \cdot \kappa$, let μ be the least cardinal such that $\mu^\lambda = \kappa^\lambda$, and by Theorem 5.20 (for μ and λ), $\mu^\lambda = \mu^{\operatorname{cf}\mu}$. □

In the Exercises, we list some properties of the gimel function.

A cardinal κ is a *strong limit* cardinal if

$$2^\lambda < \kappa \quad \text{for every } \lambda < \kappa.$$

Obviously, every strong limit cardinal is a limit cardinal. If the GCH holds, then every limit cardinal is a strong limit.

It is easy to see that if κ is a strong limit cardinal, then

$$\lambda^\nu < \kappa \quad \text{for all } \lambda, \nu < \kappa.$$

An example of a strong limit cardinal is $\aleph_0$. Actually, the strong limit cardinals form a proper class: If α is an arbitrary cardinal, then the cardinal

$$\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \ldots\}$$

(of cofinality ω) is a strong limit cardinal.

Another fact worth mentioning is:

(5.23) *If κ is a strong limit cardinal, then $2^\kappa = \kappa^{\operatorname{cf}\kappa}$.*

We recall that κ is weakly inaccessible if it is uncountable, regular, and limit. We say that a cardinal κ is *inaccessible* (strongly) if $\kappa > \aleph_0$, κ is regular, and κ is strong limit.

Every inaccessible cardinal is weakly inaccessible. If the GCH holds, then every weakly inaccessible cardinal κ is inaccessible.

The inaccessible cardinals owe their name to the fact that they cannot be obtained from smaller cardinals by the usual set-theoretical operations.

If κ is inaccessible and $|X| < \kappa$, then $|P(X)| < \kappa$. If $|S| < \kappa$ and if $|X| < \kappa$ for every $X \in S$, then $|\bigcup S| < \kappa$.

In fact, $\aleph_0$ has this property too. Thus we can say that in a sense an inaccessible cardinal is to smaller cardinals what $\aleph_0$ is to finite cardinals. This is one of the main themes of the theory of large cardinals.

The Singular Cardinal Hypothesis

The *Singular Cardinal Hypothesis* (SCH) is the statement: For every singular cardinal κ, if $2^{\operatorname{cf}\kappa} < \kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

Obviously, the Singular Cardinals Hypothesis follows from GCH. If $2^{\operatorname{cf}\kappa} \geq \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. If $2^{\operatorname{cf}\kappa} < \kappa$, then κ^+ is the least possible value of $\kappa^{\operatorname{cf}\kappa}$.

We shall prove later in the book that if SCH fails then a large cardinal axiom holds. In fact, the failure of SCH is equiconsistent with the existence of a certain large cardinal.

Under the assumption of SCH, cardinal exponentiation is determined by the continuum function on regular cardinals:

Theorem 5.22. *Assume that* SCH *holds.*

(i) *If κ is a singular cardinal then*
  (a) $2^\kappa = 2^{<\kappa}$ *if the continuum function is eventually constant below* κ,
  (b) $2^\kappa = (2^{<\kappa})^+$ *otherwise.*

(ii) *If κ and λ are infinite cardinals, then*:
  (a) *If* $\kappa \le 2^\lambda$ *then* $\kappa^\lambda = 2^\lambda$.
  (b) *If* $2^\lambda < \kappa$ *and* $\lambda < \operatorname{cf}\kappa$ *then* $\kappa^\lambda = \kappa$.
  (c) *If* $2^\lambda < \kappa$ *and* $\operatorname{cf}\kappa \le \lambda$ *then* $\kappa^\lambda = \kappa^+$.

Proof. (i) If κ is a singular cardinal, then by Theorem 5.16, 2^κ is either λ or $\lambda^{\operatorname{cf}\kappa}$ where $\lambda = 2^{<\kappa}$. The latter occurs if 2^α is not eventually constant below κ. Then $\operatorname{cf}\lambda = \operatorname{cf}\kappa$, and since $2^{\operatorname{cf}\kappa} < 2^{<\kappa} = \lambda$, we have $\lambda^{\operatorname{cf}\lambda} = \lambda^+$ by the Singular Cardinals Hypothesis.

(ii) We proceed by induction on κ, for a fixed λ. Let $\kappa > 2^\lambda$. If κ is a successor cardinal, $\kappa = \nu^+$, then $\nu^\lambda \le \kappa$ (by the induction hypothesis), and $\kappa^\lambda = (\nu^+)^\lambda = \nu^+ \cdot \nu^\lambda = \kappa$, by the Hausdorff formula.

If κ is a limit cardinal, then $\nu^\lambda < \kappa$ for all $\nu < \kappa$. By Theorem 5.20, $\kappa^\lambda = \kappa$ if $\lambda < \operatorname{cf}\kappa$, and $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$ if $\lambda \ge \operatorname{cf}\kappa$, In the latter case, $2^{\operatorname{cf}\kappa} \le 2^\lambda < \kappa$, and by the Singular Cardinals Hypothesis, $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

Exercises

5.1. There exists a set of reals of cardinality $2^{\aleph_0}$ without a perfect subset.

[Let $\langle P_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of all perfect sets of reals. Construct disjoint $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\aleph_0}\}$ as follows: Pick a_α such that $a_\alpha \notin \{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\}$, and b_α such that $b_\alpha \in P_\alpha - \{a_\xi : \xi \le \alpha\}$. Then A is the set.]

5.2. If X is an infinite set and S is the set of all finite subsets of X, then $|S| = |X|$.

[Use $|X| = \aleph_\alpha$.]

5.3. Let $(P, <)$ be a linear ordering and let κ be a cardinal. If every initial segment of P has cardinality $< \kappa$, then $|P| \le \kappa$.

5.4. If A can be well-ordered then $P(A)$ can be linearly ordered.

[Let $X < Y$ if the least element of $X \mathbin{\triangle} Y$ belongs to X.]

5.5. Prove the Axiom of Choice from Zorn's Lemma.

[Let S be a family of nonempty sets. To find a choice function on S, let $P = \{f : f$ is a choice function on some $Z \subset S\}$, and apply Zorn's Lemma to the partially ordered set $(P, \subset)$.]

5.6. The countable AC implies that every infinite set has a countable subset.

[If A is infinite, let $A_n = \{s : s$ is a one-to-one sequence in A of length $n\}$ for each n. Use a choice function for $S = \{A_n : n \in \boldsymbol{N}\}$ to obtain a countable subset of A.]

5.7. Use DC to prove the countable AC.

[Given $S = \{A_n : n \in \boldsymbol{N}\}$, consider the set A of all choice functions on some $S_n = \{A_i : i \le n\}$, with the binary relation $\supset$.]

5.8 (The Milner-Rado Paradox). For every ordinal $\alpha < \kappa^+$ there are sets $X_n \subset \alpha$ $(n \in \boldsymbol{N})$ such that $\alpha = \bigcup_n X_n$, and for each n the order-type of X_n is $\le \kappa^n$.

[By induction on α, choosing a sequence cofinal in α.]

5.9. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are two disjoint families such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\bigcup_{i\in I} X_i| = |\bigcup_{i\in I} Y_i|$.

[Use AC.]

5.10. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\prod_{i\in I} X_i| = |\prod_{i\in I} Y_i|$.

[Use AC.]

5.11. $\prod_{0<n<\omega} n = 2^{\aleph_0}$.

5.12. $\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$.

5.13. $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^{\aleph_0}$.

5.14. If GCH holds then

(i) $2^{<\kappa} = \kappa$ for all κ, and
(ii) $\kappa^{<\kappa} = \kappa$ for all regular κ.

5.15. If β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for every α, then $\beta < \omega$.

[Let $\beta \ge \omega$. Let α be least such that $\alpha + \beta > \beta$. We have $0 < \alpha \le \beta$, and α is limit. Let $\kappa = \aleph_{\alpha+\alpha}$; since $\operatorname{cf}\kappa = \operatorname{cf}\alpha \le \alpha < \kappa$, κ is singular. For each $\xi < \alpha$, $\xi + \beta = \beta$, and so $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\xi+\beta} = \aleph_{\alpha+\beta}$. By Corollary 5.17, $2^\kappa = \aleph_{\alpha+\beta}$, a contradiction, since $\aleph_{\alpha+\beta} < \aleph_{\alpha+\alpha+\beta}$.]

5.16. $\prod_{\alpha<\omega_1+\omega} \aleph_\alpha = \aleph_{\omega_1+\omega}^{\aleph_1}$.

[$\aleph_{\omega_1+\omega}^{\aleph_1} \le (\prod_{n=0}^\infty \aleph_{\omega_1+n})^{\aleph_1} = \prod_n \aleph_{\omega_1+n}^{\aleph_1} = \prod_n (\aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\omega_1+n}) = \aleph_{\omega_1}^{\aleph_1} \cdot \prod_n \aleph_{\omega_1+n} = \prod_{\alpha<\omega_1+\omega} \aleph_\alpha$.]

5.17. If κ is a limit cardinal and $\lambda < \operatorname{cf}\kappa$, then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.

5.18. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

5.19. If $\alpha < \omega_1$, then $\aleph_\alpha^{\aleph_1} = \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1}$.

5.20. If $\alpha < \omega_2$, then $\aleph_\alpha^{\aleph_2} = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}$.

5.21. If κ is regular and limit, then $\kappa^{<\kappa} = 2^{<\kappa}$. If κ is regular and strong limit then $\kappa^{<\kappa} = \kappa$.

5.22. If κ is singular and is not strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

5.23. If κ is singular and strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.

5.24. If $2^{\aleph_0} > \aleph_\omega$, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$.

5.25. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.

5.26. If $2^{\aleph_0} \ge \aleph_{\omega_1}$, then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

5.27. If $2^{\aleph_1} = \aleph_2$, then $\aleph_\omega^{\aleph_0} \ne \aleph_{\omega_1}$.

5.28. If κ is a singular cardinal and if $\kappa < \gimel(\lambda)$ for some $\lambda < \kappa$ such that $\operatorname{cf}\kappa \le \operatorname{cf}\lambda$ then $\gimel(\kappa) \le \gimel(\lambda)$.

5.29. If κ is a singular cardinal such that $2^{\operatorname{cf}\kappa} < \kappa \le \lambda^{\operatorname{cf}\kappa}$ for some $\lambda < \kappa$, then $\gimel(\kappa) = \gimel(\lambda)$ where λ is the least λ such that $\kappa \le \lambda^{\operatorname{cf}\kappa}$.

Historical Notes

The Axiom of Choice was formulated by Zermelo, who used it to prove the Well-Ordering Theorem in [1904]. Zorn's Lemma is as in Zorn [1935]; for a related principle, see Kuratowski [1922]. (Hausdorff in [1914], pp. 140–141, proved that every partially ordered set has a maximal linearly ordered subset.) The Principle of Dependent Choices was formulated by Bernays in [1942].

König's Theorem 5.10 appeared in J. König [1905]. Corollary 5.17 was found independently by Bukovský [1965] and Hechler. The discovery that cardinal exponentiation is determined by the gimel function was made by Bukovský; cf. [1965]. The inductive computation of κ^λ in Theorem 5.20 is as in Jech [1973a].

The Hausdorff formula (5.22): Hausdorff [1904].

Inaccessible cardinals were introduced in the paper by Sierpiński and Tarski [1930]; see Tarski [1938] for more details.

Exercise 5.1: Felix Bernstein.
Exercise 5.8: Milner and Rado [1965].
Exercise 5.15: L. Patai.
Exercise 5.17: Tarski [1925b].
Exercises 5.28–5.29: Jech [1973a].

6. The Axiom of Regularity

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

Axiom of Regularity. *Every nonempty set has an $\in$-minimal element*:

$$\forall S\,(S \ne \emptyset \to (\exists x \in S)\, S \cap x = \emptyset).$$

As a consequence, there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \ldots.$$

(Consider the set $S = \{x_0, x_1, x_2, \ldots\}$ and apply the axiom.) In particular, there is no set x such that

$$x \in x$$

and there are no "cycles"

$$x_0 \in x_1 \in \ldots \in x_n \in x_0.$$

Thus the Axiom of Regularity postulates that sets of certain type do no exist. This restriction on the universe of sets is not contradictory (i.e., the axiom is consistent with the other axioms) and is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics. However, it is extremely useful in the metamathematics of set theory, in construction of models. In particular, all sets can be assigned ranks and can be arranged in a cumulative hierarchy.

We recall that a set T is *transitive* if $x \in T$ implies $x \subset T$.

Lemma 6.1. *For every set S there exists a transitive set $T \supset S$.*

Proof. We define by induction

$$S_0 = S, \qquad S_{n+1} = \bigcup S_n$$

and

$$T = \bigcup_{n=0}^{\infty} S_n. \tag{6.1}$$

Clearly, T is transitive and $T \supset S$. □

Since every transitive set must satisfy $\bigcup T \subset T$, it follows that the set in (6.1) is the smallest transitive $T \supset S$; it is called *transitive closure* of S:

$$\mathrm{TC}(S) = \bigcap\{T : T \supset S \text{ and } T \text{ is transitive}\}.$$

Lemma 6.2. *Every nonempty class C has an $\in$-minimal element.*

Proof. Let $S \in C$ be arbitrary. If $S \cap C = \emptyset$, then S is a minimal element of C; if $S \cap C \neq \emptyset$, we let $X = T \cap C$ where $T = \mathrm{TC}(S)$. X is a nonempty set and by the Axiom of Regularity, there is $x \in X$ such that $x \cap X = \emptyset$. It follows that $x \cap C = \emptyset$; otherwise, if $y \in x$ and $y \in C$, then $y \in T$ since T is transitive, and so $y \in x \cap T \cap C = x \cap X$. Hence x is a minimal element of C. □

The Cumulative Hierarchy of Sets

We define, by transfinite induction,

$$V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha),$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

The Axiom of Regularity implies that every set is in some V_α:

Lemma 6.3. *For every x there is α such that $x \in V_\alpha$:*

$$\bigcup_{\alpha\in Ord} V_\alpha = V. \tag{6.2}$$

Proof. Let C be the class of all x that are not in any V_α. If C is nonempty, then C has an $\in$-minimal element x. That is, $x \in C$, and $z \in \bigcup_\alpha V_\alpha$ for every $z \in x$. Hence $x \subset \bigcup_{\alpha\in Ord} V_\alpha$. By Replacement, there exists an ordinal γ such that $x \subset \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x \subset V_\gamma$ and so $x \in V_{\gamma+1}$. Thus C is empty and we have (6.2). □

Since every x is in some V_α, we may define the *rank* of x:

$$\mathrm{rank}(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}. \tag{6.3}$$

Thus each V_α is the collection of all sets of rank less than α, and we have

(i) If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.

(ii) $\mathrm{rank}(\alpha) = \alpha$.

One of the uses of the rank function is a definition of equivalence classes for equivalence relations on a proper class. The basic trick is the following: Given a class C, let

$$\hat{C} = \{x \in C : (\forall z \in C)\ \mathrm{rank}\, x \le \mathrm{rank}\, z\}. \tag{6.4}$$

$\hat{C}$ is always a set, and if C is nonempty, then $\hat{C}$ is nonempty. Moreover, (6.4) can be applied uniformly.

Thus, for example, if $\equiv$ is an equivalence on a proper class C, we apply (6.4) to each equivalence class of $\equiv$, and define

$$[x] = \{y \in C : y \equiv x \text{ and } \forall z \in C\,(z \equiv x \to \mathrm{rank}\, y \le \mathrm{rank}\, z)\}$$

and

$$C/{\equiv} = \{[x] : x \in C\}.$$

In particular, this trick enables us to define isomorphism types for a given isomorphism. For instance, one can define order-types of linearly ordered sets, or cardinal numbers (even without AC).

We use the same argument to prove the following.

Collection Principle.

$$\forall X\, \exists Y\, (\forall u \in X)[\exists v\, \varphi(u, v, p) \to (\exists v \in Y)\, \varphi(u, v, p)] \tag{6.5}$$

(p is a parameter).

The Collection Principle is a schema of formulas. We can formulate it as follows:

Given a "collection of classes" C_u, $u \in X$ (X is a set), then there is a set Y such that for every $u \in X$,

$$\text{if } C_u \neq \emptyset, \quad \text{then } C_u \cap Y \neq \emptyset.$$

To prove (6.5), we let

$$Y = \bigcup_{u\in X} \hat{C}_u$$

where $C_u = \{v : \varphi(u, v, p)\}$, i.e.,

$$v \in Y \leftrightarrow (\exists u \in X)(\varphi(u, v, p) \text{ and } \forall z\,(\varphi(u, z, p) \to \mathrm{rank}\, v \le \mathrm{rank}\, z)).$$

That Y is a set follows from the Replacement Schema.

Note that the Collection Principle implies the Replacement Schema: Given a function F, then for every set X we let Y be a set such that

$$(\forall u \in X)(\exists v \in Y)\, F(u) = v.$$

Then

$$F{\restriction}X = F \cap (X \times Y)$$

is a set by the Separation Schema.

∈-Induction

The method of transfinite induction can be extended to an arbitrary transitive class (instead of *Ord*), both for the proof and for the definition by induction:

Theorem 6.4 (∈-Induction). *Let T be a transitive class, let Φ be a property. Assume that*

(i) $\Phi(\emptyset)$;
(ii) *if $x \in T$ and $\Phi(z)$ holds for every $z \in x$, then $\Phi(x)$.*

Then every $x \in T$ has property Φ.

Proof. Let C be the class of all $x \in T$ that do not have the property Φ. If C is nonempty, then it has an ∈-minimal element x; apply (i) or (ii). □

Theorem 6.5 (∈-Recursion). *Let T be a transitive class and let G be a function (defined for all x). Then there is a function F on T such that*

$$F(x) = G(F{\restriction}x) \tag{6.6}$$

for every $x \in T$.

Moreover, F is the unique function that satisfies (6.6).

Proof. We let, for every $x \in T$,

$F(x) = y \leftrightarrow$ there exists a function f such that dom(f) is a transitive subset of T and:
(i) $(\forall z \in \operatorname{dom}(f))\, f(z) = G(f{\restriction}z)$,
(ii) $f(x) = y$.

That F is a (unique) function on T satisfying (6.6) is proved by ∈-induction. □

Corollary 6.6. *Let A be a class. There is a unique class B such that*

$$B = \{x \in A : x \subset B\}. \tag{6.7}$$

Proof. Let

$$F(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } F(z) = 1 \text{ for all } z \in x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{x : F(x) = 1\}$. The uniqueness of B is proved by ∈-induction. □

We say that each $x \in B$ is *hereditarily* in A.

One consequence of the Axiom of Regularity is that the universe does not admit nontrivial ∈-automorphisms. More generally:

Theorem 6.7. *Let T_1, T_2 be transitive classes and let π be an ∈-isomorphism of T_1 onto T_2; i.e., π is one-to-one and*

$$u \in v \leftrightarrow \pi u \in \pi v. \tag{6.8}$$

Then $T_1 = T_2$ and $\pi u = u$ for every $u \in T_1$.

Proof. We show, by ∈-induction, that $\pi x = x$ for every $x \in T_1$. Assume that $\pi z = z$ for each $z \in x$ and let $y = \pi x$.

We have $x \subset y$ because if $z \in x$, then $z = \pi z \in \pi x = y$.

We also have $y \subset x$: Let $t \in y$. Since $y \subset T_2$, there is $z \in T_1$ such that $\pi z = t$. Since $\pi z \in y$, we have $z \in x$, and so $t = \pi z = z$. Thus $t \in x$.

Therefore $\pi x = x$ for all $x \in T_1$, and $T_2 = T_1$. □

Well-Founded Relations

The notion of well-founded relations that was introduced in Chapter 2 can be generalized to relations on proper classes, and one can extend the method of induction to well-founded relations.

Let E be a binary relation on a class P. For each $x \in P$, we let

$$\operatorname{ext}_E(x) = \{z \in P : z \mathrel{E} x\}$$

the *extension* of x.

Definition 6.8. A relation E on P is well-founded, if:

(6.9) (i) every nonempty set $x \subset P$ has an E-minimal element;
(ii) $\operatorname{ext}_E(x)$ is a set, for every $x \in P$.

(Condition (ii) is vacuous if P is a set.) Note that the relation ∈ is well-founded on any class, by the Axiom of Regularity.

Lemma 6.9. *If E is a well-founded relation on P, then every nonempty class $C \subset P$ has an E-minimal element.*

Proof. We follow the proof of Lemma 6.2; we are looking for $x \in C$ such that $\operatorname{ext}_E(x) \cap C = \emptyset$. Let $S \in C$ be arbitrary and assume that $\operatorname{ext}_E(S) \cap C \neq \emptyset$. We let $X = T \cap C$ where

$$T = \bigcup_{n=0}^{\infty} S_n$$

and

$$S_0 = \operatorname{ext}_E S, \qquad S_{n+1} = \bigcup\{\operatorname{ext}_E(z) : z \in S_n\}.$$

As in Lemma 6.2, it follows that an E-minimal element x of X is E-minimal in C. □

Theorem 6.10 (Well-Founded Induction). *Let E be a well-founded relation on P. Let Φ be a property. Assume that:*

(i) *every E-minimal element x has property Φ;*
(ii) *if $x \in P$ and if $\Phi(z)$ holds for every z such that $z \mathrel{E} x$, then $\Phi(x)$.*

Then every $x \in P$ has property Φ.

Proof. A modification of the proof of Theorem 6.4. □

Theorem 6.11 (Well-Founded Recursion). *Let E be a well-founded relation on P. Let G be a function (on $V \times V$). Then there is a unique function F on P such that*

$$F(x) = G(x, F \restriction \operatorname{ext}_E(x)) \tag{6.10}$$

for every $x \in P$.

Proof. A modification of the proof of Theorem 6.5. □

(Note that if $F(x) = G(F \restriction \operatorname{ext}(x))$ for some G, then $F(x) = F(y)$ whenever $\operatorname{ext}(x) = \operatorname{ext}(y)$; in particular, $F(x)$ is the same for all minimal elements.)

Example 6.12 (The Rank Function). We define, by induction, for all $x \in P$:

$$\rho(x) = \sup\{\rho(z) + 1 : z \mathrel{E} x\}$$

(compare with (2.7)). The range of ρ is either an ordinal or the class *Ord*. For all $x, y \in P$,

$$x \mathrel{E} y \to \rho(x) < \rho(y).$$ □

Example 6.13 (The Transitive Collapse). By induction, let

$$\pi(x) = \{\pi(z) : z \mathrel{E} x\}$$

for every $x \in P$. The range of π is a transitive class, and for all $x, y \in P$,

$$x \mathrel{E} y \to \pi(x) \in \pi(y).$$ □

The transitive collapse of a well-founded relation is not necessarily a one-to-one function. It is one-to-one if E satisfies an additional condition, extensionality.

Definition 6.14. A well-founded relation E on a class P is *extensional* if

$$\operatorname{ext}_E(X) \neq \operatorname{ext}_E(Y) \tag{6.11}$$

whenever X and Y are distinct elements of P.

A class M is *extensional* if the relation $\in$ on M is extensional, i.e., if for any distinct X and $Y \in M$, $X \cap M \neq Y \cap M$.

The following theorem shows that the transitive collapse of an extensional well-founded relation is one-to-one, and that every extensional class is $\in$-isomorphic to a transitive class.

Theorem 6.15 (Mostowski's Collapsing Theorem).

(i) *If E is a well-founded and extensional relation on a class P, then there is a transitive class M and an isomorphism π between (P, E) and $(M, \in)$. The transitive class M and the isomorphism π are unique.*
(ii) *In particular, every extensional class P is isomorphic to a transitive class M. The transitive class M and the isomorphism π are unique.*
(iii) *In case* (ii), *if $T \subset P$ is transitive, then $\pi x = x$ for every $x \in T$.*

Proof. Since (ii) is a special case of (i) ($E = \in$ in case (ii)), we shall prove the existence of an isomorphism in the general case.

Since E is a well-founded relation, we can define π by well-founded induction (Theorem 6.11), i.e., $\pi(x)$ can be defined in terms of the $\pi(z)$'s, where $z \mathrel{E} x$. We let, for each $x \in P$

$$\pi(x) = \{\pi(z) : z \mathrel{E} x\}. \tag{6.12}$$

In particular, in the case $E = \in$, (6.12) becomes

$$\pi(x) = \{\pi(z) : z \in x \cap P\}. \tag{6.13}$$

The function π maps P onto a class $M = \pi(P)$, and it is immediate from the definition (6.12) that M is transitive.

We use the extensionality of E to show that π is one-to-one. Let $z \in M$ be of least rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. Then $\operatorname{ext}_E(x) \neq \operatorname{ext}_E(y)$ and there is, e.g., some $u \in \operatorname{ext}_E(x)$ such that $u \notin \operatorname{ext}_E(y)$. Let $t = \pi(u)$. Since $t \in z = \pi(y)$, there is $v \in \operatorname{ext}_E(y)$ such that $t = \pi(v)$. Thus we have $t = \pi(u) = \pi(v)$, $u \neq v$, and t is of lesser rank than z (since $t \in z$). A contradiction.

Now it follows easily that

$$x \mathrel{E} y \leftrightarrow \pi(x) \in \pi(y). \tag{6.14}$$

If $x \mathrel{E} y$, then $\pi(x) \in \pi(y)$ by definition (6.12). On the other hand, if $\pi(x) \in \pi(y)$, then by (6.12), $\pi(x) = \pi(z)$ for some $z \mathrel{E} y$. Since π is one-to-one, we have $x = z$ and so $x \mathrel{E} y$.

The uniqueness of the isomorphism π, and the transitive class $M = \pi(P)$, follows from Theorem 6.7. If π_1 and π_2 are two isomorphisms of P and M_1, M_2, respectively, then $\pi_2\pi_1^{-1}$ is an isomorphism between M_1 and M_2, and therefore the identity mapping. Hence $\pi_1 = \pi_2$.

It remains to prove (iii). If $T \subset P$ is transitive, then we first observe that $x \subset P$ for every $x \in T$ and so $x \cap P = x$, and we have

$$\pi(x) = \{\pi(z) : z \in x\}$$

for all $x \in T$. It follows easily by $\in$-induction that $\pi(x) = x$ for all $x \in T$. □

The Bernays-Gödel Axiomatic Set Theory

There is an alternative axiomatization of set theory. We consider two types of objects: *sets* (for which we use lower case letters) and *classes* (denoted by capital letters).

A. 1. Extensionality: $\forall u\,(u \in X \leftrightarrow y \in Y) \rightarrow X = Y$.
   2. Every set is a class.
   3. If $X \in Y$, then X is a set.
   4. Pairing: For any sets x and y there is a set $\{x, y\}$.

B. Comprehension:

$$\forall X_1 \ldots \forall X_n \,\exists Y\; Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula in which only set variables are quantified.

C. 1. Infinity: There is an infinite set.
   2. Union: For every set x the set $\bigcup x$ exists.
   3. Power Set: For every set x the power set $P(x)$ of x exists.
   4. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.

D. Regularity.

E. Choice: There is a function F such that $F(x) \in x$ for every nonempty set x.

Let BG denote the axiomatic theory A–D and let BGC denote BG + Choice.

If a set-theoretical statement is provable in ZF (ZFC), then it is provable in BG (BGC).

On the other hand, a theorem of Shoenfield (using proof-theoretic methods) states that if a sentence involving only set variables is provable in BG, then it is provable in ZF. This result can be extended to BGC/ZFC using the method of forcing.

Exercises

6.1. $\operatorname{rank}(x) = \sup\{\operatorname{rank}(z) + 1 : z \in x\}$.

6.2. $|V_\omega| = \aleph_0$, $|V_{\omega+\alpha}| = \beth_\alpha$.

6.3. If κ is inaccessible, then $|V_\kappa| = \kappa$.

6.4. If x and y have rank $\le \alpha$ then $\{x, y\}$, $\langle x, y\rangle$, $x \cup y$, $\bigcup x$, $P(x)$, and x^y have rank $< \alpha + \omega$

6.5. The sets $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are in $V_{\omega+\omega}$.

6.6. Let B be the class of all x that are hereditarily in the class A. Show that

(i) $x \in B$ if and only if $\operatorname{TC}(x) \subset A$,
(ii) B is the largest transitive class $B \subset A$.

Historical Notes

The Axiom of Regularity was introduced by von Neumann in [1925], although a similar principle had been considered previously by Skolem (see [1970], pp. 137–152). The concept of rank appears first in Mirimanov [1917]. The transitive collapse is defined in Mostowski [1949]. Induction on well-founded relations (Theorems 6.10, 6.11) was formulated by Montague in [1955].

The axiomatic system BG was introduced by Bernays in [1937]. Shoenfield's result was published in [1954].

For more references on the history of axioms of set theory consult Fraenkel *et al.* [1973].

7. Filters, Ultrafilters and Boolean Algebras

Filters and Ultrafilters

Filters and ideals play an important role in several mathematical disciplines (algebra, topology, logic, measure theory). In this chapter we introduce the notion of filter (and ideal) on a given set. The notion of ideal extrapolates the notion of small sets: Given an ideal I on S, a set $X \subset S$ is considered small if it belongs to I.

Definition 7.1. A *filter* on a nonempty set S is a collection F of subsets of S such that

(7.1) (i) $S \in F$ and $\emptyset \notin F$,
(ii) if $X \in F$ and $Y \in F$, then $X \cap Y \in F$,
(iii) If $X, Y \subset S$, $X \in F$, and $X \subset Y$, then $Y \in F$.

An *ideal* on a nonempty set S is a collection I of subsets of S such that:

(7.2) (i) $\emptyset \in I$ and $S \notin I$,
(ii) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$,
(iii) if $X, Y \subset S$, $X \in I$, and $Y \subset X$, then $Y \in I$.

If F is a filter on S, then the set $I = \{S - X : X \in F\}$ is an ideal on S; and conversely, if I is an ideal, then $F = \{S - X : X \in I\}$ is a filter. If this is the case we say that F and I are *dual* to each other.

Examples. 1. A *trivial* filter: $F = \{S\}$.

2. A *principal* filter. Let X_0 be a nonempty subset of S. The filter $F = \{X \subset S : X \supset X_0\}$ is a *principal* filter. Note that every filter on a finite set is principal.

The dual notions are a *trivial* ideal and a *principal* ideal.

3. The *Fréchet filter*. Let S be an infinite set, and let I be the ideal of all finite subsets of S. The dual filter $F = \{X \subset S : S - X \text{ is finite}\}$ is called the Fréchet filter on S. Note that the Fréchet filter is not principal.

4. Let A be an infinite set and let $S = [A]^{<\omega}$ be the set of all finite subsets of A. For each $P \in S$, let $\hat{P} = \{Q \in S : P \subset Q\}$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$. Then F is a nonprincipal filter on S.

5. A set $A \subset \boldsymbol{N}$ has *density* 0 if $\lim_{n\to\infty} |A \cap n|/n = 0$. The set of all A of density 0 is an ideal on $\boldsymbol{N}$.

A family G of sets has the *finite intersection property* if every finite $H = \{X_1, \ldots, X_n\} \subset G$ has a nonempty intersection $X_1 \cap \ldots \cap X_n \neq \emptyset$. Every filter has the finite intersection property.

Lemma 7.2.

(i) *If $\mathcal{F}$ is a nonempty family of filters on S, then $\bigcap \mathcal{F}$ is a filter on S.*
(ii) *If $\mathcal{C}$ is a $\subset$-chain of filters on S, then $\bigcup \mathcal{C}$ is a filter on S.*
(iii) *If $G \subset P(S)$ has the finite intersection property, then there is a filter F on S such that $G \subset F$.*

Proof. (i) and (ii) are easy to verify.

(iii) Let F be the set of all $X \subset S$ such that there is a finite $H = \{X_1, \ldots, X_n\} \subset G$ with $X_1 \cap \ldots \cap X_n \subset X$. Then F is a filter and $F \supset G$. □

Since every filter $F \supset G$ must contain all finite intersections of sets in G, it follows that the filter F constructed in the proof of Lemma 7.2(iii) is the smallest filter on S that extends G:

$$F = \bigcap \{D : D \text{ is a filter on } S \text{ and } G \subset D\}.$$

We say that the filter F is *generated* by G.

Definition 7.3. A filter U on a set S is an *ultrafilter* if

(7.3) for every $X \subset S$, either $X \in U$ or $S - X \in U$.

The dual notion is a *prime ideal*: For every $X \subset S$, either $X \in I$ or $S - X \in I$. Note that $I = P(S) - U$.

A filter F on S is *maximal* if there is no filter F' on S such that $F \subset F'$ and $F \neq F'$.

Lemma 7.4. *A filter F on S is an ultrafilter if and only if it is maximal.*

Proof. (a) An ultrafilter U is clearly a maximal filter: Assume that $U \subset F$ and $X \in F - U$. Then $S - X \in U$, and so both $S - X \in F$ and $X \in F$, a contradiction.

(b) Let F be a filter that is not an ultrafilter. We will show that F is not maximal. Let $Y \subset S$ be such that neither Y nor $S - Y$ is in F. Consider the family $G = F \cup \{Y\}$; we claim that G has the finite intersection property. If $X \in F$, then $X \cap Y \neq \emptyset$, for otherwise we would have $S - Y \supset X$ and $S - Y \in F$. Thus, if $X_1, \ldots, X_n \in F$, we have $X_1 \cap \ldots \cap X_n \in F$ and so $Y \cap X_1 \cap \ldots \cap X_n \neq \emptyset$. Hence G has the finite intersection property, and by Lemma 7.2(iii) there is a filter $F' \supset G$. Since $Y \in F' - F$, F is not maximal. □

Theorem 7.5 (Tarski). *Every filter can be extended to an ultrafilter.*

Proof. Let F_0 be a filter on S. Let P be the set of all filters F on S such that $F \supset F_0$ and consider the partially ordered set $(P, \subset)$. If $\mathcal{C}$ is a chain in P, then by Lemma 7.2(ii), $\bigcup \mathcal{C}$ is a filter and hence an upper bound of $\mathcal{C}$ in P. By Zorn's Lemma there exists a maximal element U in P. This U is an ultrafilter by Lemma 7.4. □

For every $a \in S$, the principal filter $\{X \subset S : a \in X\}$ is an ultrafilter. If S is finite, then every ultrafilter on S is principal.

If S is infinite, then there is a nonprincipal ultrafilter on S: If U extends the Fréchet filter, then U is nonprincipal.

The proof of Theorem 7.5 uses the Axiom of Choice. We shall see later that the existence of nonprincipal ultrafilters cannot be proved without AC.

If S is an infinite set of cardinality κ, then because every ultrafilter on S is a subset of $P(S)$, there are at most 2^{2^κ} ultrafilters on S. The next theorem shows that the number of ultrafilters on κ is exactly 2^{2^κ}. To get a slightly stronger result, let us call an ultrafilter D on κ *uniform* if $|X| = \kappa$ for all $X \in D$.

Theorem 7.6 (Pospíšil). *For every infinite cardinal κ, there exist 2^{2^κ} uniform ultrafilters on κ.*

We prove first the following lemma. Let us call a family $\mathcal{A}$ of subsets of κ *independent* if for any distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ in $\mathcal{A}$, the intersection

$$X_1 \cap \ldots \cap X_n \cap (\kappa - Y_1) \cap \ldots \cap (\kappa - Y_m) \tag{7.4}$$

has cardinality κ.

Lemma 7.7. *There exists an independent family of subsets of κ of cardinality 2^κ.*

Proof. Let us consider the set P of all pairs $(F, \mathcal{F})$ where F is a finite subset of κ and $\mathcal{F}$ is a finite set of finite subsets of κ. Since $|P| = \kappa$, it suffices to find an independent family $\mathcal{A}$ of subsets of P, of size 2^κ.

For each $u \subset \kappa$, let

$$X_u = \{(F, \mathcal{F}) \in P : F \cap u \in \mathcal{F}\}$$

and let $\mathcal{A} = \{X_u : u \subset \kappa\}$. If u and v are distinct subsets of κ, then $X_u \neq X_v$: For example, if $\alpha \in u$ but $\alpha \notin v$, then let $F = \{\alpha\}$, $\mathcal{F} = \{F\}$, and $(F, \mathcal{F}) \in X_u$ while $(F, \mathcal{F}) \notin X_v$. Hence $|\mathcal{A}| = 2^\kappa$.

To show that $\mathcal{A}$ is independent, let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of κ. For each $i \le n$ and each $j \le m$, let $\alpha_{i,j}$ be some element of κ such that either $\alpha_{i,j} \in u_i - v_j$ or $\alpha_{i,j} \in v_j - u_i$. Now let F be any finite subset of κ such that $F \supset \{\alpha_{i,j} : i \le n, j \le m\}$ (note that there are κ many such finite sets). Clearly, we have $F \cap u_i \neq F \cap v_j$ for any $i \le n$ and $j \le m$. Thus if we let $\mathcal{F} = \{F \cap u_i : i \le n\}$, we have $(F, \mathcal{F}) \in X_{u_i}$ for all $i \le n$ and $(F, \mathcal{F}) \notin X_{v_j}$ for all $j \le m$. Consequently, the intersection

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap (P - X_{v_1}) \cap \ldots \cap (P - X_{v_m})$$

has cardinality κ. □

Proof of Theorem 7.6. Let $\mathcal{A}$ be an independent family of subsets of κ. For every function $f : \mathcal{A} \to \{0, 1\}$, consider this family of subsets of κ:

$$G_f = \{X : |\kappa - X| < \kappa\} \cup \{X : f(X) = 1\} \cup \{\kappa - X : f(X) = 0\}. \tag{7.5}$$

By (7.4), the family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supset G_f$. If follows from (7.5) that D_f is uniform. If $f \neq g$, then for some $X \in \mathcal{A}$, $f(X) \neq g(X)$; e.g., $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$, while $\kappa - X \in D_g$. Thus we obtain 2^{2^κ} distinct uniform ultrafilters on κ. □

Ultrafilters on ω

We present two properties of ultrafilters on ω that are frequently used in set-theoretic topology.

Let D be a nonprincipal ultrafilter on ω. D is called a *p-point* if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ is finite, for all $n \in \omega$.

First we notice that it is easy to find a nonprincipal ultrafilter that is not a p-point: Let $\{A_n : n \in \omega\}$ be any partition of ω into $\aleph_0$ infinite pieces, and let F be the following filter on ω:

(7.6) $X \in F$ if and only if except for finitely many n, $X \cap A_n$ contains all but finitely many elements of A_n.

If D is any ultrafilter extending F, then D is not a p-point.

Theorem 7.8 below shows that existence of p-points follows from the Continuum Hypothesis. By a result of Shelah there exists a model of ZFC in which there are no p-points.

A nonprincipal ultrafilter D on ω is a *Ramsey* ultrafilter if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ has one element for all $n \in \omega$.

Every Ramsey ultrafilter is a p-point.

Theorem 7.8. *If $2^{\aleph_0} = \aleph_1$, then a Ramsey ultrafilter exists.*

Proof. Let $\mathcal{A}_\alpha$, $\alpha < \omega_1$, enumerate all partitions of ω and let us construct an ω_1-sequence of infinite subsets of ω as follows: Given X_α, let $X_{\alpha+1} \subset X_\alpha$ be such that either $X_{\alpha+1} \subset A$ for some $A \in \mathcal{A}_\alpha$, or that $|X_{\alpha+1} \cap A| \le 1$ for all $A \in \mathcal{A}_\alpha$. If α is a limit ordinal, let X_α be such that $X_\alpha - X_\beta$ is finite for all $\beta < \alpha$. (Such a set X_α exists because α is countable.) Then $D = \{X : X \supset X_\alpha \text{ for some } \alpha < \omega_1\}$ is a Ramsey ultrafilter. □

κ-Complete Filters and Ideals

A filter F on S is *countably complete* (*σ-complete*) if whenever $\{X_n : n \in \boldsymbol{N}\}$ is a countable family of subsets of S and $X_n \in F$ for every n, then

$$\bigcap_{n=0}^{\infty} X_n \in F. \tag{7.7}$$

A *countably complete* ideal (a *σ-ideal*) is such that if $X_n \in I$ for every n, then

$$\bigcup_{n=0}^{\infty} X_n \in I.$$

More generally, if κ is a regular uncountable cardinal, and F is a filter on S, then F is called *κ-complete* if F is closed under intersection of less than κ sets, i.e., if whenever $\{X_\alpha : \alpha < \gamma\}$ is a family of subsets of S, $\gamma < \kappa$, and $X_\alpha \in F$ for every $\alpha < \gamma$, then

$$\bigcap_{\alpha<\gamma} X_\alpha \in F. \tag{7.8}$$

The dual notion is a κ-complete ideal.

An example of a κ-complete ideal is $I = \{X \subset S : |X| < \kappa\}$, on any set S such that $|S| \ge \kappa$.

A σ-complete filter is the same as an $\aleph_1$-complete filter.

There is no nonprincipal σ-complete filter on a countable set S. If S is uncountable, then

$$\{X \subset S : |X| \le \aleph_0\}$$

is a σ-ideal on S.

Similarly, if $\kappa > \omega$ is regular and $|S| \ge \kappa$, then

$$\{X \subset S : |X| < \kappa\}$$

is the smallest κ-complete ideal on S containing all singletons $\{a\}$.

The question whether a nonprincipal *ultrafilter* on a set can be σ-complete gives rise to deep investigations of the foundations of set theory. In particular, if such ultrafilters exist, then there exist large cardinals (inaccessible, etc.).

Boolean Algebras

An algebra of sets (see Definition 4.9) is a collection of subsets of a given nonempty set that is closed under unions, intersections and complements. These properties of algebras of sets are abstracted in the notion of Boolean algebra:

Definition 7.9. A *Boolean algebra* is a set B with at least two elements, 0 and 1, endowed with binary operations $+$ and $\cdot$ and a unary operation $-$.

The Boolean operations satisfy the following axioms:

(7.9)

$$u + v = v + u, \qquad u \cdot v = v \cdot u, \qquad \text{(commutativity)}$$

$$u + (v + w) = (u + v) + w, \qquad u \cdot (v \cdot w) = (u \cdot v) \cdot w, \qquad \text{(associativity)}$$

$$u \cdot (v + w) = u \cdot v + u \cdot w, \qquad u + (v \cdot w) = (u + v) \cdot (u + w), \qquad \text{(distributivity)}$$

$$u \cdot (u + v) = u, \qquad u + (u \cdot v) = u, \qquad \text{(absorption)}$$

$$u + (-u) = 1, \qquad u \cdot (-u) = 0. \qquad \text{(complementation)}$$

An algebra of sets $\mathcal{S}$, with $\bigcup \mathcal{S} = S$, is a Boolean algebra, with Boolean operations $X \cup Y$, $X \cap Y$ and $S - X$, and with $\emptyset$ and S being 0 and 1. If follows from Stone's Representation Theorem below that every Boolean algebra is isomorphic to an algebra of sets.

From the axioms (7.9) one can derive additional Boolean algebraic rules that correspond to rules for the set operations $\cup$, $\cap$ and $-$. Among others, we have

$$u + u = u, \quad u \cdot u = u, \quad u + 0 = u, \quad u \cdot 0 = 0, \quad u + 1 = 1, \quad u \cdot 1 = u$$

and the De Morgan laws

$$-(u + v) = -u \cdot -v, \qquad -(u \cdot v) = -u + -v.$$

Two elements $u, v \in B$ are *disjoint* if $u \cdot v = 0$. Let us define

$$u - v = u \cdot (-v),$$

and

$$u \le v \quad \text{if and only if} \quad u - v = 0. \tag{7.10}$$

It is easy to see that $\le$ is a partial ordering of B and that

$$u \le v \quad \text{if and only if} \quad u + v = v \quad \text{if and only if} \quad u \cdot v = u.$$

Moreover, 1 is the greatest element of B and 0 is the least element. Also, for any $u, v \in B$, $u + v$ is the least upper bound of $\{u, v\}$ and $u \cdot v$ is the greatest lower bound of $\{u, v\}$. Since $-u$ is the unique v such that $u + v = 1$ and $u \cdot v = 0$, it follows that all Boolean-algebraic operations can be defined in terms of the partial ordering of B.

We shall now give an example showing the relation between Boolean algebras and logic:

Let $\mathcal{L}$ be a first order language and let S be the set of all sentences of $\mathcal{L}$. We consider the equivalence relation $\vdash \varphi \leftrightarrow \psi$ on S. The set B of all equivalence classes $[\varphi]$ is a Boolean algebra under the following operations:

$$\begin{aligned} [\varphi] + [\psi] &= [\varphi \vee \psi], & 0 &= [\varphi \wedge \neg\varphi], \\ [\varphi] \cdot [\psi] &= [\varphi \wedge \psi], & 1 &= [\varphi \vee \neg\varphi]. \\ -[\varphi] &= [\neg\varphi], \end{aligned}$$

This algebra is called the *Lindenbaum algebra*.

A subset A of a Boolean algebra B is a *subalgebra* if it contains 0 and 1 and is closed under the Boolean operations:

(7.11) (i) $0 \in A$, $1 \in A$;
(ii) if $u, v \in A$, then $u + v \in A$, $u \cdot v \in A$, $-u \in A$.

If $X \subset B$, then there is a smallest subalgebra A of B that contains X; A can be described either as $\bigcap\{A : X \subset A \subset B$ and A is a subalgebra$\}$, or as the set of all Boolean combinations in B of elements of X. The subalgebra A is *generated* by X. If X is infinite, then $|A| = |X|$. See Exercises 7.18–7.20.

If B is a Boolean algebra, let $B^+ = B - \{0\}$ denote the set of all nonzero elements of B. If $a \in B^+$, the set $B{\restriction}a = \{u \in B : u \le a\}$ with the partial order inherited from B, is a Boolean algebra; its $+$ and $\cdot$ are the same as in B, and the complement of u is $a - u$. An element $a \in B$ is called an *atom* if it is a minimal element of B^+; equivalently, if there is no x such that $0 < x < a$. A Boolean algebra is *atomic* if for every $u \in B^+$ there is an atom $a \le u$; B is *atomless* if it has no atoms.

Let B and C be two Boolean algebras. A mapping $h : B \to C$ is a *homomorphism* if it preserves the operations:

(7.12) (i) $h(0) = 0$, $h(1) = 1$,
(ii) $h(u + v) = h(u) + h(v)$, $h(u \cdot v) = h(u) \cdot h(v)$, $h(-u) = -h(u)$.

Note that the range of a homomorphism is a subalgebra of C and that $h(u) \le h(v)$ whenever $u \le v$. A one-to-one homomorphism of B onto C is called an *isomorphism*. An *embedding* of B in C is an isomorphism of B onto a subalgebra of C. Note that if $h : B \to C$ is a one-to-one mapping such that $u \le v$ if and only if $h(u) \le h(v)$, then h is an isomorphism. An isomorphism of a Boolean algebra onto itself is called an *automorphism*.

Ideals and Filters on Boolean Algebras

The definition of filter (and ideal) given earlier in this chapter generalizes to arbitrary Boolean algebras. Let B be a Boolean algebra. An *ideal* on B is a subset I of B such that:

(7.13) (i) $0 \in I$, $1 \notin I$;
(ii) if $u \in I$ and $v \in I$, then $u + v \in I$;
(iii) if $u, v \in B$, $u \in I$ and $v \le u$, then $v \in I$.

A *filter* on B is a subset F of B such that:

(7.14) (i) $1 \in F$, $0 \notin F$;
(ii) if $u \in F$ and $v \in F$, then $u \cdot v \in F$;
(iii) if $u, v \in B$, $u \in F$ and $u \le v$, then $v \in F$.

The *trivial* ideal is the ideal $\{0\}$; an ideal is *principal* if $I = \{u \in B : u \le u_0\}$ for some $u_0 \ne 1$. Similarly for filters.

A subset G of $B - \{0\}$ has the *finite intersection property* if for every finite $\{u_1, \ldots, u_n\} \subset G$, $u_1 \cdot \ldots \cdot u_n \ne 0$. Every $G \subset B$ that has the finite intersection property generates a filter on B; this and the other two clauses of Lemma 7.2 hold also for Boolean algebras.

There is a relation between ideals and homomorphisms. If $h : B \to C$ is a homomorphism, then

(7.15) $$I = \{u \in B : h(u) = 0\}$$

is an ideal on B (the *kernel* of the homomorphism). On the other hand, let I be an ideal on B. Let us consider the following equivalence relation on B:

(7.16) $$u \sim v \quad \text{if and only if} \quad u \mathbin{\triangle} v \in I$$

where

$$u \mathbin{\triangle} v = (u - v) + (v - u).$$

Let C be the set of all equivalence classes, $C = B/{\sim}$, and endow C with the following operations:

(7.17) $$\begin{aligned} [u] + [v] &= [u + v], & 0 &= [0], \\ [u] \cdot [v] &= [u \cdot v], & 1 &= [1]. \\ -[u] &= [-u], \end{aligned}$$

Then C is a Boolean algebra, the *quotient* of B mod I, and is a homomorphic image of B under the homomorphism

(7.18) $$h(u) = [u].$$

The quotient algebra is denoted B/I.

An ideal I on B is a *prime ideal* if

(7.19) for every $u \in B$, either $u \in I$ or $-u \in I$.

The dual of a prime ideal is an *ultrafilter*.

Lemma 7.4 holds in general: An ideal is a prime ideal (and a filter is an ultrafilter) if and only if it is maximal. Also, an ideal I on B is prime if and only if the quotient of B mod I is the trivial algebra $\{0, 1\}$.

Tarski's Theorem 7.5 easily generalizes to Boolean algebras:

Theorem 7.10 (The Prime Ideal Theorem). *Every ideal on B can be extended to a prime ideal.* □

The proof of the Prime Ideal Theorem uses the Axiom of Choice. It is known that the theorem cannot be proved without using the Axiom of Choice. However, it is also known that the Prime Ideal Theorem is weaker than the Axiom of Choice.

Theorem 7.11 (Stone's Representation Theorem). *Every Boolean algebra is isomorphic to an algebra of sets.*

Proof. Let B be a Boolean algebra. We let

(7.20) $$S = \{p : p \text{ is an ultrafilter on } B\}.$$

For every $u \in B$, let X_u be the set of all $p \in S$ such that $u \in p$. Let

(7.21) $$\mathcal{S} = \{X_u : u \in B\}.$$

Let us consider the mapping $\pi(u) = X_u$ from B onto $\mathcal{S}$. Clearly, $\pi(1) = S$ and $\pi(0) = \emptyset$. It follows from the definition of ultrafilter that

$$\pi(u \cdot v) = \pi(u) \cap \pi(v), \qquad \pi(u + v) = \pi(u) \cup \pi(v), \qquad \pi(-u) = S - \pi(u).$$

Thus π is a homomorphism of B onto the algebra of sets $\mathcal{S}$. It remains to show that π is one-to-one.

If $u \neq v$, then using the Prime Ideal Theorem, one can find an ultrafilter p on B containing one of these two elements but not the other. Thus π is an isomorphism. □

Complete Boolean Algebras

The partial ordering $\leq$ of a Boolean algebra can be used to define infinitary operations on B, generalizing $+$ and $\cdot$. Let us recall that $u + v = \sup\{u, v\}$

and $u \cdot v = \inf\{u, v\}$ in the partial ordering of B. Thus for any nonempty $X \subset B$, we define

(7.22) $$\sum\{u : u \in X\} = \sup X \quad \text{and} \quad \prod\{u : u \in X\} = \inf X,$$

provided that the least upper bound (the greatest lower bound) exists. We also define $\sum \emptyset = 0$ and $\prod \emptyset = 1$.

If the infinitary sum and product is defined for all $X \subset B$, the Boolean algebra is called *complete*. Similarly, we call B *κ-complete* (where κ is a regular uncountable cardinal) if sums and products exist for all X of cardinality $< \kappa$. An $\aleph_1$-complete Boolean algebra is called *σ-complete* or *countably complete*.

An algebra of sets $\mathcal{S}$ is *κ-complete* if it is closed under unions and intersections of $< \kappa$ sets. A κ-complete algebra of sets is a κ-complete Boolean algebra and for every $X \subset \mathcal{S}$ such that $|X| < \kappa$, $\sum X = \bigcup X$.

An ideal I on a κ-complete Boolean algebra is *κ-complete* if

$$\sum\{u : u \in X\} \in I$$

whenever $X \subset I$ and $|X| < \kappa$. A *κ-complete filter* is the dual notion.

If I is a κ-complete ideal on a κ-complete Boolean algebra B, then B/I is κ-complete, and

$$\sum\{[u] : u \in X\} = [\sum\{u : u \in X\}]$$

for every $X \subset B$, $|X| < \kappa$. Similarly for products.

An $\aleph_1$-complete ideal is called a *σ-ideal*.

There are two important examples of σ-ideals on the Boolean algebra of all Borel sets of reals: the σ-ideal of Borel sets of Lebesgue measure 0, and the σ-ideal of meager Borel sets. (Exercises 7.14 and 7.15.)

Let A be a subalgebra of a Boolean algebra B. A is a *dense* subalgebra of B if for every $u \in B^+$ there is a $v \in A^+$ such that $v \leq u$.

A *completion* of a Boolean algebra B is a complete Boolean algebra C such that B is a dense subalgebra of C.

Lemma 7.12. *The completion of a Boolean algebra B is unique up to isomorphism.*

Proof. Let C and D be completions of B. We define an isomorphism $\pi : C \to D$ by

(7.23) $$\pi(c) = \sum\nolimits^D \{u \in B : u \leq c\}.$$

To verify that π is an isomorphism, one uses the fact that B is a dense subalgebra of both C and D. For example, to show that $\pi(c) \neq 0$ whenever $c \neq 0$: There is $u \in B$ such that $0 < u \leq c$, and we have $0 < u \leq \pi(c)$. □

Theorem 7.13. *Every Boolean algebra has a completion.*

Proof. We use a construction similar to the method of Dedekind cuts. Let A be a Boolean algebra. Let us call a set $U \subset A^+$ a *cut* if

$$p \leq q \text{ and } q \in U \text{ implies } p \in U. \tag{7.24}$$

For every $p \in A^+$, let U_p denote the cut $\{x : x \leq p\}$.

A cut U is *regular* if

$$\text{whenever } p \notin U, \text{ then there exists } q \leq p \text{ such that } U_q \cap U = \emptyset. \tag{7.25}$$

Note that every U_p is regular, and that every cut includes some U_p.

We let B be the set of all regular cuts in A^+. We claim that B, under the partial ordering by inclusion, is a complete Boolean algebra. Note that the intersection of any collection of regular cuts is a regular cut, and hence each cut U is included in a least regular cut $\overline{U}$. In fact,

$$\overline{U} = \{p : (\forall q \leq p)\, U \cap U_q \neq \emptyset\}.$$

Thus for $u, v \in B$ we have

$$u \cdot v = u \cap v, \qquad u + v = \overline{u \cup v}.$$

The complement of $u \in B$ is the regular cut

$$-u = \{p : U_p \cap u = \emptyset\}.$$

And, of course, $\emptyset$ and A^+ are the zero and the unit of B. It is not difficult to verify that B is a complete Boolean algebra, and we leave the verification to the reader.

Furthermore, for all $p, q \in A^+$ we have $U_p + U_q = U_{p+q}$, $U_p \cdot U_q = U_{p \cdot q}$ and $-U_p = U_{-p}$. Thus A embeds in B as a dense subalgebra. □

Complete and Regular Subalgebras

Let B be a complete Boolean algebra. A subalgebra A of B is a *complete subalgebra* if $\sum X \in A$ and $\prod X \in A$ for all $X \subset A$. (Caution: A subalgebra A of B that is itself complete is not necessarily a complete subalgebra of B.) Similarly, a *complete homomorphism* is a homomorphism h of B into C such that for all $X \subset B$,

$$h(\textstyle\sum X) = \sum h(X), \qquad h(\prod X) = \prod h(X). \tag{7.26}$$

A *complete embedding* is an embedding that satisfies (7.26). Note that every isomorphism is complete.

Since the intersection of any collection of complete subalgebras of B is a complete subalgebra, every $X \subset B$ is included in a smallest complete subalgebra of B. This algebra is called the complete subalgebra of B *completely generated* by X.

Definition 7.14. A set $W \subset B^+$ is an *antichain* in a Boolean algebra B if $u \cdot v = 0$ for all distinct $u, v \in W$.

If W is an antichain and if $\sum W = u$ then we say that W is a *partition* of u. A partition of 1 is just a *partition*, or a *maximal antichain*.

If B is a Boolean algebra and A is a subalgebra of B then an antichain in A that is maximal in A need not be maximal in B. If every maximal antichain in A is also maximal in B, then A is called a *regular subalgebra* of B.

If A is a complete subalgebra of a complete Boolean algebra B then A is a regular subalgebra of B. Also, if A is a dense subalgebra of B then A is a regular subalgebra. See also Exercise 7.31.

Saturation

Let κ be an infinite cardinal. A Boolean algebra B is *κ-saturated* if there is no partition W of B such that $|W| = \kappa$, and

$$\operatorname{sat}(B) = \text{the least } \kappa \text{ such that } B \text{ is } \kappa\text{-saturated.} \tag{7.27}$$

B is also said to satisfy the *κ-chain condition*; this is because if B is complete, B is κ-saturated if and only if there exists no descending κ-sequence $u_0 > u_1 > \ldots > u_\alpha > \ldots$, $\alpha < \kappa$, of elements of B. The $\aleph_1$-chain condition is called the *countable chain condition* (c.c.c.).

Theorem 7.15. *If B is an infinite complete Boolean algebra, then* $\operatorname{sat}(B)$ *is a regular uncountable cardinal.*

Proof. Let $\kappa = \operatorname{sat}(B)$. It is clear that κ is uncountable. Let us assume that κ is singular; we shall obtain a contradiction by constructing a partition of size κ.

For $u \in B$, $u \neq 0$, let $\operatorname{sat}(u)$ denote $\operatorname{sat}(B_u)$. Let us call $u \in B$ *stable* if $\operatorname{sat}(v) = \operatorname{sat}(u)$ for every nonzero $v \leq u$. The set S of stable elements is dense in B; otherwise, there would be a descending sequence $u_0 > u_1 > u_2 > \ldots$ with decreasing cardinals $\operatorname{sat}(u_0) > \operatorname{sat}(u_1) > \ldots$. Let T be a maximal set of pairwise disjoint elements of S. Thus T is a partition of B, and $|T| < \kappa$.

First we show that $\sup\{\operatorname{sat}(u) : u \in T\} = \kappa$. For every regular $\lambda < \kappa$ such that $\lambda > |T|$, consider a partition W of B of size λ. Then at least one $u \in T$ is partitioned by W into λ pieces.

Thus we consider two cases:

Case I. There is $u \in T$ such that $\operatorname{sat}(u) = \kappa$. Since $\operatorname{cf} \kappa < \kappa$, there is a partition W of u of size $\operatorname{cf} \kappa$: $W = \{u_\alpha : \alpha < \operatorname{cf} \kappa\}$. Let κ_α, $\alpha < \operatorname{cf} \kappa$, be an increasing sequence with limit κ. For each α, $\operatorname{sat}(u_\alpha) = \operatorname{sat}(u) = \kappa$ and so let W_α be a partition of u_α of size κ_α. Then $\bigcup_{\alpha < \operatorname{cf} \kappa} W_\alpha$ is a partition of u of size κ.

Case II. For all $u \in T$, $\text{sat}(u) < \kappa$, but $\sup\{\text{sat}(u) : u \in T\} = \kappa$. Again, let $\kappa_\alpha \to \kappa$, $\alpha < \text{cf}\,\kappa$. For each $\alpha < \text{cf}\,\kappa$ (by induction), we find $u_\alpha \in T$, distinct from all u_β, $\beta < \alpha$, which admits a partition W_α of size κ_α. Then $\bigcup_{\alpha<\text{cf}\,\kappa} W_\alpha$ is an antichain in B of size κ. □

Distributivity of Complete Boolean Algebras

The following distributive law holds for every complete Boolean algebra:

$$\sum_{i\in I} u_{0,i} \cdot \sum_{u\in J} u_{1,j} = \sum_{(i,j)\in I\times J} u_{0,i}\cdot u_{1,j}.$$

To formulate a general distributive law, let κ be a cardinal, and let us call B *κ-distributive* if

$$\prod_{\alpha<\kappa}\sum_{i\in I_\alpha} u_{\alpha,i} = \sum_{f\in\prod_{\alpha<\kappa} I_\alpha}\ \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}. \tag{7.28}$$

(Every complete algebra of sets satisfies (7.28).) We shall see later that distributivity plays an important role in generic models. For now, let us give two equivalent formulations of κ-distributivity.

If W and Z are partitions of B, then W is a *refinement* of Z if for every $w \in W$ there is $z \in Z$ such that $w \le z$. A set $D \subset B$ is *open dense* if it is dense in B and $0 \ne u \le v \in D$ implies $u \in D$.

Lemma 7.16. *The following are equivalent, for any complete Boolean algebra B:*

(i) *B is κ-distributive.*
(ii) *The intersection of κ open dense subsets of B is open dense.*
(iii) *Every collection of κ partitions of B has a common refinement.*

Proof. (i) → (ii). Let D_α, $\alpha < \kappa$, be open dense, $D = \bigcap_{\alpha<\kappa} D_\alpha$. D is certainly open; thus let $u \ne 0$. If we let $\{u_{\alpha,i} : i \in I_\alpha\} = \{u \cdot v : v \in D_\alpha\}$, then $\sum_i u_{\alpha,i} = u$ for every α and the left-hand side of (7.28) is u. For each $f \in \prod_\alpha I_\alpha$, let $u_f = \prod_\alpha u_{\alpha,f(\alpha)}$; clearly, each nonzero u_f is in D. However, $\sum_f u_f = u$, by (7.28), and so some u_f is nonzero.

(ii) → (iii). Let W_α, $\alpha < \kappa$ be partions of B. For each α, let $D_\alpha = \{u : u \le v$ for some $v \in W_\alpha\}$; each D_α is open dense. Let $D = \bigcap_{\alpha<\kappa} D_\alpha$, and let W be a maximal set of pairwise disjoint elements of D. Since D is dense, W is a partition of B, and clearly, W is a refinement of each W_α.

(iii) → (i). Let $\{u_{\alpha,i} : \alpha < \kappa,\ i \in I_\alpha\}$ be a collection of elements of B. First we show that the right-hand side of (7.28) is always $\le$ the left-hand side. For each $f \in \prod_{\alpha<\kappa} I_\alpha$, let $u_f = \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$; we have $u_f \le u_{\alpha,f(\alpha)}$ and so $u_f \le \sum_{i\in I_\alpha} u_{\alpha,i}$ for each α. Thus, for each α,

$$\sum_f u_f \le \sum_i u_{\alpha,i}$$

and so

$$\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = \sum_f u_f \le \prod_\alpha \sum_i u_{\alpha,i}.$$

To prove (7.28), assume that (iii) holds, and let $u = \prod_\alpha \sum_i u_{\alpha,i}$; we want to show that $\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = u$. Without loss of generality, we can assume that $u = 1$ (otherwise we argue in the algebra $B{\restriction}u$). For each α, let us replace $\{u_{\alpha,i} : i \in I_\alpha\}$ by pairwise disjoint $\{v_{\alpha,i} : i \in I_\alpha\} = W_\alpha$ such that $v_{\alpha,i} \le u_{\alpha,i}$ and $\sum_i v_{\alpha,i} = \sum_i u_{\alpha,i}$ (some of the $v_{\alpha,i}$ may be 0). Clearly $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} \le \sum_f \prod_\alpha u_{\alpha,f(\alpha)}$. Each W_α is a partition of B and so there is a partition W that is a refinement of each W_α. Now for each $w \in W$ there exists f such that $w \le \prod_\alpha v_{\alpha,f(\alpha)}$, and so $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} = 1$. □

Exercises

7.1. If F is a filter and $X \in F$, then $P(X) \cap F$ is a filter on X.

7.2. The filter in Example 4 is generated by the sets $\{a\}^\wedge$, $a \in A$.

7.3. If U is an ultrafilter and $X \cup Y \in U$, then either $X \in U$ or $Y \in U$.

7.4. Let U be an ultrafilter on S. Then the set of all $X \subset S \times S$ such that $\{a \in S : \{b \in S : (a,b) \in X\} \in U\} \in U$ is an ultrafilter on $S \times S$.

7.5. Let U be an ultrafilter on S and let $f : S \to T$. Then the set $f_*(U) = \{X \subset T : f_{-1}(X) \in U\}$ is an ultrafilter on T.

7.6. Let U be an ultrafilter on $\boldsymbol{N}$ and let $\langle a_n\rangle_{n=0}^\infty$ be a bounded sequence of real numbers. Prove that there exists a unique *U-limit* $a = \lim_U a_n$ such that for every $\varepsilon > 0$, $\{n : |a_n - a| < \varepsilon\} \in U$.

7.7. A nonprincipal ultrafilter D on ω is a p-point if and only if it satisfies the following: If $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ is a decreasing sequence of elements of D, then there exists $X \in D$ such that for each n, $X - A_n$ is finite.

7.8. If $(P,<)$ is a countable linearly ordered set and if D is a p-point on P, then there exists $X \in D$ such that the order-type of X is either ω or ω^*. (X has order-type ω^* if and only if $X = \{x_n\}_{n=0}^\infty$ and $x_0 > x_1 \ldots > x_n > \ldots$.)

7.9. An ultrafilter D on ω is Ramsey if and only if every function $f : \omega \to \omega$ is either one-to-one on a set in D, or constant on a set in D.

If D and E are ultrafilters on ω, then $D \le E$ means that for some function $f : \omega \to \omega$, $D = f_*(E)$ (the Rudin-Keisler ordering, see Exercise 7.5).

$D \equiv E$ means that there is a one-to-one function of ω onto ω such that $E = f_*(D)$.

7.10. If $D = f_*(D)$, then $\{n : f(n) = n\} \in D$.

[Let $X = \{n : f(n) < n\}$, $Y = \{n : f(n) > n\}$. For each $n \in X$, let $l(n)$ be the length of the maximal sequence such that $n > f(n) > f(f(n)) > \ldots$. Let $X_0 = \{n \in X : l(n)$ is even$\}$ and $X_1 = \{n \in X : l(n)$ is odd$\}$. Neither X_0 nor X_1 can be in D since, e.g., $X_0 \cap f_{-1}(X_0) = \emptyset$. The set Y is handled similarly, except that it remains to show that the set Z of all n such that the sequence $n < f(n) < f^2(n) < f^3(n) < \dots$ is infinite cannot be in D. For $x, y \in Z$ let $x \equiv y$ if $f^k(x) = f^m(y)$ for some k and m. For each $x \in Z$, let a_x be a fixed representative of the class $\{y : y \equiv x\}$; let $l(x)$ be the least $k + m$ such that $f^k(x) = f^m(a_x)$. Let $Z_0 = \{x \in Z : l(x) \text{ is even}\}$ and $Z_1 = \{x \in Z : l(x) \text{ is odd}\}$. Clearly $f_{-1}(Z_1) \cap Z = Z_0$.]

7.11. If $D \le E$ and $E \le D$, then $D \equiv E$.
[Use Exercise 7.10.]

Thus $\le$ is a partial ordering of ultrafilters on ω. A nonprincipal ultrafilter D is *minimal* if there is no nonprincipal E such that $E \le D$ and $E \not\equiv D$.

7.12. An ultrafilter D on ω is minimal if and only if it is Ramsey.
[If D is Ramsey and $E = f_*(D)$ is nonprincipal, then f is unbounded mod D, hence one-to-one mod D and consequently, $E \equiv D$. If D is minimal and f is unbounded mod D, then $D \le f_*(D)$ and hence $D = g_*(f_*(D))$ for some g. It follows, by Exercise 7.10, that f is one-to-one mod D.]

7.13. If ω_α is singular, then there is no nonprincipal ω_α-complete ideal on ω_α.

7.14. The set of all sets $X \subset \boldsymbol{R}$ that have Lebesgue measure 0 is a σ-ideal.

A set $X \subset \boldsymbol{R}$ is *meager* if it is the union of a countable collection of nowhere dense sets.

7.15. The set of all meager sets $X \subset \boldsymbol{R}$ is a σ-ideal.
[By the Baire Category Theorem, $\boldsymbol{R}$ is not meager.]

7.16. Let κ be a regular uncountable cardinal, let $|A| \ge \kappa$ and let $S = P_\kappa(A)$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$, where $\hat{P} = \{Q \in S : P \subset Q\}$. Then F is a κ-complete filter on S.

7.17. Let B be a Boolean algebra and define

$$u \oplus v = (u - v) + (v - u).$$

Then B with operations $\oplus$ and $\cdot$ is a ring (with zero 0 and unit 1).

7.18. Every element of the subalgebra generated by X is equal to $u_1 + \dots + u_n$ where each u_s is of the form $u_s = \pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_k$ with $x_i \in X$.

7.19. If A is a subalgebra of B and $u \in B$, then the subalgebra generated by $A \cup \{u\}$ is equal to $\{a \cdot u + (b - u) : a, b \in A\}$.

7.20. A finitely generated Boolean algebra is finite. If A has k generators, then $|A| \le 2^{2^k}$.

7.21. Every finite Boolean algebra is atomic. If $A = \{a_1, \dots, a_n\}$ are the atoms of B, then B is isomorphic to the field of sets $P(A)$. Hence B has 2^n elements.

7.22. Any two countable atomless Boolean algebras are isomorphic.

7.23. $B{\restriction}a$ is isomorphic to B/I where I is the principal ideal $\{u : u \le -a\}$.

7.24. Let A be a subalgebra of a Boolean algebra B and let $u \in B - A$. Then there exist ultrafilters F, G on B such that $u \in F$, $u \notin G$, and $F \cap A = G \cap A$.

7.25. Let B be an infinite Boolean algebra, $|B| = \kappa$. There are at least κ ultrafilters on B.
[Assume otherwise. For each pair $(F, G) \in S \times S$ pick $u \in F - G$, and let these u's generate a subalgebra A. Since $|A| \le |S| < \kappa$, let $u \in B - A$. Use Exercise 7.24 to get a contradiction.]

7.26. For B to be complete it is sufficient that all the sums $\sum X$ exist.
[$\prod X = \sum\{u : u \le x \text{ for all } x \in X\}$.]

7.27. Let B be a complete Boolean algebra.
(i) Verify the distributive laws:

$$a \cdot \sum\{u : u \in X\} = \sum\{a \cdot u : u \in X\},$$
$$a + \prod\{u : u \in X\} = \prod\{a + u : u \in X\}.$$

(ii) Verify the De Morgan laws:

$$-\sum\{u : u \in X\} = \prod\{-u : u \in X\},$$
$$-\prod\{u : u \in X\} = \sum\{-u : u \in X\}.$$

7.28. Let A and B be σ-complete Boolean algebras. If A is isomorphic to $B{\restriction}b$ and B is isomorphic to $A{\restriction}a$, then A and B are isomorphic.
[Follow the proof of the Cantor-Bernstein Theorem.]

7.29. Let A be a subalgebra of a Boolean algebra B, let $u \in B$ and let $A(u)$ be the algebra generated by $A \cup \{u\}$. If h is a homomorphism from A into a complete Boolean algebra C then h extends to a homomorphism from $A(u)$ into C.
[Let $v \in C$ be such that $\sum\{h(a) : a \in A, a \le u\} \le v \le \sum\{h(b) : b \in A, u \le b\}$. Define $h(a \cdot u + b \cdot (-u)) = h(a) \cdot v + h(b) \cdot (-v)$.]

7.30 (Sikorski's Extension Theorem). Let A be a subalgebra of a Boolean algebra B and let h be a homomorphism from A into a complete Boolean algebra C. Then h can be extended to a homomorphism from B into C.
[Use Exercise 7.29 and Zorn's Lemma.]

7.31. If B is a Boolean algebra and A is a regular subalgebra of B then the inclusion mapping extends to a (unique) complete embedding of the completion of A into the completion of B.
[Use Sikorski's Extension Theorem.]

7.32. If B is an infinite complete Boolean algebra, then $|B|^{\aleph_0} = |B|$.
[First consider the case when $|B{\restriction}a| = |B|$ for all $a \ne 0$: There is a partition W such that $|W| = \aleph_0$, and $|B| = \prod\{|B{\restriction}a| : a \in W\} = |B|^{\aleph_0}$. In general, call $a \ne 0$ *stable* if $|B{\restriction}x| = |B{\restriction}a|$ for all $x \le a$, $x \ne 0$. The set of all stable $a \in B$ is dense, and $|B{\restriction}a| = 2$ or $|B{\restriction}a|^{\aleph_0} = |B{\restriction}a|$ if a is stable. Let W be a partition of B such that each $a \in W$ is stable; we have $|B| = \prod\{|B{\restriction}a| : a \in W\}$ and the theorem follows.]

7.33. If B is a κ-complete, κ-saturated Boolean algebra, then B is complete.
[It suffices to show that $\sum X$ exists for every *open* X (i.e., $u \le v \in X$ implies $u \in X$). If $X \subset B$ is open, show that $\sum X = \sum W$ where W is a maximal subset of X that is an antichain.]

Historical Notes

The notion of filter is, according to Kuratowski's book [1966], due to H. Cartan. Theorem 7.5 was first proved by Tarski in [1930].

Theorem 7.6 is due to Pospíšil [1937]; the present proof uses independent sets (Lemma 7.7); cf. Fichtenholz and Kantorovich [1935] ($\kappa = \omega$) and Hausdorff [1936b].

W. Rudin [1956] proved that p-points exist if $2^{\aleph_0} = \aleph_1$, a recent result of Shelah shows that existence of p-points is unprovable in ZFC. Galvin showed that $2^{\aleph_0} = \aleph_1$ implies the existence of Ramsey ultrafilters.

Facts about Boolean algebras can be found in Handbook of Boolean algebras [1989] which also contains an extensive bibliography. The Representation Theorem for Boolean algebras as well as the existence of the completion (Theorems 7.11 and 7.13) are due to Stone [1936]. Theorem 7.15 on saturation was proved by Erdős and Tarski [1943].

Exercise 7.8: Booth [1970/71].

Exercise 7.10: Frolík [1968], M. E. Rudin [1971].

The Rudin-Keisler equivalence was first studied by W. Rudin in [1956]; the study of the Rudin-Keisler ordering was initiated by M. E. Rudin [1966].

Exercise 7.25: Makinson [1969].

Exercises 7.29 and 7.30: Sikorski [1964].

Exercise 7.32: Pierce [1958]. The assumption can be weakened to "σ-complete," see Comfort and Hager [1972].

8. Stationary Sets

In this chapter we develop the theory of closed unbounded and stationary subsets of a regular uncountable cardinal, and its generalizations.

Closed Unbounded Sets

If X is a set of ordinals and $\alpha > 0$ is a limit ordinal then α is a *limit point* of X if $\sup(X \cap \alpha) = \alpha$.

Definition 8.1. Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is a *closed unbounded* subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.

A set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ.

An unbounded set $C \subset \kappa$ is closed if and only if for every sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots$ $(\xi < \gamma)$ of elements of C, of length $\gamma < \kappa$, we have $\lim_{\xi \to \gamma} \alpha_\xi \in C$.

Lemma 8.2. *If C and D are closed unbounded, then $C \cap D$ is closed unbounded.*

Proof. It is immediate that $C \cap D$ is closed. To show that $C \cap D$ is unbounded, let $\alpha < \kappa$. Since C is unbounded, there exists an $\alpha_1 > \alpha$ with $\alpha_1 \in C$. Similarly there exists an $\alpha_2 > \alpha_1$ with $\alpha_2 \in D$. In this fashion, we construct an increasing sequence

$$\alpha < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots \tag{8.1}$$

such that $\alpha_1, \alpha_3, \alpha_5, \ldots \in C$, $\alpha_2, \alpha_4, \alpha_6, \ldots \in D$. If we let β be the limit of the sequence (8.1), then $\beta < \kappa$, and $\beta \in C$ and $\beta \in D$. □

The collection of all closed unbounded subsets of κ has the finite intersection property. The filter generated by the closed unbounded sets consists of all $X \subset \kappa$ that contain a closed unbounded subset. We call this filter the *closed unbounded filter* on κ.

The set of all limit ordinals $\alpha < \kappa$ is closed unbounded in κ. If A is an unbounded subset of κ, then the set of all limit points $\alpha < \kappa$ of A is closed unbounded.

A function $f : \kappa \to \kappa$ is *normal* if it is increasing and continuous ($f(\alpha) = \lim_{\xi\to\alpha} f(\xi)$ for every nonzero limit $\alpha < \kappa$). The range of a normal function is a closed unbounded set. Conversely, if C is closed unbounded, there is a unique normal function that enumerates C.

The closed unbounded filter on κ is κ-complete:

Theorem 8.3. *The intersection of fewer than κ closed unbounded subsets of κ is closed unbounded.*

Proof. We prove, by induction on $\gamma < \kappa$, that the intersection of a sequence $\langle C_\alpha : \alpha < \gamma\rangle$ of closed unbounded subsets of κ is closed unbounded. The induction step works at successor ordinals because of Lemma 8.2. If γ is a limit ordinal, we assume that the assertion is true for every $\alpha < \gamma$; then we can replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$ and obtain a decreasing sequence with the same intersection. Thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \gamma)$$

are closed unbounded, and let $C = \bigcap_{\alpha<\gamma} C_\alpha$.

It is easy to see that C is closed. To show that C is unbounded, let $\alpha < \kappa$. We construct a γ-sequence

$$\beta_0 < \beta_1 < \ldots \beta_\xi < \ldots \qquad (\xi < \gamma) \tag{8.2}$$

as follows: We let $\beta_0 \in C_0$ be such that $\beta_0 > \alpha$, and for each $\xi < \gamma$, let $\beta_\xi \in C_\xi$ be such that $\beta_\xi > \sup\{\beta_\nu : \nu < \xi\}$. Since κ is regular and $\gamma < \kappa$, such a sequence (8.2) exists and its limit β is less than κ. For each $\eta < \gamma$, β is the limit of a sequence $\langle\beta_\xi : \eta \le \xi < \gamma\rangle$ in C_η, and so $\beta \in C_\eta$. Hence $\beta \in C$. □

Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of subsets of κ. The *diagonal intersection* of X_α, $\alpha < \kappa$, is defined as follows:

$$\mathop{\triangle}_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}. \tag{8.3}$$

Note that $\triangle X_\alpha = \triangle Y_\alpha$ where $Y_\alpha = \{\xi \in X_\alpha : \xi > \alpha\}$. Note also that $\triangle X_\alpha = \bigcap_\alpha (X_\alpha \cup \{\xi : \xi \le \alpha\})$.

Lemma 8.4. *The diagonal intersection of a κ-sequence of closed unbounded sets is closed unbounded.*

Proof. Let $\langle C_\alpha : \alpha < \kappa\rangle$ be a sequence of closed unbounded sets. It is clear from the definition that if we replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$, the diagonal

intersection is the same. In view of Theorem 8.3 we may thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \kappa).$$

Let $C = \triangle_{\alpha<\kappa} C_\alpha$. To show that C is closed, let α be a limit point of C. We want to show that $\alpha \in C$, or that $\alpha \in C_\xi$ for all $\xi < \alpha$. If $\xi < \alpha$, let $X = \{\nu \in C : \xi < \nu < \alpha\}$. Every $\nu \in X$ is in C_ξ, by (8.3). Hence $X \subset C_\xi$ and $\alpha = \sup X \in C_\xi$. Therefore $\alpha \in C$ and C is closed.

To show that C is unbounded, let $\alpha < \kappa$. We construct a sequence $\langle\beta_n : n < \omega\rangle$ as follows: Let $\beta_0 > \alpha$ be such that $\beta_0 \in C_0$, and for each n, let $\beta_{n+1} > \beta_n$ be such that $\beta_{n+1} \in C_{\beta_n}$. Let us show that $\beta = \lim_n \beta_n$ is in C: If $\xi < \beta$, let us show that $\beta \in C_\xi$. Since $\xi < \beta$, there is an n such that $\xi < \beta_n$. Each β_k, $k > n$, belongs to C_{β_n} and so $\beta \in C_{\beta_n}$. Therefore $\beta \in C_\xi$. Thus $\beta \in C$, and C is unbounded. □

Corollary 8.5. *The closed unbounded filter on κ is closed under diagonal intersections.* □

The dual of the closed unbounded filter is the ideal of nonstationary sets, the *nonstationary ideal* I_{NS}. I_{NS} is κ-complete and is closed under *diagonal unions*:

$$\sum_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcup_{\alpha<\xi} X_\alpha\}.$$

The quotient algebra $B = P(\kappa)/I_{\mathrm{NS}}$ is a κ-complete Boolean algebra, where the Boolean operations $\sum_{\alpha<\gamma}$ and $\prod_{\alpha<\gamma}$ for $\gamma < \kappa$ are induced by $\bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$. As a consequence of Lemma 8.4, B is κ^+-complete: If $\{X_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ then the equivalence classes of $\triangle_{\alpha<\kappa} X_\alpha$ and $\sum_{\alpha<\kappa} X_\alpha$ are, respectively, the greatest lower bound and the least upper bound of the equivalence classes $[X_\alpha]$ in B. It also follows that if $\langle X_\alpha : \alpha < \kappa\rangle$ and $\langle Y_\alpha : \alpha < \kappa\rangle$ are two enumerations of the same collection, then $\triangle_{\alpha<\kappa} X_\alpha$ and $\triangle_{\alpha<\kappa} Y_\alpha$ differ only by a nonstationary set.

Definition 8.6. An ordinal function f on a set S is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in S$, $\alpha > 0$.

Theorem 8.7 (Fodor). *If f is a regressive function on a stationary set $S \subset \kappa$, then there is a stationary set $T \subset S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Proof. Let us assume that for each $\gamma < \kappa$, the set $\{\alpha \in S : f(\alpha) = \gamma\}$ is nonstationary, and choose a closed unbounded set C_γ such that $f(\alpha) \ne \gamma$ for each $\alpha \in S \cap C_\gamma$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$. The set $S \cap C$ is stationary and if $\alpha \in S \cap C$, we have $f(\alpha) \ne \gamma$ for every $\gamma < \alpha$; in other words, $f(\alpha) \ge \alpha$. This is a contradiction. □

For a regular uncountable cardinal κ and a regular $\lambda < \kappa$, let

$$E^\kappa_\lambda = \{\alpha < \kappa : \operatorname{cf}\alpha = \lambda\}. \tag{8.4}$$

It is easy to see that each E^κ_λ is a stationary subset of κ.

The closed unbounded filter on κ is not an ultrafilter. This is because there is a stationary subset of κ whose complement is stationary. If $\kappa > \omega_1$, this is clear: The sets E^κ_ω and $E^\kappa_{\omega_1}$ are disjoint. If $\kappa = \omega_1$, the decomposition of ω_1 into disjoint stationary sets uses the Axiom of Choice.

The use of AC is necessary: It is consistent (relative to large cardinals) that the closed unbounded filter on ω_1 is an ultrafilter.

In Theorem 8.10 below we show that every stationary subset of κ is the union of κ disjoint stationary sets. In the following lemma we prove a weaker result that illustrates a typical use of Fodor's Theorem.

Lemma 8.8. *Every stationary subset of E^κ_ω is the union of κ disjoint stationary sets.*

Proof. Let $W \subset \{\alpha < \kappa : \operatorname{cf}\alpha = \omega\}$ be stationary. For every $\alpha \in W$, we choose an increasing sequence $\langle a^\alpha_n : n \in \boldsymbol{N}\rangle$ such that $\lim_n a^\alpha_n = \alpha$. First we show that there is an n such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a^\alpha_n \ge \eta\} \tag{8.5}$$

is stationary. Otherwise there is η_n and a closed unbounded set C_n such that $a^\alpha_n < \eta_n$ for all $\alpha \in C_n \cap W$, for every n. If we let η be the supremum of the η_n and C the intersection of the C_n, we have $a^\alpha_n < \eta$ for all n and all $\alpha \in C \cap W$. This is a contradiction. Now let n be such that (8.5) is stationary for every $\eta < \kappa$. Let f be the following function on W: $f(\alpha) = a^\alpha_n$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.5) and $\gamma_\eta \ge \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \ne \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$, and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

The proof easily generalizes to the case when $\lambda > \omega$: Every stationary subset of E^κ_λ the union of κ stationary sets. From that it follows that every stationary subset W of the set $\{\alpha < \kappa : \operatorname{cf}\alpha < \alpha\}$ admits such a decomposition: By Fodor's Theorem, there exists some $\lambda < \kappa$ such that $W \cap E^\kappa_\lambda$ is stationary. The remaining case in Theorem 8.10 is when the set $\{\alpha < \kappa : \alpha$ is a regular cardinal$\}$ is stationary and the following lemma plays the key role.

Lemma 8.9. *Let S be a stationary subset of κ and assume that every $\alpha \in S$ is a regular uncountable cardinal. Then the set $T = \{\alpha \in S : S \cap \alpha$ is not a stationary subset of $\alpha\}$ is stationary.*

Proof. We prove that T intersects every closed unbounded subset of κ. Let C be closed unbounded. The set C' of all limit points of C is also closed unbounded, and hence $S \cap C' \ne \emptyset$. Let α be the least element of $S \cap C'$. Since α is regular and a limit point of C, $C \cap \alpha$ is a closed unbounded subset of α, and so is $C' \cap \alpha$. As α is the least element of $S \cap C'$, $C' \cap \alpha$ is disjoint from $S \cap \alpha$ and so $S \cap \alpha$ is a nonstationary subset of α. Hence $\alpha \in T \cap C$. □

Theorem 8.10 (Solovay). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

Proof. We follow the proof of Lemma 8.8 as much as possible. Let A be a stationary subset of κ. By Lemma 8.8, by the subsequent discussion and by Lemma 8.9, we may assume that the set W of all $\alpha \in A$ such that α is a regular cardinal and $A \cap \alpha$ is not stationary, is stationary. There exists for each $\alpha \in W$ a continuous increasing sequence $\langle a^\alpha_\xi : \xi < \alpha\rangle$ such that $a^\alpha_\xi \notin W$, for all α and ξ, and $\alpha = \lim_{\xi\to\alpha} a^\alpha_\xi$.

First we show that there is ξ such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a^\alpha_\xi \ge \eta\} \tag{8.6}$$

is stationary. Otherwise, there is for each ξ some $\eta(\xi)$ and a closed unbounded set C_ξ such that $a^\alpha_\xi < \eta(\xi)$ for all $\alpha \in C_\xi \cap W$ if a^α_ξ is defined. Let C be the diagonal intersection of the C_ξ. Thus if $\alpha \in C \cap W$, then $a^\alpha_\xi < \eta(\xi)$ for all $\xi < \alpha$. Now let D be the closed unbounded set of all $\gamma \in C$ such that $\eta(\xi) < \gamma$ for all $\xi < \gamma$. Since W is stationary, $W \cap D$ is also stationary; let $\gamma < \alpha$ be two ordinals in $W \cap D$. Now if $\xi < \gamma$, then $a^\alpha_\xi < \eta(\xi) < \gamma$ and it follows that $a^\alpha_\gamma = \gamma$. This is a contradiction since $\gamma \in W$ and $a^\alpha_\gamma \notin W$.

Once we have found ξ such that (8.6) is stationary for all $\eta < \kappa$, we proceed as in Lemma 8.8. Let f be the function on W defined by $f(\alpha) = a^\alpha_\xi$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.6) and $\gamma_\eta \ge \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \ne \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$; and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

Mahlo Cardinals

Let κ be an inaccessible cardinal. The set of all cardinals below κ is a closed unbounded subset of κ, and so is the set of its limit points, the set of all limit cardinals. In fact, the set of all strong limit cardinals below κ is closed unbounded.

If κ is the least inaccessible cardinal, then all strong limit cardinals below κ are singular, and so the set of all singular strong limit cardinals below κ is closed unbounded. If κ is the αth inaccessible, where $\alpha < \kappa$, then still the set of all regular cardinals below κ is nonstationary.

An inaccessible cardinal κ is called a *Mahlo cardinal* if the set of all regular cardinals below κ is stationary.

(Then the set of all inaccessibles below κ is stationary, and κ is the κth inaccessible cardinal.)

Similarly, we define a *weakly Mahlo* cardinal as a cardinal κ that is weakly inaccessible and the set of all regular cardinals bellow κ is stationary (then the set of all weakly inaccessibles is stationary in κ).

Normal Filters

Let F be a filter on a cardinal κ; F is *normal* if it is closed under diagonal intersections:

(8.7) if $X_\alpha \in F$ for all $\alpha < \kappa$, then $\triangle_{\alpha<\kappa} X_\alpha \in F$.

An ideal I on κ is *normal* if the dual filter is normal.

The closed unbounded filter is κ-complete and normal, and contains all complements of bounded sets. It is the smallest such filter on κ:

Lemma 8.11. *If κ is regular and uncountable and if F is a normal filter on κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$, then F contains all closed unbounded sets.*

Proof. First we note that the set C_0 of all limit ordinals is in F: C_0 is the diagonal intersection of the sets $X_\alpha = \{\xi : \alpha + 1 < \xi < \kappa\}$. Now let C be a closed unbounded set, and let $C = \{a_\alpha : \alpha < \kappa\}$ be its increasing enumeration. We let $X_\alpha = \{\xi : a_\alpha < \xi < \kappa\}$. Then $C \supset C_0 \cap \triangle_{\alpha<\kappa} X_\alpha$. □

Silver's Theorem

We shall now apply the techniques using ultrafilters and stationary sets to prove the following theorems.

Theorem 8.12 (Silver). *Let κ be a singular cardinal such that $\operatorname{cf}\kappa > \omega$. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

Theorem 8.13 (Silver). *If the Singular Cardinals Hypothesis holds for all singular cardinals of cofinality ω, then it holds for all singular cardinals.*

The proofs of both theorems use the following lemma:

Lemma 8.14. *Let κ be a singular cardinal, let $\operatorname{cf}\kappa > \omega$, and assume that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$. If $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ is a normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$, and if the set $\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$ is stationary in $\operatorname{cf}\kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.*

If GCH holds below κ then the assumptions of Lemma 8.14 are satisfied, and $2^\kappa = \kappa^{\operatorname{cf}\kappa}$. Thus Theorem 8.12 follows from Lemma 8.14.

Proof of Theorem 8.13. We prove by induction on the cofinality of κ that $2^{\operatorname{cf}\kappa} < \kappa$ implies $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. The assumption of the theorem is that this holds for each κ of cofinality ω. Thus let κ be of uncountable cofinality and let $2^{\operatorname{cf}\kappa} < \kappa$. Using the induction hypothesis and the proof of Theorem 5.22(ii) one verifies, by induction on λ, that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$.

Let $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ be any normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$. The set $S = \{\alpha < \operatorname{cf}\kappa : \operatorname{cf}\kappa_\alpha = \omega \text{ and } 2^{\aleph_0} < \kappa_\alpha\}$ is clearly stationary in $\operatorname{cf}\kappa$, and for every $\alpha \in S$, $\kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+$ by the assumption. Hence $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

We now proceed toward a proof of Lemma 8.14. To simplify the notation, we shall consider the special case when

$$\kappa = \aleph_{\omega_1}.$$

The general case is proved in a similar way.

Let f and g be two functions on ω_1. We say that f and g are *almost disjoint* if there is $\alpha_0 < \omega_1$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \alpha_0$. A family F of functions on ω_1 is an *almost disjoint family* if any two distinct $f, g \in F$ are almost disjoint.

Lemma 8.14 follows from

Lemma 8.15. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha,$$

such that the set

(8.8) $$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\}$$

is stationary. Then $|F| \leq \aleph_{\omega_1+1}$.

[In the general case, we consider almost disjoint functions on $\operatorname{cf}\kappa$.]

Proof of Lemma 8.14 *from Lemma* 8.15. We assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ and that $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = \aleph_{\alpha+1}$ for a stationary set of α's; we want to show that $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+1}$. For every $h : \omega_1 \to \aleph_{\omega_1}$, we let $f_h = \langle h_\alpha : \alpha < \omega_1 \rangle$, where $\operatorname{dom} h_\alpha = \omega_1$ and

$$h_\alpha(\xi) = \begin{cases} h(\xi) & \text{if } h(\xi) < \aleph_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and let $F = \{f_h : h \in {\aleph_{\omega_1}}^{\omega_1}\}$. If $h \neq g$, then f_h and f_g are almost disjoint. Moreover,

$$F \subset \prod_{\alpha<\omega_1} {\aleph_\alpha}^{\omega_1}.$$

Since for a stationary set of α's, $\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}$ (namely for all α such that $\aleph_\alpha > 2^{\aleph_1}$ and $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}$), we have $|F| \le \aleph_{\omega_1+1}$, and so $|{\aleph_{\omega_1}}^{\omega_1}| = \aleph_{\omega_1+1}$.

[In the general case of Lemma 8.14 we have to show that

$$\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$$

is stationary. Note that the set

$$C = \{\alpha : \alpha \text{ is a limit ordinal and } (\forall \lambda < \kappa_\alpha)\, \lambda^{\operatorname{cf}\kappa} < \kappa_\alpha\}$$

is closed unbounded in $\operatorname{cf}\kappa$; if $\alpha \in C$, then $\operatorname{cf}\kappa_\alpha < \operatorname{cf}\kappa$ and we have $\kappa_\alpha^{\operatorname{cf}\kappa} = \kappa_\alpha^{\operatorname{cf}\alpha}$.] □

The first step in the proof of Lemma 8.15 is

Lemma 8.16. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \le \aleph_\alpha\} \tag{8.9}$$

is stationary. Then $|F| \le \aleph_{\omega_1}$.

(The assumption (8.8) is replaced by (8.9) and the bound for $|F|$ is $\aleph_{\omega_1}$ rather than $\aleph_{\omega_1+1}$.)

Proof. We may as well assume that each A_α is a set of ordinals and that $A_\alpha \subset \omega_\alpha$ for all α in some stationary subset of $\aleph_1$. Let

$$S_0 = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal and } A_\alpha \subset \omega_\alpha\}.$$

Thus if $f \in F$, then $f(\alpha) < \omega_\alpha$ for all $\alpha \in S_0$. Given $f \in F$, we can find for each $\alpha > 0$ in S_0 some $\beta < \alpha$ such that $f(\alpha) < \omega_\beta$; call this $\beta = g(\alpha)$. The function g is regressive on S, and by Fodor's Theorem there is a stationary $S \subset S_0$ such that g is constant on S. In other words, the function f is bounded on S, by some $\omega_\gamma < \omega_{\omega_1}$.

We assign to each f a pair $(S, f{\restriction}S)$ where $S \subset S_0$ is a stationary set and $f{\restriction}S$ is a bounded function. For any S, if $f{\restriction}S = g{\restriction}S$, then $f = g$ since any two distinct functions in F are almost disjoint. Thus the correspondence

$$f \mapsto (S, f{\restriction}S)$$

is one-to-one.

For a given S, the number of bounded functions on S is at most

$$\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|} = \sup_{\gamma<\omega_1} \aleph_\gamma^{\aleph_1} = \aleph_{\omega_1}.$$

Since $|P(\omega_1)| = 2^{\aleph_1} < \aleph_{\omega_1}$, the number of pairs $(S, f{\restriction}S)$ is at most $\aleph_{\omega_1}$. Hence $|F| \le \aleph_{\omega_1}$. □

Proof of Lemma 8.15. Let U be an ultrafilter on ω_1 that extends the closed unbounded filter. Every $S \in U$ is stationary.

We may assume that each A_α is a subset of $\omega_{\alpha+1}$. For every $f, g \in F$, let

$$f < g \text{ if and only if } \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in U. \tag{8.10}$$

Since U is a filter, the relation $f < g$ is transitive. Since U is an ultrafilter, and $\{\alpha : f(\alpha) = g(\alpha)\} \notin U$ for distinict $f, g \in F$, the relation $f < g$ is a linear ordering of F. For every $f \in F$, let $F_f = \{g \in F : \text{for some stationary set } T,\ g(\alpha) < f(\alpha) \text{ for all } \alpha \in T\}$. By Lemma 8.16, $|F_f| \le \aleph_{\omega_1}$. If $g < f$, then $g \in F_f$, and so $|\{g \in F : g < f\}| \le \aleph_{\omega_1}$. It follows that $|F| \le \aleph_{\omega_1+1}$. □

A Hierarchy of Stationary Sets

If α is a limit ordinal of uncountable cofinality, it still makes sense to talk about closed unbounded and stationary subsets of α. Since $\operatorname{cf}\alpha > \omega$, Lemma 8.2 holds, and the closed unbounded sets generate a filter on α. The closed unbounded filter is $\operatorname{cf}\alpha$-complete. A set $S \subset \alpha$ is stationary if and only if for some (or for any) normal function $f : \operatorname{cf}\alpha \to \alpha$, $f_{-1}(S)$ is a stationary subset of $\operatorname{cf}\alpha$.

Let κ be a regular uncountable cardinal, and let us consider the following operation (the *Mahlo operation*) on stationary sets:

Definition 8.17. If $S \subset \kappa$ is stationary, the *trace* of S is the set

$$\operatorname{Tr}(S) = \{\alpha < \kappa : \operatorname{cf}\alpha > \omega \text{ and } S \cap \alpha \text{ is stationary}\}.$$

The Mahlo operation is invariant under equivalence mod I_{NS} and can thus be considered as an operation on the Boolean algebra $P(\kappa)/I_{\mathrm{NS}}$ (see Exercise 8.11).

In the context of closed unbounded and stationary sets we use the phrase *for almost all* $\alpha \in S$ to mean that the set of all contrary $\alpha \in S$ is nonstationary.

Definition 8.18. Let S and T be stationary subsets of κ.

$$S < T \quad \text{if and only if} \quad S \cap \alpha \text{ is stationary for almost all } \alpha \in T.$$

(It is implicit in the definition that almost all $\alpha \in T$ have uncountable cofinality.)

As an example, if $\lambda < \mu$ are regular, then $E^\kappa_\lambda < E^\kappa_\mu$. The following properties are easily verified:

Lemma 8.19.

(i) $A < \operatorname{Tr}(A)$,

(ii) *if* $A < B$ *and* $B < C$ *then* $A < C$,
(iii) *if* $A < B$, $A \simeq A'$ mod I_{NS} *and* $B \simeq B'$ mod I_{NS} *then* $A' < B'$. □

Thus $<$ is a transitive relation on $P(\kappa)/I_{\mathrm{NS}}$. The next theorem shows that it is a well-founded partial ordering:

Theorem 8.20 (Jech). *The relation* $<$ *is well-founded.*

Proof. Assume to the contrary that there exist stationary sets such that $A_1 > A_2 > A_3 \ldots$. Therefore there exist closed unbounded sets C_n such that $A_n \cap C_n \subset \mathrm{Tr}(A_{n+1})$ for $n = 1, 2, 3, \ldots$. For each n, let

$$B_n = A_n \cap C_n \cap \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \cap \ldots$$

where $\mathrm{Lim}(C)$ is the set of all limit points of C.

Each B_n is stationary, and for every n, $B_n \subset \mathrm{Tr}(B_{n+1})$. Let $\alpha_n = \min B_n$. Since $B_{n+1} \cap \alpha_n$ is stationary, we have $\alpha_{n+1} < \alpha_n$ and therefore, a decreasing sequence $\alpha_1 > \alpha_2 > \ldots$. A contradiction. □

The rank of a stationary set $A \subset \kappa$ in the well-founded relation $<$ is called the *order* of the set A, and the height of $<$ is the *order* of the cardinal κ:

$$o(A) = \sup\{o(X) + 1 : X < A\},$$
$$o(\kappa) = \sup\{o(A) + 1 : A \subset \kappa \text{ is stationary}\}.$$

We also define $o(\aleph_0) = 0$, and $o(\alpha) = o(\mathrm{cf}\,\alpha)$ for every limit ordinal α. Note that $o(E^\kappa_\omega) = 0$, $o(E^\kappa_{\omega_1}) = 1$, $o(\aleph_1) = 1$, $o(\aleph_2) = 2$, etc. See Exercises 8.13 and 8.14.

The Closed Unbounded Filter on $P_\kappa(\lambda)$

We shall now consider a generalization of closed unbounded and stationary sets, to the space $P_\kappa(\lambda)$. This generalization replaces $(\kappa, <)$ with the structure $(P_\kappa(\lambda), \subset)$.

Let κ be a regular uncountable cardinal and let A be a set of cardinality at least κ.

Definition 8.21. A set $X \subset P_\kappa(A)$ is *unbounded* if for every $x \in P_\kappa(A)$ there exists a $y \supset x$ such that $y \in X$.

A set $X \subset P_\kappa(A)$ is *closed* if for any chain $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, of sets in X, with $\alpha < \kappa$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.

A set $C \subset P_\kappa(A)$ is *closed unbounded* if it is closed and unbounded.

A set $S \subset P_\kappa(A)$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded $C \subset P_\kappa(A)$.

The *closed unbounded filter* on $P_\kappa(A)$ is the filter generated by the closed unbounded sets.

When $|A| = |B|$ then $P_\kappa(A)$ and $P_\kappa(B)$ are isomorphic, with closed unbounded and stationary sets corresponding to closed unbounded and stationary sets, and so it often suffices to consider such sets in $P_\kappa(\lambda)$ where λ is a cardinal $\geq \kappa$.

When $|A| = \kappa$, then the set $\kappa \subset P_\kappa(\kappa)$ is closed unbounded, and the closed unbounded filter on κ is the restriction to κ of the closed unbounded filter on $P_\kappa(\kappa)$.

Theorem 8.22 (Jech). *The closed unbounded filter on* $P_\kappa(A)$ *is* κ*-complete.*

Proof. This is a generalization of Theorem 8.3. First we proceed as in Lemma 8.2 and show that if C and D are closed unbounded then $C \cap D$ is closed unbounded. Both proofs have straightforward generalizations from $(\kappa, <)$ to $(P_\kappa(A), \subset)$. □

Fodor's Theorem also generalizes to $P_\kappa(A)$; with regressive functions replaced by choice functions. The *diagonal intersection* of subsets of $P_\kappa(A)$ is defined as follows

$$\mathop{\triangle}_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}.$$

Lemma 8.23. *If* $\{C_a : a \in A\}$ *is a collection of closed unbounded subsets of* $P_\kappa(A)$ *then its diagonal intersection is closed unbounded.*

Proof. Let $C = \triangle_{a\in A} C_a$. First we show that C is closed. Let $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, be a chain in C, with $\alpha < \kappa$, and let x be its union. To show that $x \in C$, let $a \in x$ and let us show that $x \in C_a$. There is some $\eta < \alpha$ such that $a \in x_\xi$ for all $\xi \geq \eta$; hence $x_\xi \in C_a$ for all $\xi \geq \eta$, and so $x \in C_a$.

Now we show that C is unbounded. Let $x_0 \in P_\kappa(A)$, we shall find an $x \in C$ such that $x \supset x_0$. By induction, we find $x_0 \subset x_1 \subset \ldots \subset x_n \subset \ldots$, $n \in \boldsymbol{N}$, such that $x_{n+1} \in \bigcap_{a\in x_n} C_a$; this is possible because each $\bigcap_{a\in x_n} C_a$ is closed unbounded. Then we let $x = \bigcup_{n=0}^\infty x_n$ and show that $x \in C_a$ for all $a \in x$. But if $a \in x$ then $a \in x_k$ for some k, and then $x_n \in C_a$ for all $n \geq k+1$. Hence $x \in C_a$. □

Theorem 8.24 (Jech). *If* f *is a function on a stationary set* $S \subset P_\kappa(\lambda)$ *and if* $f(x) \in x$ *for every nonempty* $x \in S$, *then there exist a stationary set* $T \subset S$ *and some* $a \in A$ *such that* $f(x) = a$ *for all* $a \in T$.

Proof. The proof uses Lemma 8.23 and generalizes the proof of Theorem 8.7. □

Let us call a set $D \subset P_\kappa(A)$ *directed* if for all x and y in D there is a $z \in D$ such that $x \cup y \subset z$.

Lemma 8.25. *If* C *is a closed subset of* $P_\kappa(A)$ *then for every directed set* $D \subset C$ *with* $|D| < \kappa$, $\bigcup D \in C$.

Proof. By induction on $|D|$. Let $|D| = \gamma$, $D = \{x_\alpha : \alpha < \gamma\}$, and assume the lemma holds for every directed set of cardinality $< \gamma$. By induction on $\alpha < \gamma$, let D_α be a smallest directed subset of D such that $x_\alpha \in D_\alpha$ and $D_\alpha \supset \bigcup_{\beta<\alpha} D_\beta$. Letting $y_\alpha = \bigcup D_\alpha$, we have $y_\alpha \in C$ for all $\alpha < \gamma$, and $y_\beta \subset y_\alpha$ if $\beta < \alpha$. It follows that $\bigcup D = \bigcup_{\alpha<\gamma} y_\alpha \in C$. □

Consider a function $f : [A]^{<\omega} \to P_\kappa(A)$; a set $x \in P_\kappa(A)$ is a *closure point* of f if $f(e) \subset x$ whenever $e \subset x$. The set C_f of all closure points $x \in P_\kappa(A)$ is a closed unbounded set. Moreover, the sets C_f generate the closed unbounded filter:

Lemma 8.26. *For every closed unbounded set C in $P_\kappa(A)$ there exists a function $f : [A]^{<\omega} \to P_\kappa(A)$ such that $C_f \subset C$.*

Proof. By induction on $|e|$ we find for each $e \in [A]^{<\omega}$ an infinite set $f(e) \in C$ such that $e \subset f(e)$ and that $f(e_1) \subset f(e_2)$ whenever $e_1 \subset e_2$. We will show that $C_f \subset C$. Let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of a directed subset of C (of cardinality $< \kappa$), by Lemma 8.25 we have $x \in C$. □

Let $A \subset B$ (and $|A| \ge \kappa$). For $X \in P_\kappa(B)$, the *projection* of X to A is the set

$$X \restriction A = \{x \cap A : x \in X\}.$$

For $Y \in P_\kappa(A)$, the *lifting* of Y to B is the set

$$Y^B = \{x \in P_\kappa(B) : x \cap A \in Y\}.$$

Theorem 8.27 (Menas). *Let $A \subset B$.*

(i) *If S is stationary in $P_\kappa(B)$, then $S \restriction A$ is stationary in $P_\kappa(A)$.*
(ii) *If S is stationary in $P_\kappa(A)$, then S^B is stationary in $P_\kappa(B)$.*

Proof. (i) holds because if C is a closed unbounded set in $P_\kappa(A)$, then C^B is closed unbounded in $P_\kappa(B)$. For (ii), it suffices to prove that if C is closed unbounded in $P_\kappa(B)$, then $C \restriction A$ contains a closed unbounded set.

If $C \subset P_\kappa(B)$ is closed unbounded, then by Lemma 8.26, $C \supset C_f$ for some $f : [B]^{<\omega} \to P_\kappa(B)$. Let $g : [A]^{<\omega} \to P_\kappa(A)$ be the following function: For $e \in [A]^{<\omega}$, let x be the smallest closure point of f such that $x \supset e$, and let $g(e) = x \cap A$. Then $C_f \restriction A = C_g$ (where C_f is defined in $P_\kappa(B)$ and C_g in $P_\kappa(A)$), and we have $C_g \subset C \restriction A$. □

When $\kappa = \omega_1$, Lemma 8.26 can be improved to give the following basis theorem for $[A]^\omega = \{x \subset A : |x| = \aleph_0\}$. An *operation* on A is a function $F : [A]^{<\omega} \to A$. A set x is *closed under* F if $f(e) \in x$ for all $e \in [x]^{<\omega}$.

Theorem 8.28 (Kueker). *For every closed unbounded set $C \subset [A]^\omega$ there is an operation F on A such that $C \supset C_F = \{x \in [A]^\omega : x$ is closed under $F\}$.*

Proof. We may assume that $A = \lambda$ is an infinite cardinal, and let C be a closed unbounded subset of $[\lambda]^\omega$. As in the proof of Lemma 8.26 there exists a function $f : [\lambda]^{<\omega} \to C$ such that $e \subset f(e)$ and $f(e_1) \subset f(e_2)$ if $e_1 \subset e_2$. As each $f(e)$ is countable, there exist functions f_k, $k \in \boldsymbol{N}$, such that $f(e) = \{f_k(e) : k \in \boldsymbol{N}\}$ for all e. Let $n \mapsto (k_n, m_n)$ be a pairing function.

Now we define an operation F on λ as follows: Let $F(\{\alpha\}) = \alpha + 1$, and if $\alpha_1 < \ldots < \alpha_n$, let $F(\{\alpha_1, \ldots, \alpha_n\}) = f_{k_n}(\{\alpha_1, \ldots, \alpha_{m_n}\})$. It is enough to show that if $x \in [\lambda]^\omega$ is closed under F then x is a closure point of f, and so $C_F \subset C_f \subset C$.

Let x be closed under F, let $k \in \boldsymbol{N}$ and let $e \in [x]^{<\omega}$; we want to show that $f_k(e) \in x$. If $e = \{\alpha_1, \ldots, \alpha_m\}$ with $\alpha_1 < \ldots < \alpha_m$, let $n \ge m$ be such that $k = k_n$ and $m = m_n$. As x does not have a greatest element (because $F(\{\alpha\}) = \alpha + 1$), there are $\alpha_{m+1}, \ldots, \alpha_n \in x$ such that $f_k(\{\alpha_1, \ldots, \alpha_m\}) = F(\{\alpha_1, \ldots, \alpha_n\}) \in x$. □

Theorem 8.28 does not generalize outright to $P_\kappa(A)$ for $\kappa > \omega_1$ (see Exercise 8.18); we shall return to the subject in Part III.

Exercises

8.1. The set of all fixed points (i.e., $f(\alpha) = \alpha$) of a normal function is closed unbounded.

8.2. If $f : \kappa \to \kappa$, then the set of all $\alpha < \kappa$ such that $f(\xi) < \alpha$ for all $\xi < \alpha$ is closed unbounded.

8.3. If S is stationary and C is closed unbounded, then $S \cap C$ is stationary.

8.4. If $X \subset \kappa$ is nonstationary, then there exists a regressive function f on X such that $\{\alpha : f(\alpha) \le \gamma\}$ is bounded, for every $\gamma < \kappa$.

[Let $C \cap X = \emptyset$, and let $f(\alpha) = \sup(C \cap \alpha)$.]

8.5. For every stationary $S \subset \omega_1$ and every $\alpha < \omega_1$ there is a closed set of ordinals A of length α such that $A \subset S$.

[By induction on α: $\forall\gamma$ $\exists$closed $A \subset S$ of length α such that $\gamma < \min A$. The nontrivial step: If true for a limit α, find a closed $A \subset S$ of length α such that $\sup A \in S$. Let A_ξ, $\xi < \omega_1$, be closed subsets of S, of length α, such that $\lambda_\xi = \sup \bigcup_{\nu<\xi} A_\nu < \min A_\xi$. There is ξ such that $\lambda_\xi \in S$. Let $\xi = \lim_n \xi_n$. Pick initial segments $B_{\xi_n} \subset A_{\xi_n}$ of length $\alpha_n + 1$ where $\lim_n \alpha_n = \alpha$. Let $A = \bigcup_{n=0}^\infty B_{\xi_n}$.]

Exercise 8.5 does not generalize to closed sets of uncountable length. It is not provable in ZFC that given $X \subset \omega_2$, either X or $\omega_2 - X$ contains a closed set of length ω_1. On the other hand, this statement is consistent, relative to large cardinals.

8.6. Let κ be the least inaccessible cardinal such that κ is the κth inaccessible cardinal. Then κ is not Mahlo.

[Use $f(\lambda) = \alpha$ where λ is the αth inaccessible.]

8.7. If κ is a limit (weakly inaccessible, weakly Mahlo) cardinal and the set of all strong limit cardinals below κ is unbounded in κ, then κ is a strong limit (inaccessible, Mahlo) cardinal.

8.8. A κ-complete ideal I on κ is normal if and only if for every $S_0 \notin I$ and any regressive f on S_0 there is $S \subset S_0$, $S \notin I$, such that f is constant on S.

[One direction is like Fodor's Theorem. For the other direction, let $X_\alpha \in F$ for each $\alpha < \kappa$. If $\triangle X_\alpha \notin F$, let $S_0 = \kappa - \triangle X_\alpha$ and let $f(\alpha) =$ some $\xi < \alpha$ such that $\alpha \notin X_\xi$. If $f(\alpha) = \gamma$ for all $\alpha \in S$, then $X_\gamma \cap S = \emptyset$, a contradiction.]

8.9. There is no normal nonprincipal filter on ω.

[Use the regressive function $f(n+1) = n$.]

8.10. If κ is singular, then there is no normal ideal on κ that contains all bounded subsets of κ.

8.11. (i) If $S \subset T$ then $\mathrm{Tr}(S) \subset \mathrm{Tr}(T)$,

(ii) $\mathrm{Tr}(S \cup T) = \mathrm{Tr}(S) \cup \mathrm{Tr}(T)$,

(iii) $\mathrm{Tr}(\mathrm{Tr}(S)) \subset \mathrm{Tr}(S)$,

(iv) if $S \simeq T \bmod I_{\mathrm{NS}}$ then $\mathrm{Tr}(S) \simeq \mathrm{Tr}(T) \bmod I_{\mathrm{NS}}$.

8.12. Show that $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}\,\alpha \geq \lambda^+\}$.

8.13. If $\lambda < \kappa$ is the αth regular cardinal cardinal, then $o(E^\kappa_\lambda) = \alpha$.

8.14. $o(\kappa) \geq \kappa$ if and only if κ is weakly inaccessible; $o(\kappa) \geq \kappa + 1$ if and only if κ is weakly Mahlo.

8.15. For each $a \in P_\kappa(A)$, the set $\{x \in P_\kappa(A) : x \supset a\}$ is closed unbounded.

A κ-complete filter F on $P_\kappa(A)$ is *normal* if for every $a \in A$, $\{x \in P_\kappa(A) : a \in x\} \in F$, and if F is closed under diagonal intersections. A set $X \subset P_\kappa(A)$ is *F-positive* if its complement is not in F.

8.16. Let F be a normal κ-complete filter on $P_\kappa(A)$. If g is a function on an F-positive set such that $g(x) \in [x]^{<\omega}$ for all x, then g is constant on an F-positive set.

8.17. If F is a normal κ-complete filter on $P_\kappa(A)$ then F contains all closed unbounded sets.

[Use Lemma 8.26 and Exercise 8.16.]

8.18. If $\kappa > \omega_1$ then the set $\{x \in P_\kappa(A) : |x| \geq \aleph_1\}$ is closed unbounded.

Contrast this with the fact that for every $F : [A]^{<\omega} \to A$ there exists a countable x closed under A.

8.19. The set $\{x \in P_\kappa(\lambda) : x \cap \kappa \in \kappa\}$ is closed unbounded.

Historical Notes

The definition of stationary set is due to Bloch [1953], and the fundamental theorem (Theorem 8.7) was proved by Fodor [1956]. (A precursor of Fodor's Theorem appeared in Aleksandrov-Urysohn [1929].) The concept of stationary sets is implicit in Mahlo [1911].

Theorem 8.10 was proved by Solovay [1971] using the technique of saturated ideals.

Mahlo cardinals are named after P. Mahlo, who in 1911–1913 investigated what is now called weakly Mahlo cardinals. Theorems 8.12 and 8.13 are due to Silver [1975]. Silver's proof uses generic ultrapowers; the elementary proof given here is as in Baumgartner-Prikry [1976, 1977]. Lemma 8.16: Erdös, Hajnal, and Milner [1968].

Definition 8.18 and Theorem 8.20 are due to Jech [1984]. The generalization of closed unbounded and stationary sets (Definition 8.21 and Theorems 8.22 and 8.24) was given by Jech [1971b] and [1972/73]; Kueker [1972, 1977] also formulated these concepts for $\kappa = \omega_1$ and proved Theorem 8.28. Theorem 8.27 is due to Menas [1974/75].

Exercise 8.5: Friedman [1974].

Exercise 8.17: Carr [1982].

9. Combinatorial Set Theory

In this chapter we discuss topics in infinitary combinatorics such as trees and partition properties.

Partition Properties

Let us consider the following argument (the *pigeonhole principle*): If seven pigeons occupy three pigeonholes, then at least one pigenhole is occupied by three pigeons. More generally: If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.

Recall that a *partition* of a set S is a pairwise disjoint family $P = \{X_i : i \in I\}$ such that $\bigcup_{i \in I} X_i = S$. With the partition P we can associate a function $F : S \to I$ such that $F(x) = F(y)$ if and only if x and y are in the same $X \in P$. Conversely, any function $F : S \to I$ determines a partition of S. (We shall sometimes say that F is a partition of S.)

For any set A and any natural number $n > 0$,

(9.1) $$[A]^n = \{X \subset A : |X| = n\}$$

is the set of all subsets of A that have exactly n elements. It is sometimes convenient, when A is a set of ordinals, to identify $[A]^n$ with the set of all sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$ in A such that $\alpha_1 < \ldots < \alpha_n$. We shall consider partitions of sets $[A]^n$ for various infinite sets A and natural numbers n. Our starting point is the theorem of Ramsey dealing with finite partitions of $[\omega]^n$.

If $\{X_i : i \in I\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition if for some i, $[H]^n$ is included in X_i; that is, if all the n-element subsets of H are in the same piece of the partition.

Theorem 9.1 (Ramsey). *Let n and k be natural numbers. Every partition $\{X_1, \ldots, X_k\}$ of $[\omega]^n$ into k pieces has an infinite homogeneous set.*

Equivalently, for every $F : [\omega]^n \to \{1, \ldots, k\}$ there exists an infinite $H \subset \omega$ such that F is constant on $[H]^n$.

Proof. By induction on n. If $n = 1$, the theorem is trivial, so we assume that it holds for n and prove for $n+1$. Let F be a function from $[\omega]^{n+1}$ into

$\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined as follows:

$$F_a(X) = F(\{a\} \cup X).$$

By the induction hypothesis, there exists for each $a \in \omega$ and each infinite $S \subset \omega$ an infinite set $H_a^S \subset S - \{a\}$ such that F_a is constant on $[H_a^S]^n$. We construct an infinite sequence $\langle a_i : i = 0, 1, 2, \ldots \rangle$: We let $S_0 = \omega$ and $a_0 = 0$, and

$$S_{i+1} = H_{a_i}^{S_i}, \qquad a_{i+1} = \text{the least element of } S_{i+1} \text{ greater than } a_i.$$

It is clear that for each $i \in \omega$, the function F_{a_i} is constant on $[\{a_m : m > i\}]^n$; let $G(a_i)$ be its value. Now there is an infinite subset $H \subset \{a_i : i \in \omega\}$ such that G is constant on H. It follows that F is constant on $[H]^{n+1}$; this is because for $x_1 < \ldots < x_{n+1}$ in H we have $F(\{x_1, \ldots, x_{n+1}\}) = F_{x_1}(\{x_2, \ldots, x_{n+1}\})$. □

The following lemma explains the terminology introduced in Chapter 7 where Ramsey ultrafilters were defined:

Lemma 9.2. *Let D be a nonprincipal ultrafilter on ω. D is Ramsey if and only if for all natural numbers n and k, every partition $F : [\omega]^n \to \{1, \ldots, k\}$ has a homogeneous set $H \in D$.*

Proof. First assume that D has the partition property stated in the lemma. Let $\mathcal{A}$ be a partition of ω such that $A \notin D$ for all $A \in \mathcal{A}$; we shall find $X \in D$ such that $|X \cap A| \le 1$ for all $A \in \mathcal{A}$. Let $F : [\omega]^2 \to \{0, 1\}$ be as follows: $F(x, y) = 1$ if x and y are in different members of $\mathcal{A}$. If $H \in D$ is homogeneous for F, then clearly H has at most one element common with each $A \in \mathcal{A}$.

Now let us assume that D is a Ramsey ultrafilter. We shall first prove that D has the following property:

(9.2) if $X_0 \supset X_1 \supset X_2 \supset \ldots$ are sets in D, then there is a sequence $a_0 < a_1 < a_2 < \ldots$ such that $\{a_n\}_{n=0}^{\infty} \in D$, $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for all n.

Thus let $X_0 \supset X_1 \supset \ldots$ be sets in D. Since D is a p-point, there exists $Y \in D$ such that each $Y - X_n$ is finite. Let us define a sequence $y_0 < y_1 < \ldots$ in Y as follows:

$$\begin{aligned}
y_0 &= \text{the least } y_0 \in Y \text{ such that } \{y \in Y : y > y_0\} \subset X_0,\\
y_1 &= \text{the least } y_1 \in Y \text{ such that } y_1 > y_0 \text{ and } \{y \in Y : y > y_1\} \subset X_{y_0},\\
&\ldots\\
y_n &= \text{the least } y_n \in Y \text{ such that } y_n > y_{n-1} \text{ and } \{y \in Y : y > y_n\} \subset X_{y_{n-1}}.
\end{aligned}$$

For each n, let $A_n = \{y \in Y : y_n < y \le y_{n+1}\}$. Since D is Ramsey, there exists a set $\{z_n\}_{n=0}^{\infty} \in D$ such that $z_n \in A_n$ for all n.

We observe that for each n, $z_{n+2} \in X_{z_n}$: Since $z_{n+2} > y_{n+2}$, we have $z_{n+2} \in X_{y_{n+1}}$, and since $y_{n+1} \geq z_n$, we have $X_{y_{n+1}} \subset X_{z_n}$ and hence $z_{n+2} \in X_{z_n}$.

Thus if we let $a_n = z_{2n}$ and $b_n = z_{2n+1}$, for all n, then either $\{a_n\}_{n=0}^{\infty} \in D$ or $\{b_n\}_{n=0}^{\infty} \in D$; and in either case we get a sequence that satisfies (9.2).

Now we use the property (9.2) to prove the partition property; we proceed by induction on n and follow closely the proof of Ramsey's Theorem. Let F be a function from $[\omega]^{n+1}$ into $\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined by $F_a(x) = F(x \cup \{a\})$. By the induction hypothesis, there exists for each $a \in \omega$ a set $H_a \in D$ such that F_a is constant on $[H_a]^n$. There exists $X \in D$ such that the constant value of F_a is the same for all $a \in X$; say $F_a(x) = r$ for all $a \in X$ and all $x \in [H_a]^n$.

For each n, let $X_n = X \cap H_0 \cap H_1 \cap \ldots \cap H_n$. By (9.2) there exists a sequence $a_0 < a_1 < a_2 < \ldots$ such that $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for each n, and that $\{a_n\}_{n=0}^{\infty} \in D$. Let $H = \{a_n\}_{n=0}^{\infty}$. It is clear that for each $i \in \omega$, $a_i \in X$ and $\{a_m : m > i\} \subset H_{a_i}$. Hence $F_{a_i}(x) = r$ for all $x \in [\{a_m : m > i\}]^n$, and it follows that F is constant on $[H]^{n+1}$. □

To facilitate our investigation of generalizations of Ramsey's Theorem, we shall now introduce the *arrow notation*. Let κ and λ be infinite cardinal numbers, let n be a natural number and let m be a (finite or infinite) cardinal. The symbol

$$\kappa \to (\lambda)^n_m \tag{9.3}$$

(read: κ *arrows* λ) denotes the following *partition property*: Every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ. In other words, every $F : [\kappa]^n \to m$ is constant on $[H]^n$ for some $H \subset \kappa$ such that $|H| = \lambda$. Using the arrow notation, Ramsey's Theorem is expressed as follows:

$$\aleph_0 \to (\aleph_0)^n_k \qquad (n, k \in \omega). \tag{9.4}$$

The subscript m in (9.3) is usually deleted when $m = 2$, and so

$$\kappa \to (\lambda)^n$$

is the same as $\kappa \to (\lambda)^n_2$.

The relation $\kappa \to (\lambda)^n_m$ remains true if κ is made larger or if λ or m are made smaller. A moment's reflection is sufficient to see that the relation also remains true when n is made smaller.

Obviously, the relation (9.3) makes sense only if $\kappa \geq \lambda$ and $\kappa > m$; if $m = \kappa$, then it is clearly false. Thus we always assume $2 \leq m < \kappa$ and $\lambda \leq \kappa$. If $n = 1$, then (9.3) holds just in case either $\kappa > \lambda$, or $\kappa = \lambda$ and $\operatorname{cf} \kappa > m$. We shall concentrate on the nontrivial case: $n \geq 2$.

We start with two negative partition relations.

Lemma 9.3. *For all* κ,

$$2^\kappa \not\to (\omega)^2_\kappa.$$

In other words, there is a partition of 2^κ *into* κ *pieces that does not have an infinite homogeneous set.*

Proof. In fact, we find a partition that has no homogeneous set of size 3. Let $S = \{0,1\}^\kappa$ and let $F : [S]^2 \to \kappa$ be defined by $F(\{f, g\}) =$ the least $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. If f, g, h are distinct elements of S, it is impossible to have $F(\{f, g\}) = F(\{f, h\}) = F(\{g, h\})$. □

Lemma 9.4. *For every* κ,

$$2^\kappa \not\to (\kappa^+)^2.$$

(Thus the obvious generalization of Ramsey's Theorem, namely $\aleph_1 \to (\aleph_1)^2_2$, is false.)

To construct a partition of $[2^\kappa]^2$ that violates the partition property, let us consider the linearly ordered set $(P, <)$ where $P = \{0,1\}^\kappa$, and $f < g$ if and only if $f(\alpha) < g(\alpha)$ where α is the least α such that $f(\alpha) \neq g(\alpha)$ (the lexicographic ordering of P).

Lemma 9.5. *The lexicographically ordered set* $\{0,1\}^\kappa$ *has no increasing or decreasing* κ^+*-sequence.*

Proof. Assume that $W = \{f_\alpha : \alpha < \kappa^+\} \subset \{0,1\}^\kappa$ is such that $f_\alpha < f_\beta$ whenever $\alpha < \beta$ (the decreasing case is similar). Let $\gamma \leq \kappa$ be the least γ such that the set $\{f_\alpha \restriction \gamma : \alpha < \kappa^+\}$ has size κ^+, and let $Z \subset W$ be such that $|Z| = \kappa^+$ and $f \restriction \gamma \neq g \restriction \gamma$ for $f, g \in Z$. We may as well assume that $Z = W$, so let us do so.

For each $\alpha < \kappa^+$, let ξ_α be such that $f_\alpha \restriction \xi_\alpha = f_{\alpha+1} \restriction \xi_\alpha$ and $f_\alpha(\xi_\alpha) = 0$, $f_{\alpha+1}(\xi_\alpha) = 1$; clearly $\xi_\alpha < \gamma$. Hence there exists $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+ elements f_α of W. However, if $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha \restriction \xi = f_\beta \restriction \xi$, then $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$; hence $f_\alpha = f_\beta$. Thus the set $\{f_\alpha \restriction \xi : \alpha < \kappa^+\}$ has size κ^+, contrary to the minimality assumption on γ. □

Proof of Lemma 9.4. Let $2^\kappa = \lambda$ and let $\{f_\alpha : \alpha < \lambda\}$ be an enumeration of the set $P = \{0,1\}^\kappa$. Let $\prec$ be a linear ordering of λ induced by the lexicographic ordering of P: $\alpha \prec \beta$ if $f_\alpha < f_\beta$.

Now we define a partition $F : [\lambda]^2 \to \{0,1\}$ by letting $F(\{\alpha, \beta\}) = 1$ when the ordering $\prec$ of $\{\alpha, \beta\}$ agrees with the natural ordering; and letting $F(\{\alpha, \beta\}) = 0$ otherwise. If $H \subset \lambda$ is a homogeneous set of order-type κ^+, then $\{f_\alpha : \alpha \in H\}$ constitutes an increasing or decreasing κ^+-sequence in $(P, <)$; a contradiction. □

By Lemma 9.4, the relation $\kappa \to (\aleph_1)^2$ is false if $\kappa \leq 2^{\aleph_0}$. On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2$ is true, as follows from this more general theorem:

Theorem 9.6 (Erdős-Rado).

$$\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}.$$

In particular, $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$.

Proof. We shall first prove the case $n = 1$ since the induction step parallels closely this case. Thus let $\kappa = (2^{\aleph_0})^+$ and let $F : [\kappa]^2 \to \omega$ be a partition of $[\kappa]^2$ into $\aleph_0$ pieces. We want to find a homogeneous $H \subset \kappa$ of size $\aleph_1$.

For each $a \in \kappa$, let F_a be a function on $\kappa - \{a\}$ defined by $F_a(x) = F(\{a, x\})$. We shall first prove the following claim: There exists a set $A \subset \kappa$ such that $|A| = 2^{\aleph_0}$ and such that for every countable $C \subset A$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on C.

To prove the claim, we construct an ω_1-sequence $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \omega_1$, of subsets of κ, each of size $2^{\aleph_0}$, as follows: Let A_0 be arbitrary, and for each limit α, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α, there exists a set $A_{\alpha+1} \supset A_\alpha$ of size $2^{\aleph_0}$ such that for each countable $C \subset A_\alpha$ and every $u \in \kappa - C$ there exists $v \in A_{\alpha+1} - C$ such that F_v agrees with F_u on C (because the number of such functions is $\le 2^{\aleph_0}$). Then we let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, and clearly A has the required property.

Next we choose some $a \in \kappa - A$, and construct a sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ in A as follows: Let x_0 be arbitrary, and given $\{x_\beta : \beta < \alpha\} = C$, let x_α be some $v \in A - C$ such that F_v agrees with F_a on C. Let $X = \{x_\alpha : \alpha < \omega_1\}$.

Now we consider the function $G : X \to \omega$ defined by $G(x) = F_a(x)$. It is clear that if $\alpha < \beta$, then $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$. Since the range of G is countable, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on H. It follows that F is constant on $[H]^2$.

Thus we have proved the theorem for $n = 1$. The general case is proved by induction. Let us assume that $\beth_{n-1}^+ \to (\aleph_1)^n_{\aleph_0}$ and let $F : [\kappa]^{n+1} \to \omega$, where $\kappa = \beth_n^+$. For each $a \in \kappa$, let $F_a : [\kappa - \{a\}]^n \to \omega$ be defined by $F_a(x) = F(x \cup \{a\})$. As in the case $n = 1$, there exists a set $A \subset \kappa$ of size $\beth_n$ such that for every $C \subset A$ of size $|C| \le \beth_{n-1}$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on $[C]^n$.

Next we choose $a \in \kappa - A$ and construct a set $X = \{x_\alpha : \alpha < \beth_{n-1}^+\} \subset A$ such that for each α, F_{x_α} agrees with F_a on $[\{x_\beta : \beta < \alpha\}]^n$.

Then we consider $G : [X]^n \to \omega$ where $G(x) = F_a(x)$. As before, if $\alpha_1 < \ldots < \alpha_{n+1}$, then $F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n+1}}\}) = G(\{x_{\alpha_1}, \ldots, x_{\alpha_n}\})$. By the induction hypothesis, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on $[H]^n$. It follows that F is constant on $[H]^{n+1}$. □

Erdős and Rado proved that for each n, the partition property $\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}$ is best possible. The property also generalizes easily to larger cardinals.

A natural generalization of the partition property (9.3) is when we allow λ to be a limit ordinal, not just a cardinal. Let κ, n and m be as in (9.3) and

let $\alpha > 0$ be a limit ordinal. The symbol

$$\kappa \to (\alpha)^n_m \tag{9.5}$$

stands for: For every $F : [\kappa]^n \to m$ there exists an $H \subset \kappa$ of order-type α such that F is constant on $[H]^n$.

There are various results about the partition relation (9.5). For instance, Baumgartner and Hajnal proved in [1973] that $\aleph_1 \to (\alpha)^2$ for all $\alpha < \omega_1$. The analogous case for $\aleph_2$ is different: If $2^{\aleph_0} = \aleph_1$, then $\aleph_2 \to (\omega_1)^2$ (by Erdős-Rado), but it is consistent (with $2^{\aleph_0} = \aleph_1$) that $\aleph_2 \not\to (\omega_1 + \omega)^2$.

Among other generalizations of (9.3), we mention the following:

$$\kappa \to (\alpha, \beta)^n \tag{9.6}$$

means that for every $F : [\kappa]^n \to \{0, 1\}$, either there exists an $H_1 \subset \kappa$ of order-type α such that $F = 0$ on $[H_1]^n$ or there exists and $H_2 \subset \kappa$ of order-type β such that $F = 1$ on $[H_2]^n$.

Theorem 9.7 (Dushnik-Miller). *For every infinite cardinal* κ,

$$\kappa \to (\kappa, \omega)^2.$$

Proof. Let $\{A, B\}$ be a partition of $[\kappa]^2$. For every $x \in \kappa$, let $B_x = \{y \in \kappa : x < y \text{ and } \{x, y\} \in B\}$. First let us assume that in every set $X \subset \kappa$ of cardinality κ there exists an $x \in X$ such that $|B_x \cap X| = \kappa$. In this case, we construct an infinite H with $[H]^2 \subset B$ as follows:

Let $X_0 = \kappa$ and $x_0 \in X_0$ such that $|B_{x_0} \cap X_0| = \kappa$. For each n, let $X_{n+1} = B_{x_n} \cap X_n$ and let $x_{n+1} \in X_{n+1}$ be such that $x_{n+1} > x_n$ and $|B_{x_{n+1}} \cap X_{n+1}| = \kappa$. Then let $H = \{x_n\}_{n=0}^\infty$; it is clear that $[H]^2 \subset B$.

Thus let us assume, for the rest of the proof, that there exists a set $S \subset \kappa$ of cardinality κ such that

$$\text{for every } x \in S,\ |B_x \cap S| < \kappa. \tag{9.7}$$

If κ is regular, then we construct (by induction) an increasing κ-sequence $\langle x_\alpha : \alpha < \kappa\rangle$ in S such that $\{x_\alpha, x_\beta\} \in A$ for all $\alpha < \beta$; this is possible by (9.7).

Thus let us assume that κ is singular, let $\lambda = \operatorname{cf}\kappa$ and let $\langle \kappa_\xi : \xi < \lambda\rangle$ be an increasing sequence of regular cardinals $> \lambda$ with limit κ. Furthermore, we assume that there is no infinite H with $[H]^2 \subset B$, and that $\kappa_\xi \to (\kappa_\xi, \omega)^2$ holds for every $\xi < \lambda$. We shall find a set $H \subset \kappa$ of cardinality κ such that $[H]^2 \subset A$.

Let $\{S_\xi : \xi < \lambda\}$ be a partition of S into disjoint sets such that $S_\xi = \kappa_\xi$. It follows from our assumptions that there exist sets $K_\xi \subset S_\xi$, $|K_\xi| = \kappa_\xi$, such that $[K_\xi]^2 \subset A$.

For every $x \in K_\xi$ there exists, by (9.7), some $\alpha < \lambda$ such that $|B_x \cap S| < \kappa_\alpha$; since $\lambda < \kappa_\xi$, there exists an $\alpha(\xi)$ such that the set $Z_\xi = \{x \in K_\xi : |B_x \cap S| < \kappa_{\alpha(\xi)}\}$ has cardinality κ_ξ.

Let $\langle \xi_\nu : \nu < \lambda \rangle$ be an increasing sequence of ordinals $< \lambda$ such that if $\nu_1 < \nu_2$ then $\alpha(\xi_{\nu_1}) < \xi_{\nu_2}$. We define, by induction on ν,

$$H_\nu = Z_{\xi(\nu)} - \bigcup\{B_x : x \in \bigcup_{\eta<\nu} Z_{\xi(\eta)}\}.$$

Clearly, $|H_\nu| = \kappa_{\xi(\nu)}$, and $[H_\nu]^2 \subset A$.

Finally, we let $H = \bigcup_{\nu<\lambda} H_\nu$. It follows from the construction of H that $[H]^2 \subset A$. □

Weakly Compact Cardinals

In the positive results given by the Erdős-Rado Theorem, the size of the homogeneous set is smaller than the size of the set being partitioned. A natural question arises, whether the relation $\kappa \to (\kappa)^2$ can hold for cardinals other than $\kappa = \omega$.

Definition 9.8. A cardinal κ is *weakly compact* if it is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$.

The reason for the name "weakly compact" is that these cardinals satisfy a certain compactness theorem for infinitary languages; we shall investigate weakly compact cardinals further in Part II.

Lemma 9.9. *Every weakly compact cardinal is inaccessible.*

Proof. Let κ be a weakly compact cardinal. To show that κ is regular, let us assume that κ is the disjoint union $\bigcup\{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F : [\kappa]^2 \to \{0,1\}$ as follows: $F(\{\alpha,\beta\}) = 0$ just in case α and β are in the same A_γ. Obviously, this partition does not have a homogeneous set $H \subset \kappa$ of size κ.

That κ is a strong limit cardinal follows from Lemma 9.4: If $\kappa \le 2^\lambda$ for some $\lambda < \kappa$, then because $2^\lambda \not\to (\lambda^+)^2$, we have $\kappa \not\to (\lambda^+)^2$ and hence $\kappa \not\to (\kappa)^2$. □

We shall prove in Chapter 17 that every weakly compact cardinal κ is the κth inaccessible cardinal.

Trees

Many problems in combinatorial set theory can be formulated as problems about trees.

In this chapter we discuss Suslin's Problem as well as the use of trees in partition calculus and large cardinals.

Definition 9.10. A *tree* is a partially ordered set $(T, <)$ with the property that for each $x \in T$, the set $\{y : y < x\}$ of all predecessors of x is well-ordered by $<$.

The αth *level* of T consists of all $x \in T$ such that $\{y : y < x\}$ has order-type α. The height of T is the least α such that the αth level of T is empty; in other words, it is the height of the well-founded relation $<$:

$$\begin{aligned} o(x) &= \text{the order-type of } \{y : y < x\}, \\ \alpha\text{th level} &= \{x : o(x) = \alpha\}, \\ \text{height}(T) &= \sup\{o(x) + 1 : x \in T\}. \end{aligned} \tag{9.8}$$

A *branch* in T is a maximal linearly ordered subset of T. The *length* of a branch b is the order-type of b. An α-*branch* is a branch of length α.

We shall now turn our attention to Suslin's Problem introduced in Chapter 4. In Lemma 9.14 below we show that the problem can be restated as a question about the existence of certain trees of height ω_1.

Suslin's Problem asks whether the real line is the only complete dense unbounded linearly ordered set that satisfies the countable chain condition. An equivalent question is whether every dense linear ordering that satisfies the countable chain condition is *separable*, i.e., has a countable dense subset.

Definition 9.11. A *Suslin line* is a dense linearly ordered set that satisfies the countable chain condition and is not separable.

Thus Suslin's Problem asks whether a Suslin line exists. We shall show that the existence of a Suslin line is equivalent to the existence of a Suslin tree.

Let T be a tree. An *antichain* in T is a set $A \subset T$ such that any two distinct elements x, y of A are incomparable, i.e., neither $x < y$ nor $y < x$.

Definition 9.12. A tree T is a *Suslin tree* if

(i) the height of T is ω_1;
(ii) every branch in T is at most countable;
(iii) every antichain in T is at most countable.

For the formulation of Suslin's Problem in terms of trees it is useful to consider Suslin trees that are called *normal*.

Let α be an ordinal number, $\alpha \le \omega_1$. A *normal α-tree* is a tree T with the following properties:

(9.9)
(i) height$(T) = \alpha$;
(ii) T has a unique least point (the root);
(iii) each level of T is at most countable;
(iv) if x is not maximal in T, then there are infinitely many $y > x$ at the next level (immediate successors of x);

(v) for each $x \in T$ there is some $y > x$ at each higher level less than α;
(vi) if $\beta < \alpha$ is a limit ordinal and x, y are both at level β and if $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

See Exercise 9.6 for a representation of normal trees.

Lemma 9.13. *If there exists a Suslin tree then there exists a normal Suslin tree.*

Proof. Let T be a Suslin tree. T has height ω_1, and each level of T is countable. We first discard all points $x \in T$ such that $T_x = \{y \in T : y \geq x\}$ is at most countable, and let $T_1 = \{x \in T : T_x \text{ is uncountable}\}$. Note that if $x \in T_1$ and $\alpha > o(x)$, then $|T_y| = \aleph_1$ for some $y > x$ at level α. Hence T_1 satisfies condition (v). Next, we satisfy property (vi): For every chain $C = \{z : z < y\}$ in T_1 of limit length we add an extra node a_C and stipulate that $z < a_C$ for all $z \in C$, and $a_C < x$ for every x such that $x > z$ for all $z \in C$. The resulting tree T_2 satisfies (iii), (v) and (vi). For each $x \in T_2$ there are uncountably many branching points $z > x$, i.e., points that have at least two immediate successors (because there is no uncountable chain and T_2 satisfies (v)). The tree $T_3 = \{\text{the branching points of } T_2\}$ satisfies (iii), (v) and (vi) and each $x \in T_3$ is a branching point. To get property (iv), let T_4 consists of all $z \in T_3$ at limit levels of T_3. The tree T_4 satisfies (i), (iii), (iv), and (v); and then $T_5 \subset T_4$ satisfying (ii) as well is easily obtained. □

Lemma 9.14. *There exists a Suslin line if and only if there exists a Suslin tree.*

Proof. (a) Let S be a Suslin line. We shall construct a Suslin tree. The tree will consist of closed (nondegenerate) intervals on the Suslin line S. The partial ordering of T is by inverse inclusion: If $I, J \in T$, then $I \leq J$ if and only if $I \supset J$.

The collection T of intervals is constructed by induction on $\alpha < \omega_1$. We let $I_0 = [a_0, b_0]$ be arbitrary (such that $a_0 < b_0$). Having constructed I_β, $\beta < \alpha$, we consider the countable set $C = \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ of endpoints of the intervals I_β, $\beta < \alpha$. Since S is a Suslin line, C is not dense in S and so there exists an interval $[a, b]$ disjoint from C; we pick some such $[a_\alpha, b_\alpha] = I_\alpha$. The set $T = \{I_\alpha : \alpha < \omega_1\}$ is uncountable and partially ordered by $\supset$. If $\alpha < \beta$, then either $I_\alpha \supset I_\beta$ or I_α is disjoint from I_β. It follows that for each α, $\{I \in T : I \supset I_\alpha\}$ is well-ordered by $\supset$ and thus T is a tree.

We shall show that T has no uncountable branches and no uncountable antichains. Then it is immediate that the height of T is at most ω_1; and since every level is an antichain and T is uncountable, we have height$(T) = \omega_1$.

If $I, J \in T$ are incomparable, then they are disjoint intervals of S; and since S satisfies the countable chain condition, every antichain in T is at most countable. To show that T has no uncountable branch, we note first that if b is a branch of length ω_1, then the left endpoints of the intervals $I \in B$ form an increasing sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of S. It is clear that the intervals $(x_\alpha, x_{\alpha+1})$, $\alpha < \omega_1$, form a disjoint uncountable collection of open intervals in S, contrary to the assumption that S satisfies the countable chain condition.

(b) Let T be a normal Suslin tree. The line S will consist of all branches of T (which are all countable). Each $x \in T$ has countably many immediate successors, and we order these successors as rational numbers. Then we order the elements of S lexicographically: If α is the least level where two branches $a, b \in S$ differ, then α is a successor ordinal and the points $a_\alpha \in a$ and $b_\alpha \in b$ are both successors of the same point at level $\alpha - 1$; we let $a < b$ or $b < a$ according to whether $a_\alpha < b_\alpha$ or $b_\alpha < a_\alpha$.

It is easy to see that S is linearly ordered and dense. If (a, b) is an open interval in S, then one can find $x \in T$ such that $I_x \subset (a, b)$, where I_x is the interval $I_x = \{c \in S : x \in c\}$. And if I_x and I_y are disjoint, then x and y are incomparable points of T. Thus every disjoint collection of open intervals of S must be at most countable, and so S satisfies the countable chain condition.

The line S is not separable: If C is a countable set of branches of T, let α be a countable ordinal bigger than the length of any branches $b \in C$. Then if x is any point at level α, the interval I_x does not contain any $b \in C$, and so C is not dense in S. □

Lemma 9.14 reduces Suslin's Problem to a purely combinatorial problem. In Part II we shall return to it and show that the problem is independent of the axioms of set theory.

We now turn our attention to the following problem, related to Suslin trees.

Definition 9.15. An *Aronszajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches.

Theorem 9.16 (Aronszajn). *There exists an Aronszajn tree.*

Proof. We construct a tree T whose elements are some bounded increasing transfinite sequences of rational numbers. If $x, y \in T$ are two such sequences, then we let $x \leq y$ just in case y extends x, i.e., $x \subset y$. Also, if $y \in T$ and x is an initial segment of y, then we let $x \in T$; thus the αth level of T will consist of all those $x \in T$ whose length is α.

It is clear that an uncountable branch in T would yield an increasing ω_1-sequence of rational numbers, which is impossible. Thus T will be an Aronszajn tree, provided we arrange that T has $\aleph_1$ levels, all of them at most countable. We construct T by induction on levels. For each $\alpha < \omega_1$ we construct a set U_α of increasing α-sequences of rationals; U_α will be the αth level of T. We construct the U_α so that for each α, $|U_\alpha| \leq \aleph_0$, and that:

(9.10) For each $\beta < \alpha$, each $x \in U_\beta$ and each $q > \sup x$ there is $y \in U_\alpha$ such that $x \subset y$ and $q \geq \sup y$.

Condition (9.10) enables us to continue the construction at limit steps.

To start, we let $U_0 = \{\emptyset\}$. The successor steps of the construction are also fairly easy. Given U_α, we let $U_{\alpha+1}$ be the set of all extensions $x^\frown r$ of sequences in U_α such that $r > \sup x$. It is clear that since U_α satisfies condition (9.10), $U_{\alpha+1}$ satisfies it also (for $\alpha + 1$), and it is equally clear that $U_{\alpha+1}$ is at most countable.

Thus let α be a limit ordinal ($\alpha < \omega_1$) and assume that we have constructed all levels U_γ, $\gamma < \alpha$, of T below α, and that all the U_γ satisfy (9.10); we shall construct U_α. The points $x \in T$ below level α form a (normal) tree T_α of length α. We claim that T_α has the following property:

(9.11) For each $x \in T_\alpha$ and each $q > \sup x$ there is an increasing α-sequence of rationals y such that $x \subset y$ and $q \geq \sup y$ and that $y{\restriction}\beta \in T_\alpha$ for all $\beta < \alpha$.

The last condition means that $\{y{\restriction}\beta : \beta < \alpha\}$ is a branch in T_α. To prove the claim, we let α_n, $n = 0, 1, \ldots$, be an increasing sequence of ordinals such that $x \in U_{\alpha_0}$ and $\lim_n \alpha_n = \alpha$, and let $\{q_n\}_{n=0}^\infty$ be an increasing sequence of rational numbers such that $q_0 > \sup x$ and $\lim_n q_n \leq q$. Using repeatedly condition (9.10), for all α_n ($n = 0, 1, \ldots$), we can construct a sequence $y_0 \subset y_1 \subset \ldots \subset y_n \ldots$ such that $y_0 = x$, $y_n \in U_{\alpha_n}$, and $\sup y_n \leq q_n$ for each n. Then we let $y = \bigcup_{n=0}^\infty y_n$; clearly, y satisfies (9.11).

Now we construct U_α as follows: For each $x \in T_\alpha$ and each rational number q such that $q > \sup x$, we choose a branch y in T_α that satisfies (9.11), and let U_α consists of all these $y : \alpha \to \boldsymbol{Q}$. The set U_α so constructed is countable and satisfies condition (9.10).

Then $T = \bigcup_{\alpha<\omega_1} U_\alpha$ is an Aronszajn tree. □

The Aronszajn tree constructed in Theorem 9.16 has the property that there exists a function $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$ (Exercise 9.8). With a little more care, one can construct T so that there is a function $f : T \to \boldsymbol{Q}$ such that $f(x) < f(y)$ if $x < y$. Such trees are called *special Aronszajn trees*. In Part II we'll show that it is consistent that all Aronszajn trees are special.

Almost Disjoint Sets and Functions

In combinatorial set theory one often consider families of sets that are as much different as possible; a typical example is an almost disjoint family of infinite sets—any two intersect in a finite set. Here we present a sample of results and problems of this kind.

Definition 9.17. A collection of finite sets Z is called a Δ-*system* if there exists a finite set S such that $X \cap Y = S$ for any two distinct sets $X, Y \in Z$.

The following theorem is often referred to as the Δ-*Lemma*:

Theorem 9.18 (Shanin). *Let W be an uncountable collection of finite sets. Then there exists an uncountable $Z \subset W$ that is a Δ-system.*

Proof. Since W is uncountable, it is clear that uncountably many $X \in W$ have the same size; thus we may assume that for some n, $|X| = n$ for all $X \in W$. We prove the theorem by induction on n. If $n = 1$, the theorem is trivial. Thus assume that the theorem holds for n, and let W be such that $|X| = n + 1$ for all $X \in W$.

If there is some element a that belongs to uncountably many $X \in W$, we apply the induction hypothesis to the collection $\{X - \{a\} : X \in W$ and $a \in X\}$, and obtain $Z \subset W$ with the required properties.

Otherwise, each a belongs to at most countably many $X \in W$, and we construct a disjoint collection $Z = \{X_\alpha : \alpha < \omega_1\}$ as follows, by induction on α. Given X_ξ, $\xi < \alpha$, we find $X = X_\alpha \in W$ that is disjoint from all X_ξ, $\xi < \alpha$. □

For an alternative proof, using Fodor's Theorem, see Exercise 9.10. Theorem 9.18 generalizes to greater cardinals, under the assumption of GCH:

Theorem 9.19. *Assume $\kappa^{<\kappa} = \kappa$. Let W be a collection of sets of cardinality less than κ such that $|W| = \kappa^+$. Then there exist a collection $Z \subset W$ such that $|Z| = \kappa^+$ and a set A such that $X \cap Y = A$ for any two distinct elements $X, Y \in Z$.* □

Definition 9.20. If X and Y are infinite subsets of ω then X and Y are *almost disjoint* if $X \cap Y$ is finite.

Let κ be a regular cardinal. If $X \cap Y$ are subsets of κ of cardinality κ then X and Y are *almost disjoint* if $|X \cap Y| < \kappa$.

An *almost disjoint family* of sets is a family of pairwise almost disjoint sets.

Lemma 9.21. *There exists an almost disjoint family of $2^{\aleph_0}$ subsets of ω.*

Proof. Let S be the set of all finite 0–1 sequences: $S = \bigcup_{n=0}^\infty \{0,1\}^n$. For every $f : \omega \to \{0,1\}$, let $A_f \subset S$ be the set $A_f = \{s \in S : s \subset f\} = \{f{\restriction}n : n \in \omega\}$. Clearly, $A_f \cap A_g$ is finite if $f \neq g$; thus $\{A_f : f \in \{0,1\}^\omega\}$ is a family of $2^{\aleph_0}$ almost disjoint subsets of the countable set S, and the lemma follows. □

A generalization from ω to arbitrary regular κ is not provable in ZFC (although under GCH the generalization is straightforward; see Exercise 9.11). Without assuming the GCH, the best one can do is to find an almost disjoint family of κ^+ subsets of κ; this follows from Lemma 9.23 below.

Definition 9.22. Let κ be a regular cardinal. Two functions f and g on κ are *almost disjoint* if $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$.

Lemma 9.23. *For every regular cardinal κ, there exists an almost disjoint family of κ^+ functions from κ to κ.*

Proof. It suffices to show that given κ almost disjoint functions $\{f_\nu : \nu < \kappa\}$, then there exists $f : \kappa \to \kappa$ almost disjoint from all f_ν, $\nu < \kappa$; this we do by diagonalization: Let $f(\alpha) \neq f_\nu(\alpha)$ for all $\nu < \alpha$. □

Let us consider the special case when $\kappa = \omega_1$.

Definition 9.24. A tree $(T, <)$ is a *Kurepa tree* if:

(i) height$(T) = \omega_1$;
(ii) each level of T is at most countable;
(iii) T has at least $\aleph_2$ uncountable branches.

If T is a Kurepa tree, then the family of all ω_1-branches is an almost disjoint family of uncountable subsets of T. In fact, since the levels of T are countable, we can identify the ω_1-branches with the functions from ω_1 into ω and get the following result: *There exists an almost disjoint family of $\aleph_2$ functions $f : \omega_1 \to \omega$.*

Lemma 9.25. *A Kurepa tree exists if and only if there exists a family $\mathcal{F}$ of subsets of ω_1 such that:*

(9.12) (i) $|\mathcal{F}| \geq \aleph_2$;
(ii) *for each* $\alpha < \omega_1$, $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq \aleph_0$.

Proof. (a) Let $(T, <_T)$ be a Kurepa tree. Since T has size $\aleph_1$, we may assume that $T = \omega_1$, and moreover that $\alpha < \beta$ whenever $\alpha <_T \beta$. If we let $\mathcal{F}$ be the family of all ω_1-branches of T, then $\mathcal{F}$ satisfies (9.12).

(b) Let $\mathcal{F}$ be a family of subsets of ω_1 such that (9.12) holds. For each $X \in \mathcal{F}$, let f_X be the functions on ω_1 defined by

$$f_X(\alpha) = X \cap \alpha \qquad (\alpha < \omega_1).$$

For each $\alpha < \omega_1$, let $U_\alpha = \{f_X \restriction \alpha : X \in \mathcal{F}\}$ and let $T = \bigcup_{\alpha<\omega_1} U_\alpha$. Then $(T, \subset)$ is a tree, the U_α are the levels of T and the functions f_X correspond to branches of T. By (9.12)(ii), every U_α is countable, and it follows that T is a Kurepa tree. □

The existence of a Kurepa tree is independent of the axioms of set theory. In fact, the nonexistence of Kurepa trees is equiconsistent with an inaccessible cardinal.

The Tree Property and Weakly Compact Cardinals

Generalizing the concept of Aronszajn tree to cardinals $> \omega_1$ we say that a regular uncountable cardinal κ has the *tree property* if every tree of height κ whose levels have cardinality $< \kappa$ has a branch of cardinality κ.

Lemma 9.26.

(i) *If κ is weakly compact, then κ has the tree property.*
(ii) *If κ is inaccessible and has the tree property, then κ is weakly compact, and in fact $\kappa \to (\kappa)^2_m$ for every $m < \kappa$.*

Proof. (i) Let κ be weakly compact and let $(T, <_T)$ be a tree of height κ such that each level of T has size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$ and we may assume that $T = \kappa$. We extend the partial ordering $<_T$ of κ to a linear ordering $\prec$: If $\alpha <_T \beta$, then we let $\alpha \prec \beta$; if α and β are incomparable and if ξ is the first level where the predecessors α_ξ, β_ξ of α and β are distinct, we let $\alpha \prec \beta$ if and only if $\alpha_\xi < \beta_\xi$.

Let $F : [\kappa]^2 \to \{0, 1\}$ be the partition defined by $F(\{\alpha, \beta\}) = 1$ if and only if $\prec$ agrees with $<$ on $\{\alpha, \beta\}$. By weak compactness, let $H \subset \kappa$ be homogeneous for F, $|H| = \kappa$.

We now consider the set $B \subset \kappa$ of all $x \in \kappa$ such that $\{\alpha \in H : x <_T \alpha\}$ has size κ. Since every level has size $< \kappa$, it is clear that at each level there is at least one x in B. Thus if we show that any two elements of B are $<_T$-comparable, we shall have proved that B is a branch in T of size κ.

Thus assume that x, y are incomparable elements of B; let $x \prec y$. Since both x and y have κ successors in H, there exist $\alpha < \beta < \gamma$ in H such that $x <_T \alpha$, $y <_T \beta$, and $x <_T \gamma$. By the definition of $\prec$, we have $\alpha \prec \beta$ and $\gamma \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\gamma, \beta\}) = 0$, contrary to the homogeneity of H.

(ii) Let κ be an inaccessible cardinal with the tree property, and let $F : [\kappa]^2 \to I$ be a partition such that $|I| < \kappa$. We shall find a homogeneous $H \subset \kappa$ of size κ.

We construct a tree $(T, \subset)$ whose elements are some functions $t : \gamma \to I$, $\gamma < \kappa$. We construct T by induction: At step $\alpha < \kappa$, we put into T one more element t, calling it t_α. Let $t_0 = \emptyset$. Having constructed $t_0, \ldots, t_\beta, \ldots, \beta < \alpha$, let us construct t_α as follows, by induction on ξ. Having constructed $t_\alpha \restriction \xi$, we look first whether $t_\alpha \restriction \xi$ is among the t_β, $\beta < \alpha$ (note that for $\xi = 0$ we have $t_\alpha \restriction 0 = t_0$). If not, then we consider t_α constructed: $t_\alpha = t_\alpha \restriction \xi$. If $t_\alpha \restriction \xi = t_\beta$ for some $\beta < \alpha$, then we let $t_\alpha(\xi) = i$ where $i = F(\{\beta, \alpha\})$.

$(T, \subset)$ is a tree of size κ; and since κ is inaccessible, each level of T has size $< \kappa$ and the height of T is κ. It follows from the construction that if $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\text{length}(t_\beta))$. By the assumption, T has a branch B of size κ. If we now let, for each $i \in I$,

$$H_i = \{\alpha : t_\alpha \in B \text{ and } t_\alpha{}^\frown i \in B\}, \tag{9.13}$$

then each H_i is homogeneous for the partition F, and at least one H_i has size κ. □

It should be mentioned that an argument similar to the one above, only more complicated, shows that if κ is inaccessible and has the tree property, then $\kappa \to (\kappa)^n_m$ for all $n \in \omega$, $m < \kappa$.

Ramsey Cardinals

Let us consider one more generalization of Ramsey's Theorem. Let κ be an infinite cardinal, let α be an infinite limit ordinal, $\alpha \le \kappa$, and let m be a cardinal, $2 \le m < \kappa$. The symbol

$$\kappa \to (\alpha)^{<\omega}_m \tag{9.14}$$

denotes the property that for every partition F of the set $[\kappa]^{<\omega} = \bigcup_{n=0}^{\infty} [\kappa]^n$ into m pieces, there exists a set $H \subset \kappa$ of order-type α such that for each $n \in \omega$, F is constant on $[H]^n$. (Again, the subscript m is deleted when $m = 2$.)

It is not difficult to see that the partition property $\omega \to (\omega)^{<\omega}$ is false (see Exercise 9.13).

A cardinal κ is a *Ramsey cardinal* if $\kappa \to (\kappa)^{<\omega}$. Clearly, every Ramsey cardinal is weakly compact. We shall investigate Ramsey cardinals and property (9.14) in general in Part II.

Exercises

9.1. (i) Every infinite partially ordered set either has an infinite chain or has an infinite set of mutually incomparable elements.
(ii) Every infinite linearly ordered set either has an infinite increasing sequence of elements or has an infinite decreasing sequence of elements.
[Use Ramsey's Theorem.]

For each κ, let $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

9.2. For every κ, $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$. In particular, we have $(2^\kappa)^+ \to (\kappa^+)^2$.

9.3. $\omega_1 \to (\omega_1, \omega+1)^2$.
[Let $\{A, B\}$ be a partition of $[\omega_1]^2$. For every limit ordinal α let K_α be a maximal subset of α such that $[K_\alpha \cup \{\alpha\}]^2 \subset B$. If K_α is finite for each α, use Fodor's Theorem to find a stationary set S such that all K_α, $\alpha \in S$, are the same. Then $[S]^2 \subset A$.]

If A is an infinite set of ordinals and α an ordinal, let $[A]^\alpha$ denote the set of all increasing α-sequences in A. The symbol

$$\kappa \to (\lambda)^\alpha$$

stands for: For every partition $F : [\kappa]^\alpha \to \{0, 1\}$ of $[\kappa]^\alpha$ into two pieces, there exists a set H of order-type λ such that F is constant on $[H]^\alpha$.

9.4. For all infinite cardinals κ, $\kappa \not\to (\omega)^\omega$.
[For $s, t \in [\kappa]^\omega$ let $s \equiv t$ if and only if $\{n : s(n) \ne t(n)\}$ is finite. Pick a representative in each equivalence class. Let $F(s) = 0$ if s differs from the representative of its class at an even number of places; let $F(s) = 1$ otherwise. F has no infinite homogeneous set.]

9.5 (König's Lemma). If T is a tree of height ω such that each level of T is finite, then T has an infinite branch.
[To construct a branch $\{x_0, x_1, \ldots, x_n, \ldots\}$ in T, pick x_0 at level 0 such that $\{y : y > x_0\}$ is infinite. Then pick x_1, x_2, ... similarly.]

9.6. If T is a normal α-tree, then T is isomorphic to a tree $\overline{T}$ whose elements are β-sequences ($\beta < \alpha$), ordered by extension; if $t \subset s$ and $s \in \overline{T}$, then $t \in \overline{T}$, and the βth level of $\overline{T}$ is the set $\{t \in \overline{T} : \operatorname{dom} t = \beta\}$.

9.7. If T is a normal ω_1-tree and if T has uncountable branch, then T has an uncountable antichain.
[For each x in the branch B pick a successor z_x of x such that $z_x \notin B$. Let $A = \{z_x : x \in B\}$.]

9.8. Show that if T is the tree in Theorem 9.16 then there exists some $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$.

9.9. An Aronszajn tree is special if and only if T is the union of ω antichains.
[If $T = \bigcup_{n=0}^{\infty} A_n$, where each A_n is an antichain, define $\pi : T \to \boldsymbol{Q}$ by induction on n, constructing $\pi \restriction A_n$ at stage n, so that the range of π remains finite.]

9.10. Prove Theorem 9.18 using Fodor's Theorem.
[Let $W = \{X_\alpha : \alpha < \omega_1\}$ with $X_\alpha \subset \omega_1$. For each α, let $f(\alpha) = X_\alpha \cap \alpha$. By Fodor's Theorem, f is constant on a stationary set S; by induction construct a Δ-system $W \subset \{X_\alpha : \alpha \in S\}$.]

9.11. If $2^{<\kappa} = \kappa$, then there exists an almost disjoint family of 2^κ subsets of κ.
[As in Lemma 9.21, let $S = \bigcup_{\alpha<\kappa} \{0,1\}^\alpha$; $|S| = \kappa$.]

9.12. Given a family $\mathcal{F}$ of $\aleph_2$ almost disjoint functions $f : \omega_1 \to \omega$, there exists a collection $\mathcal{S}$ of $\aleph_2$ pairwise disjoint *stationary* subsets of ω_1.
[Each $f \in \mathcal{F}$ is constant on a stationary set S_f with value n_f. There is $\mathcal{G} \subset \mathcal{F}$ of size $\aleph_2$ such that n_f is the same for all $f \in \mathcal{G}$. Let $\mathcal{S} = \{S_f : f \in \mathcal{G}\}$.]

9.13. $\omega \not\to (\omega)^{<\omega}$.
[For $x \in [\omega]^{<\omega}$, let $F(x) = 1$ if $|x| \in x$, and $F(x) = 0$ otherwise. If $H \subset \omega$ is infinite, pick $n \in H$ and show that F is not constant on $[H]^n$.]

Historical Notes

Theorem 9.1 is due to Ramsey [1929/30]. Ramsey ultrafilters are investigated in Booth [1970/71]. The theory of partition relations has been developed by Erdős, who has written a number of papers on the subject, some coauthored by Rado, Hajnal, and others. The arrow notation is introduced in Erdős and Rado [1956]. Other major comprehensive articles on partition relations are Erdős, Hajnal, and Rado [1965] and Erdős and Hajnal [1971].

Theorem 9.6 appears in Erdős and Rado [1956]. Lemma 9.4 is due to Sierpiński [1933]. Theorem 9.7 is in Dushnik-Miller [1941].

Weakly compact cardinals (as in Definition 9.8 as well as the tree property) were introduced by Erdős and Tarski in [1961].

The equivalence of Suslin's Problem with the tree formulation (Lemma 9.14) is due to Kurepa [1935]; this paper also presents Aronszajn's construction and Kurepa trees, with Lemma 9.25.

Theorems 9.18 and 9.19: Shanin [1946] and Erdős-Rado [1960].

Ramsey cardinals were first studied by Erdős and Hajnal in [1962].

Exercise 9.2: Erdős-Rado [1956], Exercises 9.4 and 9.13: Erdős-Rado [1952].

Exercise 9.5: D. König [1927].

Exercise 9.9: Galvin.

10. Measurable Cardinals

The theory of large cardinals owes its origin to the basic problem of measure theory, the Measure Problem of H. Lebesgue.

The Measure Problem

Let S be an infinite set. A (*nontrivial σ-additive probabilistic*) *measure* on S is a real-valued function μ on $P(S)$ such that:

(10.1)
(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
(ii) if $X \subset Y$, then $\mu(X) \le \mu(Y)$;
(iii) $\mu(\{a\}) = 0$ for all $a \in S$ (nontriviality);
(iv) if X_n, $n = 0, 1, 2, \ldots$, are pairwise disjoint, then

$$\mu\Big(\bigcup_{n=0}^{\infty} X_n\Big) = \sum_{n=0}^{\infty} \mu(X_n) \qquad (\sigma\text{-additivity}).$$

It follows from (ii) that $\mu(X)$, the *measure of* X, is nonnegative for every $X \subset S$; in a special case of (iv) we get $\mu(X \cup Y) = \mu(X) + \mu(Y)$ whenever $X \cap Y = \emptyset$ (finite additivity).

More generally, let $\mathcal{A}$ be a σ-complete algebra of sets. A measure on $\mathcal{A}$ is a real-valued function μ on $\mathcal{A}$ satisfying (i)–(iv). Thus a measure on S is a measure on $P(S)$.

An example of a measure on a σ-complete algebra of sets is the Lebesgue measure on the algebra of all Lebesgue measurable subsets of the unit interval $[0,1]$. The Lebesgue measure has, in addition to (i)–(iv), the following property:

(10.2) If X is congruent by translation to a measurable set Y, then X is measurable and $\mu(X) = \mu(Y)$.

It is well known that there exist sets of reals that are not Lebesgue measurable, and in fact that there is no measure on $[0,1]$ with the property (10.2) (*translation invariant measure*); see Exercise 10.1.

The natural question to ask is whether the Lebesgue measure can be extended to some measure (not translation invariant) such that all subsets of $[0,1]$ are measurable, or whether there exists any measure on $[0,1]$. Or, whether there exists a measure on some set S.

The investigation of this problem has lead to important discoveries in set theory, opening up a new field, the theory of large cardinal numbers, which has far-reaching consequences both in pure set theory and in descriptive set theory.

A measure μ on S is *two-valued* if $\mu(X)$ is either 0 or 1 for all $X \subset S$. If μ is a two-valued measure on S, let

$$U = \{X \subset S : \mu(X) = 1\}. \tag{10.3}$$

It is easy to verify that U is an ultrafilter on S. (For instance, if $X \in U$ and $Y \in U$, then $X \cap Y \in U$. If $\mu(X) = \mu(Y) = 1$, then $X = (X-Y) \cup (X \cap Y)$ and $Y = (Y-X) \cup (X \cap Y)$. If $\mu(X \cap Y)$ were not 1, then $\mu(X-Y) = \mu(Y-X) = 1$, and we would have $\mu(X \cup Y) = 2$.)

Next we note that the ultrafilter U is σ-complete. This is so because μ is σ-additive, and an ultrafilter U on S is σ-complete if and only if there is no partition of S into countably many disjoint parts $S = \bigcup_{n=0}^{\infty} X_n$ such that $X_n \notin U$, for all n.

Thus if μ is a two-valued measure on S, U is a σ-complete ultrafilter on S. Conversely, if U is a σ-complete ultrafilter on S, then the following function is a two-valued measure on S:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{if } X \notin U. \end{cases} \tag{10.4}$$

Let μ be a measure on S. A set $A \subset S$ is an *atom* of μ if $\mu(A) > 0$ and if for every $X \subset A$, we have either $\mu(X) = 0$ or $\mu(X) = \mu(A)$.

If μ has an atom A, then

$$U = \{X \subset S : \mu(X \cap A) = \mu(A)\} \tag{10.5}$$

is again a σ-complete ultrafilter on S.

A measure μ on S is *atomless* if it has no atoms. Then every set $X \subset S$ of positive measure can be split into two disjoint sets of positive measure: $X = Y \cup Z$, and $\mu(Y) > 0$, $\mu(Z) > 0$.

We shall eventually prove various strong consequences of the existence of a nontrivial σ-additive measure and establish the relationship between the Measure Problem and large cardinals. Our starting point is the following theorem which shows that if a measure exists, then there exists at least a weakly inaccessible cardinal.

Theorem 10.1 (Ulam). *If there is a σ-additive nontrivial measure on S, then either there exists a two-valued measure on S and $|S|$ is greater than or equal to the least inaccessible cardinal, or there exists an atomless measure on $2^{\aleph_0}$ and $2^{\aleph_0}$ is greater than or equal to the least weakly inaccessible cardinal.*

Theorem 10.1 will be proved in a sequence of lemmas, which will also provide additional information on the Measure Problem and introduce basic notions and methods of the theory of large cardinals. First we make the following observation. Let κ be the least cardinal that carries a nontrivial σ-additive two-valued measure. Clearly, κ is uncountable and is also the least cardinal that has a nonprincipal countably complete ultrafilter. And we observe that such an ultrafilter is in fact κ-complete:

Lemma 10.2. *Let κ be the least cardinal with the property that there is a nonprincipal σ-complete ultrafilter on κ, and let U be such an ultrafilter. Then U is κ-complete.*

Proof. Let U be a σ-complete ultrafilter on κ, and let us assume that U is not κ-complete. Then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ of κ such that $\gamma < \kappa$, and $X_\alpha \notin U$ for all $\alpha < \gamma$. We shall now use this partition to construct a nonprincipal σ-complete ultrafilter on γ, thus contradicting the choice of κ as the least cardinal that carries such an ultrafilter.

Let f be the mapping of κ onto γ defined as follows:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in \kappa).$$

The mapping f induces a σ-complete ultrafilter on γ: we define $D \subset P(\gamma)$ by

$$Z \in D \quad \text{if and only if} \quad f_{-1}(Z) \in U. \tag{10.6}$$

The ultrafilter D is nonprincipal: Assume that $\{\alpha\} \in D$ for some $\alpha < \gamma$. Then $X_\alpha \in U$, contrary to our assumption on X_α. Thus γ carries a σ-complete nonprincipal ultrafilter. □

Measurable and Real-Valued Measurable Cardinals

We are now ready to define the central notion of this chapter.

Definition 10.3. An uncountable cardinal κ is *measurable* if there exists a κ-complete nonprincipal ultrafilter U on κ.

By Lemma 10.2, the least cardinal that carries a nontrivial two-valued σ-additive measure is measurable. Note that if U is a κ-complete nonprincipal ultrafilter on κ, then every set $X \in U$ has cardinality κ because every set of smaller size is the union of fewer than κ singletons. For similar reasons, κ is a regular cardinal because if κ is singular, then it is the union of fewer than κ small sets. The next lemma gives a first link of the Measure Problem with large cardinals.

Lemma 10.4. *Every measurable cardinal is inaccessible.*

Proof. We have just given an argument why a measurable cardinal is regular. Let us show that measurable cardinals are strong limit cardinals. Let κ be measurable, and let us assume that there exists $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; we shall reach a contradiction.

Let S be a set of functions $f : \lambda \to \{0, 1\}$ such that $|S| = \kappa$, and let U be a κ-complete nonprincipal ultrafilter on S. For each $\alpha < \lambda$, let X_α be that one of the two sets $\{f \in S : f(\alpha) = 0\}$, $\{f \in S : f(\alpha) = 1\}$ which is in U, and let ε_α be 0 or 1 accordingly. Since U is κ-complete, the set $X = \bigcap_{\alpha<\lambda} X_\alpha$ is in U. However, X has at most one element, namely the function f that has the values $f(\alpha) = \varepsilon_\alpha$. A contradiction. □

Let us now turn our attention to measures that are not necessarily two-valued. Let μ be a nontrivial σ-additive measure on a set S. In analogy with (10.3) we consider the ideal of all *null sets*:

(10.7)
$$I_\mu = \{X \subset S : \mu(X) = 0\}.$$

I_μ is a nonprincipal σ-complete ideal on S. Moreover, it has these properties:

(10.8) (i) $\{x\} \in I$ for every $x \in S$;
(ii) every family of pairwise disjoint sets $X \subset S$ that are not in I is at most countable.

To see that (ii) holds, note that if W is a disjoint family of set of positive measure, then for each integer $n > 0$, there are only finitely many sets $X \in W$ of measure $\geq 1/n$.

A σ-complete nonprincipal ideal I on S is called *σ-saturated* if it satisfies (10.8).

The following lemma is an analog of Lemma 10.2:

Lemma 10.5.

(i) *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure on κ. Then the ideal I_μ of null sets is κ-complete.*

(ii) *Let κ be the least cardinal with the property that there is a σ-complete σ-saturated ideal on κ, and let I be such an ideal. Then I is κ-complete.*

Proof. (i) Let us assume that I_μ is not κ-complete. There exists a collection of null sets $\{X_\alpha : \alpha < \gamma\}$ such that $\gamma < \kappa$ and that their union X has positive measure. We may assume without loss of generality that the sets X_α, $\alpha < \gamma$, are pairwise disjoint; let $m = \mu(X)$.

Let f be the following mapping of X onto γ:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in X).$$

The mapping f induces a measure ν on γ:

(10.9)
$$\nu(Z) = \frac{1}{m} \cdot \mu(f_{-1}(Z)).$$

The measure ν is σ-additive and is nontrivial since $\nu(\{\alpha\}) = \mu(X_\alpha) = 0$ for each $\alpha \in \gamma$. This contradicts the choice of κ as the least cardinal that carries a measure.

(ii) The proof is similar. We define an ideal J on γ by: $Z \in J$ if and only if $f_{-1}(Z) \in I$. The induced ideal J is σ-complete and σ-saturated. □

Let $\{r_i : i \in I\}$ be a collection of nonnegative real numbers. We define

(10.10)
$$\sum_{i\in I} r_i = \sup\Bigl\{\sum_{i\in E} r_i : E \text{ is a finite subset of } I\Bigr\}.$$

Note that if the sum (10.10) is not ∞, then at most countably many r_i are not equal to 0.

Let κ be an uncountable cardinal. A measure μ on S is called *κ-additive* if for every $\gamma < \kappa$ and for every disjoint collection X_α, $\alpha < \gamma$, of subsets of S,

(10.11)
$$\mu\Bigl(\bigcup_{\alpha<\gamma} X_\alpha\Bigr) = \sum_{\alpha<\gamma} \mu(X_\alpha).$$

If μ is a κ-additive measure, then the ideal I_μ of null sets is κ-complete. The converse is also true and we get a better analog of Lemma 10.2 for real-valued measures:

Lemma 10.6. *Let μ be a measure on S, and let I_μ be the ideal of null sets. If I_μ is κ-complete, then μ is κ-additive.*

Proof. Let $\gamma < \kappa$, and let X_α, $\alpha < \gamma$, be disjoint subsets of S. Since the X_α are disjoint, at most countably many of them have positive measure. Thus let us write

$$\{X_\alpha : \alpha < \gamma\} = \{Y_n : n = 0, 1, 2, \ldots\} \cup \{Z_\alpha : \alpha < \gamma\},$$

where each Z_α has measure 0. Then we have

$$\mu\Bigl(\bigcup_{\alpha<\gamma} X_\alpha\Bigr) = \mu\Bigl(\bigcup_{n=0}^{\infty} Y_n\Bigr) + \mu\Bigl(\bigcup_{\alpha<\gamma} Z_\alpha\Bigr).$$

Now first μ is σ-additive, and we have

$$\mu\Bigl(\bigcup_{n=0}^{\infty} Y_n\Bigr) = \sum_{n=0}^{\infty} \mu(Y_n),$$

and secondly I_μ is κ-complete and

$$\mu\Bigl(\bigcup_{\alpha<\gamma} Z_\alpha\Bigr) = 0 = \sum_{\alpha<\gamma} \mu(Z_\alpha).$$

Thus $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. □

Corollary 10.7. *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure. Then μ is κ-additive.* □

Definition 10.8. An uncountable cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure μ on κ.

By Corollary 10.7, the least cardinal that carries a nontrivial σ-additive measure is real-valued measurable. We shall show that if a real-valued measurable cardinal κ is not measurable, then $\kappa \le 2^{\aleph_0}$. Note that if μ is a nontrivial κ additive measure on κ, then every set of size $< \kappa$ has measure 0, and moreover κ cannot be the union of fewer than κ sets of size $< \kappa$. Thus a real-valued measurable cardinal is regular. We shall show that it is weakly inaccessible.

We shall first prove the first claim made in the preceding paragraph.

Lemma 10.9.

(i) *If there exists an atomless nontrivial σ-additive measure, then there exists a nontrivial σ-additive measure on some $\kappa \le 2^{\aleph_0}$.*

(ii) *If I is a σ-complete σ-saturated ideal on S, then either there exists $Z \subset S$, such that $I \restriction Z = \{X \subset Z : X \in I\}$ is a prime ideal, or there exists a σ-complete σ-saturated ideal on some $\kappa \le 2^{\aleph_0}$.*

Proof. (i) Let μ be such a measure on S. We construct a tree T of subsets of S, partially ordered by reverse inclusion. The 0th level of T is $\{S\}$. Each level of T consists of pairwise disjoint subsets of S of positive measure. Each $X \in T$ has two immediate successors: We choose two sets Y, Z of positive measure such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$. If α is a limit ordinal, then the αth level consists of all intersections $X = \bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is on the ξth level of T and such that X has positive measure.

We observe that every branch of T has countable length: If $\{X_\xi : \xi < \alpha\}$ is a branch in T, then the set $\{Y_\xi : \xi < \alpha\}$, where $Y_\xi = X_\xi - X_{\xi+1}$, is a disjoint collection of sets of positive measure. Consequently, T has height at most ω_1. Similarly, each level of T is at most countable, and it follows that T has at most $2^{\aleph_0}$ branches.

Let $\{b_\alpha : \alpha < \kappa\}$, $\kappa \le 2^{\aleph_0}$, be an enumeration of all branches $b = \{X_\xi : \xi < \gamma\}$ such that $\bigcap_{\xi<\gamma} X_\xi$ is nonempty; for each $\alpha < \kappa$, let $Z_\alpha = \bigcap\{X : X \in b_\alpha\}$. The collection $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ sets of measure 0.

We induce a measure ν on κ as follows: Let f be the mapping of S onto κ defined by

$$f(x) = \alpha \quad \text{if and only if} \quad x \in Z_\alpha \qquad (x \in S),$$

and let

$$\nu(Z) = \mu(f_{-1}(Z))$$

for all $Z \subset \kappa$. It follows that ν is a nontrivial σ-additive measure on κ.

(ii) The proof is similar. We define a tree T as above and then induce an ideal J on κ by letting $Z \in J$ if and only if $f_{-1}(Z) \in I$. □

The proof of Lemma 10.9 shows that if μ is atomless, then there is a partition of S into at most $2^{\aleph_0}$ null sets; in other words, μ is not $(2^{\aleph_0})^+$-additive. Hence if κ carries an atomless κ-additive measure, then $\kappa \le 2^{\aleph_0}$ and we have:

Corollary 10.10. *If κ is a real-valued measurable cardinal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$.*

More generally, if κ carries a κ-complete σ-saturated ideal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$. □

The measure ν obtained in Lemma 10.9(i) is atomless; this follows from the fact that $\kappa \le 2^{\aleph_0}$ and Lemma 10.4. If there exists an atomless σ-additive measure, then there is one on some $\kappa \le 2^{\aleph_0}$. Clearly, such a measure can be extended to a measure on $2^{\aleph_0}$: For $X \subset 2^{\aleph_0}$, we let $\mu(X) = \mu(X \cap \kappa)$. Thus we conclude that there exists an atomless σ-additive measure on the set $\boldsymbol{R}$ of all reals. It turns out that using the same assumption, we can obtain a σ-additive measure on $\boldsymbol{R}$ that extends Lebesgue measure. This can be done by a slight modification of the proof of Lemma 10.9:

Using Exercise 10.3, we construct for each finite 0–1 sequence s, a set $X_s \subset S$ such that $X_\emptyset = S$, and for every $s \in Seq$, $X_{s^\frown 0} \cup X_{s^\frown 1} = X_s$, $X_{s^\frown 0} \cap X_{s^\frown 1} = \emptyset$, and $\mu(X_{s^\frown 0}) = \mu(X_{s^\frown 1}) = \frac{1}{2} \cdot \mu(X_{s^\frown 0})$. Then we define a measure ν_1 on 2^ω by

$$\nu_1(Z) = \mu(\bigcup\{X_f : f \in Z\}),$$

where $X_f = \bigcap_{n=0}^\infty X_{f\restriction n}$ for each $f \in 2^\omega$. Using the mapping $F : 2^\omega \to [0,1]$ defined by

$$F(f) = \sum_{n=0}^\infty f(n)/2^{n+1}$$

we obtain a nontrivial σ-additive measure ν on $[0,1]$. This measure agrees with the Lebesgue measure on all intervals $[k/2^n, (k+1)/2^n]$, and hence on all Borel sets. Every set of Lebesgue measure 0 is included in a Borel (in fact, G_δ) set of Lebesgue measure 0 and hence has ν-measure 0. Every Lebesgue measurable set X can be written as $X = (B - N_1) \cup N_2$, where N_1 and N_2 have Lebesgue measure 0, and hence the Lebesgue measure of X is equal to $\nu(X)$. Thus ν agrees with the Lebesgue measure on all Lebesgue measurable subsets of $[0,1]$.

We shall now show that a real-valued measurable cardinal is weakly inaccessible. The proof is by a combinatorial argument, using matrices of sets.

Definition 10.11. An *Ulam matrix* (more precisely, an Ulam $(\aleph_1, \aleph_0)$-matrix) is a collection $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of subsets of ω_1 such that:

(10.12) (i) if $\alpha \ne \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for every $n < \omega$;
(ii) for each α, the set $\omega_1 - \bigcup_{n=0}^\infty A_{\alpha,n}$ is at most countable.

An Ulam matrix has $\aleph_1$ rows and $\aleph_0$ columns. Each column consists of pairwise disjoint sets, and the union of each row contains all but countably many elements of ω_1.

Lemma 10.12. *An Ulam matrix exists.*

Proof. For each $\xi < \omega_1$, let f_ξ be a function on ω such that $\xi \subset \operatorname{ran}(f_\xi)$. Let us define $A_{\alpha,n}$ for $\alpha < \omega_1$ and $n < \omega$ by

$$(10.13) \qquad \xi \in A_{\alpha,n} \quad \text{if and only if} \quad f_\xi(n) = \alpha.$$

If $n < \omega$, then for each $\xi \in \omega_1$ there is only one α such that $\xi \in A_{\alpha,n}$, namely $\alpha = f_\xi(n)$; and we have property (i) of (10.12). If $\alpha < \omega_1$, then for each $\xi > \alpha$ there is an n such that $f_\xi(n) = \alpha$ and hence $(\omega_1 - \bigcup_{n=0}^\infty A_{\alpha,n}) \subset \alpha + 1$; that verifies property (ii). □

Using an Ulam matrix, we can show that there is no measure on ω_1:

Lemma 10.13. *There is no nontrivial σ-additive measure on ω_1. More generally, there is no σ-complete σ-saturated ideal on ω_1.*

Proof. Let $\{A_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$ be an Ulam matrix. Assuming that we have a measure on ω_1, there is for each α some $n = n_\alpha$ such that $A_{\alpha,n}$ has positive measure (because of (10.12)(ii)). Hence there exist an uncountable set $W \subset \omega_1$ and some $n < \omega$ such that $n_\alpha = n$ for all $\alpha \in W$. Then $\{A_{\alpha,n} : \alpha \in W\}$ is an uncountable, pairwise disjoint (by (10.12)(i)) family of sets of positive measure; a contradiction. □

A straightforward generalization of Lemmas 10.12 and 10.13 gives the result mentioned above:

Lemma 10.14. *If $\kappa = \lambda^+$, then there is no κ-complete σ-saturated ideal on κ.*

Proof. For each $\xi < \lambda^+$, we let f_ξ be a function on λ such that $\xi \subset \operatorname{ran}(f_\xi)$, and let

$$\xi \in A_{\alpha,\eta} \quad \text{if and only if} \quad f_\xi(\eta) = \alpha.$$

Then $\{A_{\alpha,\eta} : \alpha < \lambda^+,\ \eta < \lambda\}$ is an Ulam (λ^+, λ)-matrix, that is a collection of subsets of λ^+ such that:

(10.14) (i) $A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset$ whenever $\alpha \neq \beta < \lambda^+$, and $\eta < \lambda$;
(ii) $|\lambda^+ - \bigcup_{\eta<\lambda} A_{\alpha,\eta}| \le \lambda$ for each $\alpha < \lambda^+$.

The proof of Lemma 10.13 generalizes to show that there is no κ-complete σ-saturated ideal on κ. □

Corollary 10.15. *Every real-valued measurable cardinal is weakly inaccessible.* □

Lemma 10.14 completes the proof of Theorem 10.1: If there is a σ-additive nontrivial measure on S, then either the measure has an atom A and we can construct a two-valued measure on S via a σ-complete nonprincipal ultrafilter on A, and then $|S| \ge$ the least measurable cardinal, which is inaccessible; or the measure on S is atomless and we construct, as in Lemma 10.9, an atomless measure on $2^{\aleph_0}$, and then $2^{\aleph_0} \ge$ the least real-valued measurable cardinal, which is weakly inaccessible. □

Prior to Ulam's work, Banach and Kuratowski proved that if the Continuum Hypothesis holds then there exists no σ-additive measure on $\boldsymbol{R}$. We present their proof below; in fact, Lemma 10.16 gives a slightly more general result.

If f and g are functions from ω to ω, let $f < g$ mean that $f(n) < g(n)$ for all but finitely many $n \in \omega$. A κ-sequence of functions $\langle f_\alpha : \alpha < \kappa \rangle$ is called a *κ-scale* if $f_\alpha < f_\beta$ whenever $\alpha < \beta$, and if for every $g : \omega \to \omega$ there exists an α such that $g < f_\alpha$.

Lemma 10.16. *If there exists a κ-scale, then κ is not a real-valued measurable cardinal.*

Proof. Let f_α, $\alpha < \kappa$, be a κ-scale. We define an $(\aleph_0, \aleph_0)$-matrix of subsets of κ as follows: For $n, k < \omega$, let

$$(10.15) \qquad \alpha \in A_{n,k} \quad \text{if and only if} \quad f_\alpha(n) = k \qquad (\alpha \in \kappa).$$

Since for each n and each α there is k such that $\alpha \in A_{n,k}$, we have

$$\bigcup_{k=0}^\infty A_{n,k} = \kappa$$

for every $n = 0, 1, 2, \ldots$.

Now assume that μ is a nontrivial κ-additive measure on κ. For each n, let k_n be such that

$$\mu(A_{n,0} \cup A_{n,1} \cup \ldots \cup A_{n,k_n}) \ge 1 - (1/2^{n+2}),$$

and let $B_n = A_{n,0} \cup \ldots \cup A_{n,k_n}$. If we let $B = \bigcap_{n=0}^\infty B_n$, then we clearly have $\mu(B) \ge 1/2$.

Let $g : \omega \to \omega$ be the function $g(n) = k_n$. If $\alpha \in B$, then by the definition of B and by (10.15), we have

$$f_\alpha(n) \le g(n)$$

for all $n = 0, 1, 2, \ldots$; hence $g \not< f_\alpha$. However, since B has positive measure, B has size κ, and therefore we have $g \not< f_\alpha$ for cofinally many $\alpha < \kappa$. This contradicts the assumption that the f_α form a scale. □

Corollary 10.17. *If there is a measure on $2^{\aleph_0}$, then $2^{\aleph_0} > \aleph_1$.*

Proof. If $2^{\aleph_0} = \aleph_1$, then there exists an ω_1-scale; a scale $\langle f_\alpha : \alpha < \omega_1 \rangle$ is constructed by transfinite induction to ω_1:

Let $\{g_\alpha : \alpha < \omega_1\}$ enumerate all functions from ω to ω. At stage α, we construct, by diagonalization, a function f_α such that for all $\beta < \alpha$, $f_\alpha > f_\beta$ and $f_\alpha > g_\beta$. Then $\langle f_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-scale. □

Measurable Cardinals

By Lemma 10.4, every measurable cardinal is inaccessible. While we shall investigate measurable cardinals extensively in Part II, we now present a few basic results that establish the relationship of measurable cardinals and the large cardinals introduced in Chapter 9.

We recall that by Lemma 9.26, a cardinal κ is weakly compact if and only if it is inaccessible and has the tree property.

Lemma 10.18. *Every measurable cardinal is weakly compact.*

Proof. Let κ be a measurable cardinal. To show that κ is weakly compact, it suffices to prove the tree property. Let $(T, <)$ be a tree of height κ with levels of size $< \kappa$. We consider a nonprincipal κ-complete ultrafilter U on T. Let B be the set of all $x \in T$ such that the set of all successors of x is in U. It is clear that B is a branch in T and it is easy to verify that each level of T has one element in B; thus B is a branch of size κ. □

Normal Measures

In Chapter 8 we defined the notion of a normal κ-complete filter, namely a filter closed under diagonal intersections (8.7).

Thus we call a normal κ-complete nonprincipal ultrafilter a *normal measure* on κ. Note that by Exercise 8.8, a measure is normal if and only if every regressive function on a set of measure one is constant on a set of measure one.

Lemma 10.19. *If D is a normal measure on κ, then every set in D is stationary.*

Proof. By Lemma 8.11, every closed unbounded set is in D, and the lemma follows. □

Theorem 10.20 below shows that if κ is measurable cardinal then a normal measure exists.

Theorem 10.20. *Every measurable cardinal carries a normal measure. If U is a nonprincipal κ-complete ultrafilter on κ then there exists a function $f : \kappa \to \kappa$ such that $f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$ is a normal measure.*

Proof. Let U be a nonprincipal κ-complete ultrafilter on κ. For f and g in κ^κ, let

$$f \equiv g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U.$$

It is easily seen that $\equiv$ is an equivalence relation on κ^κ. Let $[f]$ denote the equivalence class of $f \in \kappa^\kappa$. Furthermore, if we let

$$f < g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U,$$

then $<$ is a linear ordering of (the equivalence classes of) κ^κ.

There exists no infinite descending sequence $f_0 > f_1 > \ldots > f_n > \ldots$: Otherwise, let $X_n = \{\alpha : f_n(\alpha) > f_{n+1}(\alpha)\}$, and let $X = \bigcap_{n=0}^\infty X_n$. X is nonempty, and if $\alpha \in X$, we would have $f_0(\alpha) > f_1(\alpha) > \ldots > f_n(\alpha) > \ldots$, a contradiction.

Thus $<$ is a well-ordering of $\kappa^\kappa/{\equiv}$.

Now let $f : \kappa \to \kappa$ be the least function (in this well-ordering) with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$. Such functions exist: for instance, the *diagonal function* $d(\alpha) = \alpha$ has this property.

Let $D = f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$. We claim that D is a normal measure.

It is easy to verify that D is a κ-complete ultrafilter. For every $\gamma < \kappa$, we have $f_{-1}(\{\gamma\}) \notin U$, and so $\{\gamma\} \notin D$, and so D is nonprincipal.

In order to show that D is normal, let h be a regressive function on a set $X \in D$. We shall show that h is constant on a set in D. Let g be the function defined by $g(\alpha) = h(f(\alpha))$. As $g(\alpha) < f(\alpha)$ for all $\alpha \in f_{-1}(X)$, we have $g < f$, and it follows by the minimality of f that g is constant on some $Y \in U$. Hence h is constant on $f(Y)$ and $f(Y) \in D$. □

As an application of normal measures we show that every measurable cardinal is a Mahlo cardinal, and improve Lemma 10.18 by showing that every measurable cardinal is a Ramsey cardinal.

Lemma 10.21. *Every measurable cardinal is a Mahlo cardinal.*

Proof. Let κ be a measurable cardinal. We shall show that the set of all inaccessible cardinals $\alpha < \kappa$ is stationary. As κ is strong limit, the set of all strong limit cardinals $\alpha < \kappa$ is closed unbounded, and it suffices to show that the set of all regular cardinals $\alpha < \kappa$ is stationary.

Let D be a normal measure on κ. We claim that $\{\alpha < \kappa : \alpha \text{ is regular}\} \in D$; this will complete the proof, since every set in D is stationary, by Lemma 10.19.

Toward a contradiction, assume that $\{\alpha : \operatorname{cf} \alpha < \alpha\} \in D$. By normality, there is some $\lambda < \kappa$ such that $E_\lambda = \{\alpha : \operatorname{cf} \alpha = \lambda\} \in D$. For each $\alpha \in E_\lambda$, let $\langle x_{\alpha,\xi} : \xi < \lambda \rangle$ be an increasing sequence with limit α. For each $\xi < \lambda$ there exist y_ξ and $A_\xi \in D$ such that $x_{\alpha,\xi} = y_\xi$ for all $\alpha \in A_\xi$. Let $A = \bigcap_{\xi<\lambda} A_\xi$. Then $A \in D$, but A contains only one element, namely $\lim_{\xi\to\lambda} y_\xi$; a contradiction. □

Theorem 10.22. *Let κ be a measurable cardinal, let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into less than κ pieces. Then there exists a set $H \in D$ homogeneous for F. Hence every measurable cardinal is a Ramsey cardinal.*

Proof. Let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into fewer than κ pieces. It suffices to show that for each $n = 1, 2, \ldots,$ there is $H_n \in D$ such that F is constant on $[H_n]^n$; then $H = \bigcap_{n=1}^{\infty} H_n$ is homogeneous for F.

We prove, by induction on n, that every partition of $[\kappa]^n$ into fewer than κ pieces is constant on $[H]^n$ for some $H \in D$. The assertion is trivial for $n = 1$, so we assume that it is true for n and prove that it holds also for $n+1$. Let $F : [\kappa]^{n+1} \to I$, where $|I| < \kappa$. For each $\alpha < \kappa$, we define F_α on $[\kappa - \{\alpha\}]^n$ by $F_\alpha(x) = F(\{\alpha\} \cup x)$.

By the induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$; let i_a be its constant value. Let X be the diagonal intersection $X = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\}$. We have $X \in D$ since D is normal; also, if $\gamma < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$ and so $F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, there exist $i \in I$ and $H \subset X$ in D such that $i_\gamma = i$ for all $\gamma \in H$. It follows that $F(x) = i$ for all $x \in [H]^{n+1}$. □

Strongly Compact and Supercompact Cardinals

Among the various large cardinals that we shall investigate in more detail in Part II there are two that are immediate generalizations of measurable cardinals.

Definition 10.23. An uncountable cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Clearly, every strongly compact cardinal is measurable.

Let A be a set of size at least κ, and let us consider the filter F on $P_\kappa(A)$ generated by the sets $\hat{P} = \{Q \in P_\kappa(A) : P \subset Q\}$. F is a κ-complete filter and if κ is strongly compact, F can be extended to a κ-complete ultrafilter U. A κ-complete ultrafilter U on $P_\kappa(A)$ that extends F is called a *fine measure*. In Part II we prove that if a fine measure on $P_\kappa(A)$ exists for every A, then κ is strongly compact.

A fine measure U on $P_{<\kappa}(A)$ is *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(P) \in P$ for all P in a set in U, then f is constant on a set in U. Equivalently, U is normal if it is closed under diagonal intersections $\triangle_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}$.

Definition 10.24. An uncountable cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$ there exists a normal measure on $P_\kappa(A)$.

We return to the subject of strongly compact and supercompact cardinals in Part II.

Exercises

10.1 (Vitali). Let M be maximal (under $\subset$) subset of $[0,1]$ with the property that $x - y$ is not a rational number, for any pair of distinct $x, y \in M$. Show that M is not Lebesgue measurable.

[Consider the sets $M_q = \{x + q : x \in M\}$ where q is rational. They are pairwise disjoint and $[0,1] \subset \bigcup\{M_q : q \in \boldsymbol{Q} \cap [-1,1]\} \subset [-1,2]$.]

10.2. Prove directly that the measure ν defined in the proof of Lemma 10.9(i) is atomless.

[Assume that Z is an atom of ν, and let $Y = f_{-1}(Z)$. If $X \in T$ is such that $\mu(Y \cap X) \neq 0$ and if X_1, X_2 are the two immediate successors of X, then either $\mu(Y \cap X_1) = 0$ or $\mu(Y \cap X_2) = 0$. Prove by induction that on each level of T there is a unique X such that $\mu(Y \cap X) \neq 0$, and that these X's constitute a branch in T of length ω_1; a contradiction.]

10.3. If μ is an atomless measure on S, there exists $Z \subset S$ such that $\mu(Z) = 1/2$. More generally, given $Z_0 \subset S$, there exists $Z \subset Z_0$ such that $\mu(Z) = (1/2) \cdot \mu(Z_0)$.

[Construct a sequence $S = S_0 \supset S_1 \supset \ldots \supset S_\alpha \supset \ldots$, $\alpha < \omega_1$, such that $\mu(S_\alpha) \geq 1/2$, and if $\mu(S_\alpha) > 1/2$, then $1/2 \leq \mu(S_{\alpha+1}) < \mu(S_\alpha)$; if α is a limit ordinal, let $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. There exists $\alpha < \omega_1$ such that $\mu(S_\alpha) = 1/2$.]

10.4. Let μ be a two-valued measure and U the ultrafilter of all sets of measure one. Then μ is κ-additive if and only if U is κ-complete.

10.5. A measure U on κ is normal if and only if the diagonal function $d(\alpha) = \alpha$ is the least function f with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$.

10.6. Let D be a normal measure on κ and let $f : [\kappa]^{<\omega} \to \kappa$ be such that $f(x) = 0$ or $f(x) < \min x$ for all $x \in [\kappa]^{<\omega}$. Then there is $H \in D$ such that for each n, f is constant on $[H]^n$.

[By induction, as in Theorem 10.22. Given f on $[\kappa]^{n+1}$, let $f_\alpha(s) = f(\{\alpha\} \cup s)$ for $\alpha < \min s$; f_α is constant on $[X_\alpha]^n$ with value $\gamma_\alpha < \alpha$. Let X be the diagonal intersection of X_α, $\alpha < \kappa$, and let γ and $H \subset X$ be such that $H \in D$ and $\gamma_\alpha = \gamma$ for all $\alpha \in H$.]

10.7. If κ is measurable then there exists a normal measure on $P_\kappa(\kappa)$.

Historical Notes

The study of measurable cardinals originated around 1930 with the work of Banach, Kuratowski, Tarski, and Ulam. Ulam showed in [1930] that measurable cardinals are large, that the least measurable cardinal is at least as large as the least inaccessible cardinal.

The main result on measurable and real-valued measurable cardinals (Theorem 10.1) is due to Ulam [1930]. The fact that a measurable cardinal is inaccessible (Lemma 10.4) was discovered by Ulam and Tarski (cf. Ulam [1930]). Prior to Ulam, Banach and Kuratowski proved in [1929] that if $2^{\aleph_0} = \aleph_1$, then there is no measure on the continuum; their proof is as in Lemma 10.16. Real-valued measurable cardinals were introduced by Banach in [1930].

Lemma 10.18: Erdős and Tarski [1943]. Hanf [1963/64a] proved that the least inaccessible cardinal is not measurable. That every measurable cardinal is a Ramsey cardinal was proved by Erdős and Hajnal [1962]; the stronger version (Theorem 10.22) is due to Rowbottom [1971].

Strongly compact cardinals were introduced by Keisler and Tarski in [1963/64]; supercompact cardinals were defined by Reinhardt and Solovay, cf. Solovay *et al.* [1978].

Exercise 10.1: Vitali [1905].

11. Borel and Analytic Sets

Descriptive set theory deals with sets of reals that are described in some simple way: sets that have a simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The main theme is that questions that are difficult to answer if asked for arbitrary sets of reals, become much easier when asked for sets that have a simple description. An example of that is the Cantor-Bendixson Theorem (Theorem 4.6): Every closed set of reals is either at most countable or has size $2^{\aleph_0}$.

Since properties of definable sets can usually be established effectively, without use of the Axiom of Choice, we shall work in set theory ZF without the Axiom of Choice. When some statement depends on the Axiom of Choice, we shall explicitly say so. However, we shall assume a weak form of the Axiom of Choice. The reason is that in descriptive set theory one frequently considers unions and intersections of countably many sets of reals, and we shall often use facts like "the union of countably many countable sets is countable." Thus we shall work, throughout this chapter, in set theory ZF + the Countable Axiom of Choice.

In this chapter we develop the basic theory of Borel and analytic sets in Polish spaces. A Polish space is a topological space that is homeomorphic to a complete separable metric space (Definition 4.12).

A canonical example of a Polish space is the Baire space $\mathcal{N}$. The following lemma shows that every Polish space is a continuous image of $\mathcal{N}$:

Lemma 11.1. *Let X be a Polish space. Then there exists a continuous mapping from $\mathcal{N}$ onto X.*

Proof. Let X be a complete separable metric space; we construct a mapping f of $\mathcal{N}$ onto X as follows: It is easy to construct, by induction on the length of $s \in Seq$, a collection $\{C_s : s \in Seq\}$ of closed balls such that $C_\emptyset = X$ and

(11.1)
(i) $\text{diameter}(C_s) \le 1/n$ where $n = \text{length}(s)$,
(ii) $C_s \subset \bigcup_{k=0}^{\infty} C_{s^\frown k}$ (all $s \in Seq$),
(iii) if $s \subset t$ then $\text{center}(C_t) \in C_s$.

For each $a \in \mathcal{N}$, let $f(a)$ be the unique point in $\bigcap\{C_s : s \subset a\}$; it is easily checked that f is continuous and that $X = f(\mathcal{N})$. □

Borel Sets

Let X be a Polish space. A set $A \subset X$ is a *Borel set* if it belongs to the smallest σ-algebra of subsets of X containing all closed sets. We shall now give a more explicit description of Borel sets. For each $\alpha < \omega_1$, let us define the collections $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of subsets of X:

(11.2)
$$\begin{aligned} \mathbf{\Sigma}^0_1 &= \text{the collection of all open sets;} \\ \mathbf{\Pi}^0_1 &= \text{the collection of all closed sets;} \\ \mathbf{\Sigma}^0_\alpha &= \text{the collection of all sets } A = \textstyle\bigcup_{n=0}^\infty A_n, \text{ where each } A_n \\ &\quad \text{belongs to } \mathbf{\Pi}^0_\beta \text{ for some } \beta < \alpha; \\ \mathbf{\Pi}^0_\alpha &= \text{the collection of all complements of sets in } \mathbf{\Sigma}^0_\alpha \\ &= \text{the collection of all sets } A = \textstyle\bigcap_{n=0}^\infty A_n, \text{ where each } A_n \\ &\quad \text{belongs to } \mathbf{\Sigma}^0_\beta \text{ for some } \beta < \alpha. \end{aligned}$$

It is clear (by induction on α) that the elements of each $\mathbf{\Sigma}^0_\alpha$ and each $\mathbf{\Pi}^0_\alpha$ are Borel sets. Since every open set is the union of countably many closed sets, we have $\mathbf{\Sigma}^0_1 \subset \mathbf{\Sigma}^0_2$, and consequently, if $\alpha < \beta$, then

$$\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Sigma}^0_\beta, \qquad \mathbf{\Sigma}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Sigma}^0_\beta.$$

Hence

(11.3)
$$\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$$

and it is easy to verify that the collection (11.3) is a σ-algebra (here we use the Countable Axiom of Choice). Hence every Borel set is in some $\mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$.

Note that each $\mathbf{\Sigma}^0_\alpha$ (and each $\mathbf{\Pi}^0_\alpha$) is closed under finite unions, finite intersections, and inverse images by continuous functions (i.e., if $A \in \mathbf{\Sigma}^0_\alpha$ in Y, then $f_{-1}(A) \in \mathbf{\Sigma}^0_\alpha$ in X whenever $f : X \to Y$ is a continuous function).

If the Polish space X is countable, then of course every $A \in X$ is a Borel set, in fact an F_σ set. Uncountable Polish spaces are more interesting: Not all sets are Borel, and the collections $\mathbf{\Sigma}^0_\alpha$ form a hierarchy. We show below that for each α, $\mathbf{\Sigma}^0_\alpha \not\subset \mathbf{\Pi}^0_\alpha$, and hence $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for all $\alpha < \omega_1$.

While we prove the next lemma for the special case when X is the Baire space, the proof can be modified to prove the same result for any uncountable Polish space.

Lemma 11.2. *For each $\alpha \geq 1$ there exists a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^0_\alpha$ (in $\mathcal{N}^2$), and that for every $\mathbf{\Sigma}^0_\alpha$ set A in $\mathcal{N}$ there exists some $a \in \mathcal{N}$ such that*

(11.4)
$$A = \{x : (x, a) \in U\}.$$

U is a universal $\mathbf{\Sigma}^0_\alpha$ set.

Proof. By induction on α. To construct a universal open set in $\mathcal{N}^2$, let $G_1, \ldots,$ $G_k, \ldots$ be an enumeration of all basic open sets in $\mathcal{N}$, and let $G_0 = \emptyset$. Let

(11.5)
$$(x, y) \in U \text{ if and only if } x \in G_{y(n)} \text{ for some } n.$$

Since $U = \bigcup_{n=0}^\infty H_n$ where each $H_n = \{(x, y) : x \in G_{y(n)}\}$ is an open set in $\mathcal{N}^2$, we see that U is open. Now if G is an open set in $\mathcal{N}$, we let $a \in \mathcal{N}$ be such that $G = \bigcup_{n=0}^\infty G_{a(n)}$; then $G = \{x : (x, a) \in U\}$.

Next let U be a universal $\mathbf{\Sigma}^0_\alpha$ set, and let us construct a universal $\mathbf{\Sigma}^0_{\alpha+1}$ set V. Let us consider some continuous mapping of $\mathcal{N}$ onto the product space $\mathcal{N}^\omega$; for each $a \in \mathcal{N}$ and each n, let $a_{(n)}$ be the nth coordinate of the image of a. [For instance, let us define $a_{(n)}$ as follows: $a_{(n)}(k) = a(\Gamma(n, k))$, where Γ is the canonical one-to-one pairing function $\Gamma : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$.] Now let

(11.6)
$$(x, y) \in V \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U.$$

Since $V = \bigcup_{n=0}^\infty H_n$ where each $H_n = \{(x, y) : (x, y_{(n)}) \notin U\}$ is a $\mathbf{\Pi}^0_\alpha$ set, we see that V is $\mathbf{\Sigma}^0_{\alpha+1}$. If A is a $\mathbf{\Sigma}^0_{\alpha+1}$ set in $\mathcal{N}$, then $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi}^0_\alpha$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x, a) \in V\}$.

Finally, let α be a limit ordinal, and let U_β, $1 \leq \beta \leq \alpha$, be universal $\mathbf{\Sigma}^0_\beta$ sets. Let $1 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha_n = \alpha$. Let

(11.7)
$$(x, y) \in U \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U_{\alpha_n}$$

(where $a_{(n)}$ has the same meaning as above). The set U is $\mathbf{\Sigma}^0_\alpha$. If A is a $\mathbf{\Sigma}^0_\alpha$ set in $\mathcal{N}$ then $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi}^0_{\alpha_n}$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U_{\alpha_n}\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x, a) \in U\}$. □

Corollary 11.3. *For every $\alpha \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Let us consider the set

(11.8)
$$A = \{x : (x, x) \in U\}.$$

Clearly, A is a $\mathbf{\Sigma}^0_\alpha$ set. If A were $\mathbf{\Pi}^0_\alpha$, then its complement would be $\mathbf{\Sigma}^0_\alpha$ and there would be some a such that

$$A = \{x : (x, a) \notin U\}.$$

But this contradicts (11.8): Simply let $x = a$. □

Analytic Sets

While the collection of Borel sets of reals is closed under Boolean operations, and countable unions and intersections, it is not closed under continuous images: As we shall learn presently, the image of a Borel set by a continuous function need not be a Borel set. We shall now investigate the continuous images of Borel sets.

Definition 11.4. A subset of A of a Polish space X is *analytic* if there exists a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

Definition 11.5. The *projection* of a set $S \subset X \times Y$ (into X) is the set $P = \{x \in X : \exists y\,(x, y) \in S\}$.

The following lemma gives equivalent definitions of analytic sets.

Lemma 11.6. *The following are equivalent, for any set A in a Polish space X:*

(i) *A is the continuous image of $\mathcal{N}$.*
(ii) *A is the continuous image of a Borel set B (in some Polish space Y).*
(iii) *A is the projection of a Borel set in $X \times Y$, for some Polish space Y.*
(iv) *A is the projection of a closed set in $X \times \mathcal{N}$.*

Proof. We shall prove that every closed set (in any Polish space) is analytic and that every Borel set is the projection of a closed set in $X \times \mathcal{N}$. Then the lemma follows: Since the projection map $\pi : X \times Y \to X$ defined by $\pi(x, y) = x$ is continuous, it follows that every Borel set is analytic and that the continuous image of a Borel set is analytic. Conversely, if $A \subset X$ is an analytic set, $A = f(\mathcal{N})$, then A is the projection of the set $\{(f(x), x) : x \in \mathcal{N}\}$ which is a closed set in $X \times \mathcal{N}$.

In order to prove that every closed set is analytic, note that every closed set in a Polish space is itself a Polish space, and thus a continuous image of $\mathcal{N}$ by Lemma 11.1.

In order to prove that every Borel set in X is the projection of a closed set in $X \times \mathcal{N}$, it suffices to show that the family P of all subsets of X that are such projections contains all closed sets, all open sets, and is closed under countable unions and intersections.

Clearly, the family P contains all closed sets. Moreover, every open set is a countable union of closed sets; thus it suffices to show that P is closed under $\bigcup_{n=0}^\infty$ and $\bigcap_{n=0}^\infty$.

Recall the continuous mapping $a \mapsto \langle a_{(n)} : n \in \boldsymbol{N}\rangle$ of $\mathcal{N}$ onto $\mathcal{N}^\omega$ from Lemma 11.2, and also recall that the inverse image of a closed set under a continuous function is closed. Let A_n, $n < \omega$, be projections of closed sets in $X \times \mathcal{N}$; we shall show that $\bigcup_{n=0}^\infty A_n$ and $\bigcap_{n=0}^\infty A_n$ are projections of closed sets.

For each n, let $F_n \subset X \times \mathcal{N}$ be a closed set such that

$$A_n = \{x : \exists a\,(x, a) \in F_a\}.$$

Thus

$$\begin{aligned} x \in \bigcup_{n=0}^\infty A_n &\leftrightarrow \exists n\,\exists a\,(x, a) \in F_n \\ &\leftrightarrow \exists a\,\exists b\,(x, a) \in F_{b(0)} \\ &\leftrightarrow \exists c\,(x, c_{(0)}) \in F_{c_{(1)}(0)}, \end{aligned}$$

and

$$\begin{aligned} x \in \bigcap_{n=0}^\infty A_n &\leftrightarrow \forall n\,\exists a\,(x, a) \in F_n \\ &\leftrightarrow \exists c\,\forall n\,(x, c_{(n)}) \in F_n \\ &\leftrightarrow \exists c\,(x, c) \in \bigcap_{n=0}^\infty \{(x, c) : (x, c_{(n)}) \in F_n\}. \end{aligned}$$

Hence $\bigcup_{n=0}^\infty A_n$ is the projection of the closed set

$$\{(x, c) : (x, c_{(0)}) \in F_{c_{(1)}(0)}\}$$

and $\bigcap_{n=0}^\infty A_n$ is the projection of an intersection of closed sets. □

The Suslin Operation $\mathcal{A}$

For each $a \in \omega^\omega$, $a{\restriction}n$ is the finite sequence $\langle a_k : k < n\rangle$. For each $s \in Seq$, $O(s)$ is the basic open set $\{a \in \mathcal{N} : a{\restriction}n = s\}$ of the Baire space. $O(s)$ is both open and closed. For every set A in a Polish space, $\overline{A}$ denotes the closure of A.

Let $\{A_s : s \in Seq\}$ be a collection of sets indexed by elements of Seq. We define

$$\mathcal{A}\{A_s : s \in Seq\} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^\infty A_{a{\restriction}n} \tag{11.9}$$

Note that if $\{B_s : s \in Seq\}$ is arbitrary, then

$$\bigcup_{a \in \omega^\omega} \bigcap_{n=0}^\infty B_{a{\restriction}n} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^\infty (B_{a{\restriction}0} \cap B_{a{\restriction}1} \cap \ldots \cap B_{a{\restriction}n})$$

and hence $\mathcal{A}\{B_s : s \in Seq\} = \mathcal{A}\{A_s : s \in Seq\}$ where the sets A_s are finite intersections of the sets B_s and satisfy the following condition:

$$\text{if } s \subset t, \text{ then } A_s \supset A_t. \tag{11.10}$$

Thus we shall restrict our use of $\mathcal{A}$ to families that satisfy condition (11.10). The operation $\mathcal{A}$ is called the *Suslin operation*.

Lemma 11.7. *A set A in a Polish space is analytic if and only if A is the result of the operation $\mathcal{A}$ applied to a family of closed sets.*

Proof. First we show that if F_s, $s \in Seq$, are closed sets in a Polish space X, then $A = \mathcal{A}\{F_s : s \in Seq\}$ is analytic. We have

$$\begin{aligned} x \in A &\leftrightarrow \exists a \in \mathcal{N}\, x \in \bigcap_{n=0}^{\infty} F_{a\restriction n} \\ &\leftrightarrow \exists a\, (x,a) \in \bigcap_{n=0}^{\infty} B_n \end{aligned}$$

where $B_n = \{(x,a) : x \in F_{a\restriction n}\}$. Clearly, each B_n is a Borel set in $X \times \mathcal{N}$ and hence A is analytic.

Conversely, let $A \subset X$ be analytic. There is a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$. Notice that for every $a \in \mathcal{N}$,

$$\bigcap_{n=0}^{\infty} f(O(a\restriction n)) = \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))} = \{f(a)\}. \tag{11.11}$$

Thus

$$A = f(\mathcal{N}) = \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))},$$

and hence A is the result of the operation $\mathcal{A}$ applied to the closed sets $\overline{f(O(s))}$ (which satisfy the condition (11.10)). ◻

It follows from the preceding lemmas that the collection of all analytic sets in a Polish space is closed under countable unions and intersections, continuous images, and inverse images, and the Suslin operation (the last statement is proved like the first part of Lemma 11.7). It is however not the case that the complement of an analytic set is analytic (if X is an uncountable Polish space). In the next section we establish exactly that; we show that there exists an analytic set (in $\mathcal{N}$) whose complement is not analytic.

The Hierarchy of Projective Sets

For each $n \geq 1$, we define the collections $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ of subsets of a Polish space X as follows:

$$\begin{aligned} \mathbf{\Sigma}^1_1 &= \text{the collection of all analytic sets,} \\ \mathbf{\Pi}^1_1 &= \text{the complements of analytic sets,} \\ \mathbf{\Sigma}^1_{n+1} &= \text{the collection of the projections of all } \mathbf{\Pi}^1_n \text{ sets in } X \times \mathcal{N}, \\ \mathbf{\Pi}^1_n &= \text{the complements of the } \mathbf{\Sigma}^1_n \text{ sets in } X, \\ \mathbf{\Delta}^1_n &= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n. \end{aligned} \tag{11.12}$$

The sets belonging to one of the collections $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ are called *projective sets*. It is easily seen that for every n, $\mathbf{\Delta}^1_n \subset \mathbf{\Sigma}^1_n \subset \mathbf{\Delta}^1_{n+1}$ and $\mathbf{\Delta}^1_n \subset \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1}$.

We shall show that for each n there is a $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}$ that is not $\mathbf{\Pi}^1_n$; thus the above inclusions are proper inclusions.

Lemma 11.8. *For each $n \geq 1$, there exists a universal $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}^2$; i.e., a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^1_n$ and that for every $\mathbf{\Sigma}^1_n$ set A in $\mathcal{N}$ there exists some $v \in \mathcal{N}$ such that*

$$A = \{x : (x,v) \in U\}.$$

Proof. Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$. If $n = 1$, let V be a universal $\mathbf{\Sigma}^0_1$ set; if $n > 1$, let V be, by the induction hypothesis, a universal $\mathbf{\Sigma}^1_{n-1}$ set. Let

$$(x,y) \in U \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (h(x,a),y) \notin V. \tag{11.13}$$

Since the set $\{(x,y,a) : (h(x,a),y) \notin V\}$ is closed (if $n = 1$) or $\mathbf{\Pi}^1_{n-1}$ (if $n > 1$), U is $\mathbf{\Sigma}^1_n$.

If $A \subset \mathcal{N}$ is $\mathbf{\Sigma}^1_n$, there is a closed (or $\mathbf{\Pi}^1_{n-1}$) set B such that

$$x \in A \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (x,a) \in B. \tag{11.14}$$

The set $C = \mathcal{N} - h(B)$ is open (or $\mathbf{\Sigma}^1_{n-1}$) in $\mathcal{N}$ and since V is universal, there exists a v such that $C = \{u : (u,v) \in V\}$. Then by (11.13), we have

$$\begin{aligned} x \in A &\leftrightarrow (\exists a \in \mathcal{N})\, (x,a) \in B \leftrightarrow (\exists a \in \mathcal{N})\, h(x,a) \notin C \\ &\leftrightarrow (\exists a \in \mathcal{N})\, (h(x,a),v) \notin V \leftrightarrow (x,v) \in U. \end{aligned}$$

Hence U is a universal $\mathbf{\Sigma}^1_n$ set. ◻

Corollary 11.9. *For each $n \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Pi}^1_n$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^1_n$ set and let

$$A = \{x : (x,x) \in U\}$$ ◻

The collection of all $\mathbf{\Delta}^1_1$ sets in a Polish space is a σ-algebra and contains all Borel sets. It turns out that $\mathbf{\Delta}^1_1$ is exactly the collection of all Borel sets.

Theorem 11.10 (Suslin). *Every analytic set whose complement is also analytic is a Borel set. Thus $\mathbf{\Delta}^1_1$ is the collection of all Borel sets.*

Let X be a Polish space and let A and B be two disjoint analytic sets in X. We say that A and B are *separated* by a Borel set if there exists a Borel set D such that $A \subset D$ and $B \subset X - D$.

Lemma 11.11. *Any two disjoint analytic sets are separated by a Borel set.*

This lemma is often called "the $\boldsymbol{\Sigma}^1_1$-Separation Principle." It clearly implies Suslin's Theorem since if A is an analytic set such that $B = X - A$ is also analytic, A and B are separated by a Borel set D and we clearly have $D = A$.

Proof. First we make the following observation: If $A = \bigcup_{n=0}^\infty A_n$ and $B = \bigcup_{m=0}^\infty B_m$ are such that for all n and m, A_n and B_m are separated, then A and B are separated. This is proved as follows: For each n and each m, let $D_{n,m}$ be a Borel set such that $A_n \subset D_{n,m} \subset X - B_m$. Then A and B are separated by the Borel set $D = \bigcup_{n=0}^\infty \bigcap_{m=0}^\infty D_{n,m}$.

Let A and B be two disjoint analytic sets in X. Let f and g be continuous functions such that $A = f(\mathcal{N})$ and $B = g(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$ and $B_s = g(O(s))$; the sets A_s and B_s are all analytic sets. For each s we have $A_s = \bigcup_{n=0}^\infty A_{s^\frown n}$ and $B_s = \bigcup_{m=0}^\infty B_{s^\frown m}$. If $a \in \omega^\omega$, then

$$\{f(a)\} = \bigcap_{n=0}^\infty f(O(a{\restriction}n)) = \bigcap_{n=0}^\infty A_{a{\restriction}n},$$

and similarly for the sets B_s.

Let $a, b \in \omega^\omega$ be arbitrary. Since $f(\mathcal{N})$ and $g(\mathcal{N})$ are disjoint, we have $f(a) \neq g(b)$. Let G_a and G_b be two disjoint open neighbourhoods of $f(a)$ and $g(b)$, respectively. By the continuity of f and g there exists some n such that $A_{a{\restriction}n} \subset G_a$ and $B_{b{\restriction}n} \subset G_b$. It follows that the sets $A_{a{\restriction}n}$ and $B_{b{\restriction}n}$ are separated by a Borel set.

We shall now show, by contradiction, that the sets A and B are separated by a Borel set. If A and B are not separated, then because $A = \bigcup_{n=0}^\infty A_{\langle n\rangle}$ and $B = \bigcup_{m=0}^\infty B_{\langle m\rangle}$, there exist n_0 and m_0 such that the sets $A_{\langle n_0\rangle}$ and $B_{\langle m_0\rangle}$ are not separated. Then similarly there exist n_1 and m_1 such that the sets $A_{\langle n_0,n_1\rangle}$ and $B_{\langle m_0,m_1\rangle}$ are not separated, and so on. In other words, there exist $a = \langle n_0, n_1, n_2, \ldots\rangle$ and $b = \langle m_0, m_1, m_2, \ldots\rangle$ such that for every k, the sets $A_{\langle n_0,\ldots,n_k\rangle}$ and $B_{\langle m_0,\ldots,m_k\rangle}$ are not separated. This is a contradiction since in the preceding paragraph we proved exactly the opposite: There is k such that $A_{a{\restriction}k}$ and $B_{b{\restriction}k}$ are separated. □

Lebesgue Measure

We shall now review basic properties of Lebesgue measure on the n-dimensional Euclidean space.

The standard way of defining Lebesgue measure is to define first the *outer measure* $\mu^*(X)$ of a set $X \subset \boldsymbol{R}^n$ as the infimum of all possible sums $\sum\{v(I_k) : k \in \boldsymbol{N}\}$ where $\{I_k : k \in \boldsymbol{N}\}$ is a collection of n-dimensional intervals such that $X \subset \bigcup_{k=0}^\infty I_k$, and $v(I)$ denotes the volume of I. For each X, $\mu^*(X) \geq 0$ and possibly $= \infty$. A set X is *null* if $\mu^*(X) = 0$.

A set $A \subset \boldsymbol{R}^n$ is *Lebesgue measurable* if for each $X \subset \boldsymbol{R}^n$,

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A).$$

For a measurable set A, we write $\mu(A)$ instead of $\mu^*(A)$ and call $\mu(A)$ the *Lebesgue measure of A*.

The standard development of the theory of Lebesgue measure gives the following facts:

(11.15)
- (i) Every interval is Lebesgue measurable, and its measure is equal to its volume.
- (ii) The Lebesgue measurable sets form a σ-algebra; hence every Borel set is measurable.
- (iii) μ is σ-additive: If A_n, $n < \omega$, are pairwise disjoint and measurable, then
$$\mu\Big(\bigcup_{n=0}^\infty A_n\Big) = \sum_{n=0}^\infty \mu(A_n).$$
- (iv) μ is σ-finite: If A is measurable, then there exist measurable sets A_n, $n < \omega$, such that $A = \bigcup_{n=0}^\infty A_n$, and $\mu(A_n) < \infty$ for each n.
- (v) Every null set is measurable. The null sets form a σ-ideal and contain all singletons.
- (vi) If A is measurable, then
$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}.$$
- (vii) If A is measurable, then there is an F_σ set F and a G_δ set G such that $F \subset A \subset G$ and $G - F$ is null.

This last property gives this characterization of Lebesgue measurable sets: A set $A \subset \boldsymbol{R}^n$ is measurable if and only if there is a Borel set B such that the symmetric difference $A \triangle B = (A - B) \cup (B - A)$ is null.

One consequence of this is that if we denote by $\mathcal{B}$ the σ-algebra of Borel sets and by $\mathcal{M}$ the σ-algebra of measurable sets, and if I_μ is the ideal of all null sets, then $\mathcal{B}/I_\mu = \mathcal{M}/I_\mu$. The Boolean algebra $\mathcal{B}/I_\mu$ is σ-complete; and since a familiar argument shows that I_μ is (as an ideal in $\mathcal{M}$) σ-saturated, we conclude that $\mathcal{B}/I_\mu$ is a complete Boolean algebra. We shall return to this in Part II.

Assuming the Axiom of Choice one can show that there exists a set of reals that is not Lebesgue measurable. One such example is the Vitali set in Exercise 10.1. As another example there exists a set $X \subset \boldsymbol{R}^n$ such that neither X nor its complement has a perfect subset (see Exercise 5.1 for a construction of such a set). The set X is not measurable: Otherwise, e.g., $\mu(X) > 0$ and by (11.15)(vi) there is a closed $K \subset X$ such that $\mu(K) > 0$; thus K is uncountable and hence contains a perfect subset, a contradiction.

However, we shall show in Part II that it is consistent (with ZF + DC) that all sets or reals are Lebesgue measurable.

We conclude this review of Lebesgue measurability with two lemmas. One is the well-known Fubini Theorem, and we state it here, without proof, for the sake of completeness. The other lemma will be used in the proof of Theorem 11.18 below.

If A is a subset of the plane $\boldsymbol{R}^2$ and $x \in \boldsymbol{R}$, let A_x denote the set $\{y : (x, y) \in A\}$.

Lemma 11.12. *Let $A \subset \boldsymbol{R}^2$ be a measurable set. Then A is null if and only if for almost all x, A_x is null (i.e., the set $\{x : A_x$ is not null$\}$ is null).* □

Lemma 11.13. *For any set $X \subset \boldsymbol{R}^n$ there exists a measurable set $A \supset X$ with the property that whenever $Z \subset A - X$ is measurable, then Z is null.*

Proof. If $\mu^*(X) < \infty$, then because $\mu^*(X) = \inf\{\mu(A) : A$ is measurable and $A \supset X\}$, there is a measurable $A \supset X$ such that $\mu(A) = \mu^*(X)$; clearly such an A will do. If $\mu^*(X) = \infty$, there exist pairwise disjoint X_n such that $X = \bigcup_{n=0}^{\infty} X_n$ and that for each n, $\mu^*(X_n) < \infty$. Let $A_n \supset X_n$, $n < \omega$, be measurable sets such that $\mu(A_n) = \mu^*(X_n)$, and let $A = \bigcup_{n=0}^{\infty} A_n$. □

It should be mentioned that the main results of descriptive set theory on Lebesgue measure can be proved in a more general context, namely for reasonable σ-additive measures on Polish spaces. An example of such a measure is the product measure in the Cantor space $\{0, 1\}^\omega$.

The Property of Baire

In Chapter 4 we proved the Baire Category Theorem (Theorem 4.8): The intersection of countably many dense open sets of reals is nonempty. It is fairly easy to see that the proof works not only for the real line $\boldsymbol{R}$ but for any Polish space.

Let us consider a Polish space X. Let us call a set $A \subset X$ *nowhere dense* if the complement of A contains a dense open set. Note that A is nowhere dense just in case for every nonempty open set G, there is a nonempty open set $H \subset G$ such that $A \cap H = \emptyset$. A set A is nowhere dense if and only if its closure $\overline{A}$ is nowhere dense.

A set $A \subset X$ is *meager* (or of *first category*) if A is the union of countably many nowhere dense sets. A nonmeager set is called a set of *second category.*

The Baire Category Theorem states in effect that in a Polish space every nonempty open set is of second category.

The meager sets form a σ-ideal. Moreover, in case of $\boldsymbol{R}^n$, $\mathcal{N}$, or the Cantor space, every singleton $\{x\}$ is nowhere dense and so the ideal of meager sets contains all countable sets.

Definition 11.14. A set A has the *Baire property* if there exists an open set G such that $A \triangle G$ is meager.

Clearly, every meager set has the Baire property. Note that if G is open, then $\overline{G} - G$ is nowhere dense. Hence if $A \triangle G$ is meager then $(X - A) \triangle (X - \overline{G}) = A \triangle \overline{G}$ is meager, and it follows that the complement of a set with the Baire property also has the Baire property. It is also easy to see that the union of countably many sets with the Baire property has the Baire property and we have:

Lemma 11.15. *The sets having the Baire property form a σ-algebra; hence every Borel set has the Baire property.* □

If $\mathcal{B}$ denotes the σ-algebra of Borel sets, and if we denote by $\mathcal{C}$ the σ-algebra of sets with the Baire property, and if I is the σ-ideal of meager sets, we have $\mathcal{B}/I = \mathcal{C}/I$. Note that the algebra $\mathcal{B}/I$ is σ-saturated: Let O be a countable topology base for X. For each nonmeager set X with the Baire property there exists $G \in O$ such that $G - X$ is meager. Thus the set $D = \{[G] : G \in O\}$ of equivalence classes is a dense set in $\mathcal{B}/I$. Hence $\mathcal{B}/I$ is σ-saturated and is a complete Boolean algebra.

The Axiom of Choice implies that sets without the Baire property exist. For instance, the Vitali set (Exercise 10.1) is such, see Exercise 11.7.

If $X \subset \boldsymbol{R}^n$ is such that neither X nor its complement has a perfect subset, then X does not have the Baire property: Otherwise, e.g., X is of second category and hence X contains a G_δ subset G of second category. Now G is uncountable, and this is a contradiction since as we shall prove in Theorem 11.18, every uncountable Borel set (even analytic) has a perfect subset.

The following two lemmas are analogs of Lemmas 11.12 and 11.13. The first one, although not very difficult to prove, is again stated without proof.

Lemma 11.16. *Let $A \subset \boldsymbol{R}^2$ have the property of Baire. Then A is meager if and only if A_x is meager for all x except a meager set.* □

Lemma 11.17. *For any set S in a Polish space X, there exists a set $A \supset S$ that has the Baire property and such that whenever $Z \subset A - S$ has the Baire property, then Z is meager.*

Proof. Let us consider a fixed countable topology basis O for X. Let $S \subset X$. Let

$$D(S) = \{x \in X : \text{for every } U \in O \text{ such that } x \in U,\ U \cap S \text{ is not meager}\}.$$

Note that the complement of $D(S)$ is the union of open sets and hence open; thus $D(S)$ is closed.

The set $S - D(S)$ is the union of all $S \cap U$ where $U \in O$ and $S \cap U$ is meager; since O is countable, $X - D(S)$ is meager. Let

$$A = S \cup D(S).$$

Since $A = (S - D(S)) \cup D(S)$ is the union of a meager and a closed set, A has the Baire property.

Let $Z \subset A - S$ have the Baire property; we shall show that Z is meager. Otherwise there is $U \in O$ such that $U - Z$ is meager; hence $U \cap S$ is meager. Since $U \cap Z \neq \emptyset$ and $Z \subset D(S)$, there is $x \in U$ such that $x \in D(S)$, and hence $U \cap S$ is not meager, a contradiction. □

Although both "null" and "meager" mean in a sense "negligible," see Exercise 11.8 that shows that the real line can be decomposed into a null set and a meager set.

Analytic Sets: Measure, Category, and the Perfect Set Property

Theorem 11.18.

(i) *Every analytic set of reals is Lebesgue measurable.*
(ii) *Every analytic set has the Baire property.*
(iii) *Every uncountable analytic set contains a perfect subset.*

Corollary 11.19. *Every $\mathbf{\Pi}^1_1$ set of reals is Lebesgue measurable and has the Baire property.* □

Corollary 11.20. *Every analytic* (*and in particular every Borel*) *set is either at most countable or has cardinality* $\mathfrak{c}$. □

We prove (ii) and (iii) for an arbitrary Polish space. The proof of (i) is general enough to work for other measures (in Polish spaces) as well.

Proof. The proof of (i) and (ii) is exactly the same and uses either Lemma 11.13 or Lemma 11.17 (and basic facts on Lebesgue measure and the Baire property). We give the proof of (i) and leave (ii) to the reader.

Let A be an analytic set of reals (or a subset of $\boldsymbol{R}^n$). Let $f : \mathcal{N} \to \boldsymbol{R}$ be a continuous function such that $A = f(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$. We have

$$A = \mathcal{A}\{A_s : s \in Seq\} = \mathcal{A}\{\overline{A_s} : s \in Seq\}, \tag{11.16}$$

and for every $s \in Seq$,

$$A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n}. \tag{11.17}$$

By Lemma 11.13, there exists for each $s \in Seq$ a measurable set $B_s \supset A_s$ such that every measurable $Z \subset B_s - A_s$ is null. Since $\overline{A_s}$ is measurable, we may actually find B_s such that $A_s \subset B_s \subset \overline{A_s}$.

Let $B = B_\emptyset$. Since B is measurable, it suffices to show that $B - A$ is a null set. Notice that because $A_s \subset B_s \subset \overline{A_s}$, and because (11.16) holds, we have

$$A = \mathcal{A}\{B_s : s \in Seq\}.$$

Thus

$$B - A = B - \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} B_{a \restriction n}.$$

We claim that

$$B - \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} B_{a \restriction n} \subset \bigcup_{s \in Seq} \Bigl(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Bigr). \tag{11.18}$$

To prove (11.18), assume that $x \in B$ is such that x is not a member of the right-hand side. Then for every s, if $x \in B_s$, then $x \in B_{s^\frown k}$ for some k. Hence there is k_0 such that $x \in B_{\langle k_0 \rangle}$, then there is k_1 such that $x \in B_{\langle k_0, k_1 \rangle}$, etc. Let $a = \langle k_0, k_1, k_2, \ldots \rangle$; we have $x \in \bigcap_{n=0}^{\infty} B_{a \restriction n}$ and hence x is not a member of the left-hand side.

Thus we have

$$B - A \subset \bigcup_{s \in Seq} \Bigl(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Bigr).$$

Since Seq is a countable set, it suffices to show that each $B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}$ is null. Let $s \in Seq$, and let $Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}$. We have

$$Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \subset B_s - \bigcup_{k=0}^{\infty} A_{s^\frown k} = B_s - A_s.$$

Now because $Z \subset B_s - A_s$ and because Z is measurable, Z must be null.

(iii) The proof is a variant of the Cantor-Bendixson argument for closed sets in the Baire space. Recall that every closed set F in $\mathcal{N}$ is of the form $F = [T] = \{a : \forall n\ a \restriction n \in T\}$, where T is a tree, $T \subset Seq$. For each tree $T \subset Seq$ and each $s \in Seq$, let T_s denote the tree $\{t \in T : t \subset s \text{ or } s \subset t\}$; note that $[T_s] = [T] \cap O(s)$.

Let A be an analytic set (in a Polish space X), and let f be a continuous function such that $A = f(\mathcal{N})$. For each tree $T \subset Seq$, we define

$$T' = \{s \in T : f([T_s]) \text{ is uncountable}\}.$$

For each $\alpha < \omega_1$, we define $T^{(\alpha)}$ as follows:

$$T^{(0)} = Seq, \qquad T^{(\alpha+1)} = (T^{(\alpha)})',$$
$$T^{(\alpha)} = \bigcap_{\beta < \alpha} T^{(\beta)} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let $\alpha < \omega_1$ be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$. If $T^{(\alpha)} = \emptyset$, then

$$A = \bigcup_{\beta < \alpha} \{f([T_s^{(\beta)}]) : s \in T^{(\beta)} - T^{(\beta+1)}\},$$

and hence A is countable. Thus if A is uncountable, $T^{(\alpha)}$ is nonempty and for every $s \in T^{(\alpha)}$, $f([T_s^{(\alpha)}])$ is uncountable. In this case, we shall find a perfect subset of A.

Let $s \in T^{(\alpha)}$ be arbitrary. Since $f([T_s^{(\alpha)}])$ has at least two elements, there exist $s_{\langle 0\rangle} \supset s$ and $s_{\langle 1\rangle} \supset s$ (in $T^{(\alpha)}$) such that $f([T_{s_{\langle 0\rangle}}^{(\alpha)}])$ and $f([T_{s_{\langle 1\rangle}}^{(\alpha)}])$ are disjoint. Then there are $s_{\langle 0,0\rangle} \supset s_{\langle 0\rangle}$ and $s_{\langle 0,1\rangle} \supset s_{\langle 0\rangle}$, and $s_{\langle 1,0\rangle} \supset s_{\langle 1\rangle}$, $s_{\langle 1,1\rangle} \supset s_{\langle 1\rangle}$ such that the four sets $f([T_{s_{\langle i,j\rangle}}^{(\alpha)}])$, $i, j = 0, 1$ are pairwise disjoint. In this fashion we construct $s_t \in T^{(\alpha)}$ for each finite 0–1 sequence t. These elements s_t generate a subtree $U = \{s : s \subset s_t \text{ for some } t\}$ of $T^{(\alpha)}$ such that (1) U is perfect, (2) every s has at most two immediate successors in U (hence $[U]$ is a compact set in $\mathcal{N}$), and (3) f is one-to-one on $[U]$.

Let P be the image of $[U]$ under the function f. Since $[U]$ is compact and f is continuous, P is also compact, and hence closed. Moreover, P has no isolated points because $[U]$ is perfect and f is continuous. Thus P is a perfect subset of A. □

Exercises

11.1. The operations $\bigcup_{n=0}^\infty$ and $\bigcap_{n=0}^\infty$ are special cases of the operation $\mathcal{A}$.

11.2. Let A_s, $s \in \mathit{Seq}$, be Borel sets satisfying (11.10) and the additional condition: For each $s \in \mathit{Seq}$ and all $n \neq m$, $A_{s^\frown n} \cap A_{s^\frown m} = \emptyset$. Then $\mathcal{A}\{A_s : s \in \mathit{Seq}\}$ is a Borel set.

[$\bigcup_{a\in\omega^\omega} \bigcap_{n=0}^\infty A_{a\restriction n} = \bigcap_{n=0}^\infty \bigcup\{A_s : \text{length}(s) = n\}$.]

11.3. Let A_n, $n = 0, 1, 2, \ldots$, be pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets D_n such that $A_n \subset D_n$ for all n.

[Modify the proof of Lemma 11.11.]

11.4. If A is a null set and $a_0 \ge a_1 \ge \ldots \ge a_n \ge \ldots$ is a sequence of positive numbers with $\lim_n a_n = 0$, then there exists a sequence G_n, $n = 0, 1, \ldots$, of finite unions of open intervals such that $A \subset \bigcup_{n=0}^\infty G_n$ and $\mu(G_n) < a_n$ for each n. Moreover, the intervals can be required to have rational endpoints.

[First find a sequence of open intervals I_k such that $A \subset \bigcup_{k=0}^\infty I_k$ and $\sum_{k=0}^\infty \mu(I_k) \le a_0$.]

11.5. For every set A with the Baire property, there exist a G_δ set G and an F_σ set F such that $G \subset A \subset F$ and such that $F - G$ is meager.

[Note that every meager set is included in a meager F_σ set.]

11.6. For every set A with the Baire property, there exists a unique regular open set U such that $A \bigtriangleup U$ is meager.

[An open set U is *regular* if $U = \text{int}(\overline{U})$.]

11.7. The Vitali set M from Exercise 10.1 does not have the Baire property.

["Meager" and "Baire property" are invariant under translation. If M has the Baire property, then there is an interval (a, b) such that $(a, b) - M$ is meager. Then $(a, b) \cap M_q$ is meager for all rational $q \neq 0$, hence each $M \cap (a - q, b - q)$ is meager, hence M is meager, hence each M_q is meager; a contradiction since $\boldsymbol{R} = \bigcup_{q\in\boldsymbol{Q}} M_q$.]

11.8. There is a null set of reals whose complement is meager.

[Let $q_1, q_2, \ldots$ be an enumeration of the rationals. For each $n \ge 1$ and $k \ge 1$, let $I_{n,k}$ be the open interval with center q_n and length $1/(k \cdot 2^n)$. Let $D_k = \bigcup_{n=1}^\infty I_{n,k}$, and $A = \bigcap_{k=1}^\infty D_k$. Each D_k is open and dense, and $\mu(D_k) \le 1/k$. Hence A is null and $\boldsymbol{R} - A$ is meager.]

Historical Notes

Borel sets were introduced by Borel in [1905]. Lebesgue in [1905] proved in effect Lemma 11.2. Suslin's discovery of an error in a proof in Lebesgue's article led to a construction of an analytic non-Borel set and introduction of the operation $\mathcal{A}$. The basic results on analytic sets as well as Theorem 11.10 appeared in Suslin's article [1917].

Projective sets were introduced by Luzin [1925] and [1927a], and Sierpiński [1925] and [1927]. The present notation ($\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$) appeared first in the paper [1959] of Addison who noticed the analogy between Luzin's hierarchy of projective sets and Kleene's hierarchy of analytic predicates [1955].

Lemma 11.8: Luzin [1930].

Lemma 11.11: Luzin [1927b].

For detailed treatment of Lebesgue measure, we refer the reader to Halmos' book [1950]; Lebesgue introduced his measure and integral in his thesis [1902]. Sets of first and second category were introduced by Baire [1899].

Lemmas 11.13 and 11.17: Marczewski [1930a].

Lemma 11.16: Kuratowski and Ulam [1932].

Theorem 11.18(i) (measurability of analytic sets) is due to Luzin [1917]. Theorem 11.18(ii) (Baire property) is due to Luzin and Sierpiński [1923] and Theorem 11.18(iii) (perfect subsets) is due to Suslin; cf. Luzin [1930]. The present proof of (i) and (ii) follows Marczewski [1930a]. Prior to Suslin (and following the Cantor-Bendixson Theorem for closed sets) Young proved in [1906] the perfect subset result for G_δ and F_σ sets; and Hausdorff [1916] and Aleksandrov [1916] proved the same for Borel sets.

12. Models of Set Theory

Modern set theory uses extensively construction of models to establish relative consistency of various axioms and conjectures. As the techniques often involve standard model-theoretic concepts, we assume familiarity with basic notions of models and satisfaction, submodels and embeddings, as well as Skolem functions, direct limit and ultraproducts. We shall review the basic notions, notation and terminology of model theory.

Review of Model Theory

A *language* is a set of symbols: relation symbols, function symbols, and constant symbols:

$$\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}.$$

Each P is assumed to be an n-placed relation for some integer $n \geq 1$; each F is an m-placed function symbol for some $m \geq 1$.

Terms and *formulas* of a language $\mathcal{L}$ are certain finite sequences of symbols of $\mathcal{L}$, and of *logical symbols* (identity symbol, parentheses, variables, connectives, and quantifiers). The set of all terms and the set of all formulas are defined by recursion. If the language is countable (i.e., if $|\mathcal{L}| \leq \aleph_0$), then we may identify the symbols of $\mathcal{L}$, as well as the logical symbols, with some hereditarily finite sets (elements of V_ω); then formulas are also hereditarily finite.

A *model* for a given language $\mathcal{L}$ is a pair $\mathfrak{A} = (A, \mathcal{I})$, where A is the universe of $\mathfrak{A}$ and $\mathcal{I}$ is the *interpretation* function which maps the symbols of $\mathcal{L}$ to appropriate relations, functions, and constants in A. A model for $\mathcal{L}$ is usually written in displayed form as

$$\mathfrak{A} = (A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

By recursion on length of terms and formulas one defines the *value* of a term

$$t^{\mathfrak{A}}[a_1, \ldots, a_n]$$

and *satisfaction*

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]$$

where t is a term, φ is a formula, and $\langle a_1, \ldots, a_n \rangle$ is a finite sequence in A.

Two models $\mathfrak{A} = (A, P, \ldots, F, \ldots, c, \ldots)$ and $\mathfrak{A}' = (A', P', \ldots, F', \ldots, c', \ldots)$ are *isomorphic* if there is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{A}'$, that is a one-to-one function f of A onto A' such that

(i) $P(x_1, \ldots, x_n)$ if and only if $P'(f(x_1), \ldots, f(x_n))$,
(ii) $f(F(x_1, \ldots, x_n)) = F'(f(x_1), \ldots, f(x_n))$,
(iii) $f(c) = c'$,

for all relations, functions, and constants of $\mathfrak{A}$. If f is an isomorphism, then $f(t^{\mathfrak{A}}[a_1, \ldots, a_n]) = t^{\mathfrak{A}'}[f(a_1), \ldots, f(a_n)]$ for each term, and

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{A}' \vDash \varphi[f(a_1), \ldots, f(a_n)]$$

for each formula φ and all $a_1, \ldots, a_n \in A$.

A *submodel* of $\mathfrak{A}$ is a subset $B \subset A$ endowed with the relations $P^{\mathfrak{A}} \cap B^n$, …, functions $F^{\mathfrak{A}} \restriction B^m$, …, and constants $c^{\mathfrak{A}}$, …; all $c^{\mathfrak{A}}$ belong to B, and B is closed under all $F^{\mathfrak{A}}$ (if $(x_1, \ldots, x_m) \in B^m$, then $F^{\mathfrak{A}}(x_1, \ldots, x_m) \in B$).

An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is an isomorphism between $\mathfrak{B}$ and a submodel $\mathfrak{B}' \subset \mathfrak{A}$.

A submodel $\mathfrak{B} \subset \mathfrak{A}$ is an *elementary submodel*

$$\mathfrak{B} \prec \mathfrak{A}$$

if for every formula φ, and every $a_1, \ldots, a_n \in B$,

$$\mathfrak{B} \vDash \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]. \tag{12.1}$$

Two models $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent* if they satisfy the same sentences.

The key lemma in construction of elementary submodels is this: A subset $B \subset A$ forms an elementary submodel of $\mathfrak{A}$ if and only if for every formula $\varphi(u, x_1, \ldots, x_n)$, and every $a_1, \ldots, a_n \in B$,

(12.2) if $\exists a \in A$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in B$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n]$.

A function $h : A^n \to A$ is a *Skolem function* for φ if

$$(\exists a \in A)\, \mathfrak{A} \vDash \varphi[a, a_1, \ldots, a_n] \quad \text{implies} \quad \mathfrak{A} \vDash \varphi[h(a_1, \ldots, a_n), a_1, \ldots, a_n]$$

for every a_1, …, a_n. Using the Axiom of Choice, one can construct a Skolem function for every φ. If a subset $B \subset A$ is closed under (some) Skolem functions for all formulas, then B satisfies (12.2) and hence forms an elementary submodel of $\mathfrak{A}$.

Given a set of Skolem functions, one for each formula of $\mathcal{L}$, the closure of a set $X \subset A$ is a *Skolem hull* of X. It is clear that the Skolem hull of X is an elementary submodel of $\mathfrak{A}$, and has cardinality at most $|X| \cdot |\mathcal{L}| \cdot \aleph_0$. In particular, we have the following:

Theorem 12.1 (Löwenheim-Skolem). *Every infinite model for a countable language has a countable elementary submodel.* □

An *elementary embedding* is an embedding whose range is an elementary submodel.

A set $X \subset A$ is *definable over* $\mathfrak{A}$ if there exist a formula φ and some $a_1, \ldots, a_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \vDash \varphi[x, a_1, \ldots, a_n]\}.$$

We say that X is *definable in* $\mathfrak{A}$ *from* $a_1, \ldots, a_n$. If φ is a formula of x only, without parameters $a_1, \ldots, a_n$, then X is *definable* in $\mathfrak{A}$. An element $a \in A$ is *definable* (*from* $a_1, \ldots, a_n$) if the set $\{a\}$ is definable (from $a_1, \ldots, a_n$).

Gödel's Theorems

The cornerstone of modern logic are Gödel's theorems: the Completeness Theorem and two incompleteness theorems.

A set Σ of sentences of a language $\mathcal{L}$ is *consistent* if there is no formal proof of contradiction from Σ. The Completeness Theorem states that every consistent set of sentences has a model.

The First Incompleteness Theorem shows that no consistent (recursive) extension of Peano Arithmetic is complete: there exists a statement that is undecidable in the theory. In particular, if ZFC is consistent (as we believe), no additional axioms can prove or refute every sentence in the language of set theory.

The Second Incompleteness Theorem proves that sufficiently strong mathematical theories such as Peano Arithmetic or ZF (if consistent) cannot prove its own consistency. Gödel's Second Incompleteness Theorem implies that it is unprovable in ZF that there exists a model of ZF. This fact is significant for the theory of large cardinals, and we shall return to it later in this chapter.

Direct Limits of Models

An often used construction in model theory is the direct limit of a directed system of models. A *directed set* is a partially ordered set $(D, <)$ such that for every $i, j \in D$ there is a $k \in D$ such that $i \le k$ and $j \le k$.

First consider a system of models $\{\mathfrak{A}_i : i \in D\}$, indexed by a directed set D, such that for all $i, j \in D$, if $i < j$ then $\mathfrak{A}_i \prec \mathfrak{A}_j$. Let $\mathfrak{A} = \bigcup_{i \in D} \mathfrak{A}_i$; i.e., the universe of $\mathfrak{A}$ is the union of the universes of the $\mathfrak{A}_i$, $P^{\mathfrak{A}} = \bigcup_{i \in D} P^{\mathfrak{A}_i}$, etc. It is easily proved by induction on the complexity of formulas that $\mathfrak{A}_i \prec \mathfrak{A}$ for all i.

In general, we consider a *directed system* of models which consists of models $\{\mathfrak{A}_i : i \in D\}$ together with elementary embeddings $e_{i,j} : \mathfrak{A}_i \to \mathfrak{A}_j$ such that $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i < j < k$.

Lemma 12.2. *If $\{\mathfrak{A}_i, e_{i,j} : i, j \in D\}$ is a directed system of models, there exists a model $\mathfrak{A}$, unique up to isomorphism, and elementary embeddings $e_i : \mathfrak{A}_i \to \mathfrak{A}$ such that $\mathfrak{A} = \bigcup_{i \in D} e_i(\mathfrak{A}_i)$ and that $e_i = e_j \circ e_{i,j}$ for all $i < j$.*

The model $\mathfrak{A}$ is called the *direct limit* of $\{\mathfrak{A}_i, e_{i,j}\}_{i,j \in D}$.

Proof. Consider the set S of all pairs (i, a) such that $i \in D$ and $a \in A_i$, and define an equivalence relation on S by

$$(i, a) \equiv (j, b) \leftrightarrow \exists k\, (i \le k,\ j \le k \text{ and } e_{i,k}(a) = e_{j,k}(b)).$$

Let $A = S/\!\equiv$ be the set of all equivalence classes, and let $e_i(a) = [(i, a)]$ for all $i \in D$ and $a \in A_i$. The rest is routine. □

In set theory, a frequent application of direct limits involves the case when D is an ordinal number (and $<$ is its well-ordering).

Reduced Products and Ultraproducts

An important method in model theory uses filters and ultrafilters. Let S be a nonempty set and let $\{\mathfrak{A}_x : x \in S\}$ be a system of models (for a language $\mathcal{L}$). Let F be a filter on S. Consider the set

$$A = \prod_{x \in S} A_x / =_F$$

where $=_F$ is the equivalence relation on $\prod_{x \in x} A_x$ defined as follows:

(12.3) $$f =_F g \quad \text{if and only if} \quad \{x \in S : f(x) = g(x)\} \in F.$$

It follows easily that $=_F$ is an equivalence relation.

The model $\mathfrak{A}$ with universe A is obtained by interpreting the language as follows:

If $P(x_1, \ldots, x_n)$ is a predicate, let

(12.4) $P^{\mathfrak{A}}([f_1], \ldots, [f_n])$ if and only if $\{x \in S : P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in F$.

If $F(x_1, \ldots, x_n)$ is a function, let

(12.5) $F^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f]$ where $f(x) = F^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))$ for all $x \in S$.

If c is a constant, let

(12.6) $c^{\mathfrak{A}} = [f]$ where $f(x) = c^{\mathfrak{A}_x}$ for all $x \in S$.

(Note that (12.4) and (12.5) does not depend on the choice of representatives from the equivalence classes $[f_1], \ldots, [f_n]$).

The model $\mathfrak{A}$ is called a *reduced product* of $\{\mathfrak{A}_x : x \in S\}$ (by F).

Reduced products are particularly important in the case when the filter is an ultrafilter. If U is an ultrafilter on S then the reduced product defined in (12.3)–(12.6) is called the *ultraproduct* of $\{\mathfrak{A}_x : x \in S\}$ by U:

$$\mathfrak{A} = \mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}.$$

The importance of ultraproducts is due mainly to the following fundamental property.

Theorem 12.3 (Łoś). *Let U be an ultrafilter on S and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_x : x \in S\}$ by U.*

(i) *If φ is a formula, then for every $f_1, \ldots, f_n \in \prod_{x\in S} A_x$,*

$$\mathfrak{A} \vDash \varphi([f_1], \ldots, [f_n]) \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \ldots, f_n(x)]\} \in U.$$

(ii) *If σ is a sentence, then*

$$\mathfrak{A} \vDash \sigma \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \sigma\} \in U.$$

Part (ii) is a consequence of (i). Note that by the theorem, the satisfaction of φ at $[f_1], \ldots, [f_n]$ does not depend on the choice of representatives $f_1, \ldots, f_n$ for the equivalence classes $[f_1], \ldots, [f_n]$. Thus we may further abuse the notation and write

$$\mathfrak{A} \vDash \varphi[f_1, \ldots, f_n].$$

It will also be convenient to adopt a measure-theoretic terminology. If

$$\{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \ldots, f_n(x)]\} \in U$$

we say that $\mathfrak{A}_x$ satisfies $\varphi(f_1(x), \ldots, f_n(x))$ for *almost all* x, or that $\mathfrak{A}_x \vDash \varphi(f_1(x), \ldots, f_n(x))$ holds *almost everywhere*. In this terminology, Łoś's Theorem states that $\varphi(f_1, \ldots, f_n)$ holds in the ultraproduct if and only if for almost all x, $\varphi(f_1(x), \ldots, f_n(x))$ holds in $\mathfrak{A}_x$.

Proof. We shall prove (i) by induction on the complexity of formulas. We shall prove that (i) holds for atomic formulas, and then prove the induction step for $\neg$, $\wedge$, and $\exists$.

Atomic formulas. First we consider the formula $u = v$, and we have

$$\begin{aligned} \text{(12.7)} \qquad \mathfrak{A} \vDash [f] = [g] &\leftrightarrow [f] = [g] \\ &\leftrightarrow f =_U g \\ &\leftrightarrow \{x : f(x) = g(x)\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash f(x) = g(x)\} \in U. \end{aligned}$$

For a predicate $P(v_1, \ldots, v_n)$ we have

$$\begin{aligned} \text{(12.8)} \qquad \mathfrak{A} \vDash P([f_1], \ldots, [f_n]) &\leftrightarrow P^{\mathfrak{A}}([f_1], \ldots, [f_n]) \\ &\leftrightarrow \{x : P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash P(f_1(x), \ldots, f_n(x))\} \in U. \end{aligned}$$

Both (12.7) and (12.8) remain true if variables are replaced by terms, and so (i) holds for all atomic formulas.

Logical connectives. First we assume that (i) holds for φ and show that it also holds for $\neg\varphi$ (here we use that $X \in U$ if and only if $S - X \notin U$).

$$\begin{aligned} \mathfrak{A} \vDash \neg\varphi[f] &\leftrightarrow \text{not } \mathfrak{A} \vDash \varphi[f] \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi[f(x)]\} \notin U \\ &\leftrightarrow \{x : \mathfrak{A}_x \nvDash \varphi[f(x)]\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \neg\varphi[f(x)]\} \in U. \end{aligned}$$

Similarly, if (i) is true for φ and ψ, we have

$$\begin{aligned} \mathfrak{A} \vDash \varphi \wedge \psi &\leftrightarrow \mathfrak{A} \vDash \varphi \text{ and } \mathfrak{A} \vDash \psi \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi\} \in U \text{ and } \{x : \mathfrak{A}_x \vDash \psi\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi \wedge \psi\} \in U \end{aligned}$$

(The last equivalence uses this: $X \in U$ and $Y \in U$ if and only if $X \cap Y \in U$.)

Existential quantifier. We assume that (i) is true for $\varphi(u, v_1, \ldots, v_n)$ and show that it remains true for the formula $\exists u\, \varphi$. Let us assume first that

$$\text{(12.9)} \qquad \mathfrak{A} \vDash \exists u\, \varphi[f_1, \ldots, f_n].$$

Then there is $g \in \prod_{x\in S} A_x$ such that $\mathfrak{A} \vDash \varphi[g, f_1, \ldots, f_n]$, and therefore

$$\text{(12.10)} \qquad \{x : \mathfrak{A}_x \vDash \varphi[g(x), f_1(x), \ldots, f_n(x)]\} \in U,$$

and it clearly follows that

$$\text{(12.11)} \qquad \{x : \mathfrak{A}_x \vDash \exists u\, \varphi[u, f_1(x), \ldots, f_n(x)]\} \in U.$$

Now let us assume that (12.11) holds. For each $x \in S$, let $u_x \in A_x$ be such that $\mathfrak{A}_x \vDash [u_x, f_1(x), \ldots, f_n(x)]$ if such u_x exists, and arbitrary otherwise. If we define $g \in \prod_{x\in S} A_x$ by $g(x) = u_x$, then we have (12.10), and therefore

$$\mathfrak{A} \vDash \varphi[g, f_1, \ldots, f_n].$$

Now (12.9) follows. □

Let us consider now the special case of ultraproducts, when each $\mathfrak{A}_x$ is the same model $\mathfrak{A}$. Then the ultraproduct is called an *ultrapower* of $\mathfrak{A}$; denoted $\text{Ult}_U \mathfrak{A}$.

Corollary 12.4. *An ultrapower of a model $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$.*

Proof. By Theorem 12.3(ii) we have $\text{Ult}_U \mathfrak{A} \vDash \sigma$ if and only if $\{x : \mathfrak{A} \vDash \sigma\}$ is either S or empty, according to whether $\mathfrak{A} \vDash \sigma$ or not. □

We shall now show that a model $\mathfrak{A}$ is elementarily embeddable in its ultrapower. If U is an ultrafilter on S, we define the *canonical embedding* $j : \mathfrak{A} \to \text{Ult}_U \mathfrak{A}$ as follows: For each $a \in A$, let c_a be the *constant function* with value a:

$$c_a(x) = a \qquad \text{(for every } x \in S\text{)}, \tag{12.12}$$

and let

$$j(a) = [c_a]. \tag{12.13}$$

Corollary 12.5. *The canonical embedding $j : \mathfrak{A} \to \text{Ult}_U \mathfrak{A}$ is an elementary embedding.*

Proof. Let $a \in A$. By Łoś's Theorem, $\text{Ult}_U \mathfrak{A} \vDash \varphi[j(a)]$ if and only if $\text{Ult}_U \mathfrak{A} \vDash \varphi[c_a]$ if and only if $\mathfrak{A} \vDash \varphi[a]$ for almost all x if and only if $\mathfrak{A} \vDash \varphi[a]$. □

Models of Set Theory and Relativization

The language of set theory consists of one binary predicate symbol $\in$, and so models of set theory are given by its universe M and a binary relation E on M that interprets $\in$.

We shall also consider models of set theory that are proper classes. However, due to Gödel's Second Incompleteness Theorem, we have to be careful how the generalization is formulated.

Definition 12.6. Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a formula of the language of set theory. The *relativization* of φ to M, E is the formula

$$\varphi^{M,E}(x_1, \ldots, x_n) \tag{12.14}$$

defined inductively as follows:

$$\begin{aligned}
(x \in y)^{M,E} &\leftrightarrow x \mathrel{E} y \\
(x = y)^{M,E} &\leftrightarrow x = y \\
(\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\
(\varphi \wedge \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \wedge \psi^{M,E} \\
(\exists x\, \varphi)^{M,E} &\leftrightarrow (\exists x \in M)\, \varphi^{M,E}
\end{aligned} \tag{12.15}$$

and similarly for the other connectives and $\forall$.

When E is $\in$, we write φ^M instead of $\varphi^{M,\in}$.

When using relativization $\varphi^{M,E}(x_1, \ldots, x_n)$ it is implicitly assumed that the variables $x_1, \ldots, x_n$ range over M. We shall often write

$$(M, E) \vDash \varphi(x_1, \ldots, x_n)$$

instead of (12.14) and say that the *model* (M, E) satisfies φ. We point out however that while this is a legitimate statement in every particular case of φ, the general satisfaction relation is formally undefinable in ZF.

Let *Form* denote the set of all formulas of the language $\{\in\}$. As with any actual (metamathematical) natural number we can associate the corresponding element of $\boldsymbol{N}$, we can similarly associate with any given formula of set theory the corresponding element of the set *Form*. To make the distinction, if φ is a formula, let $\ulcorner\varphi\urcorner$ denote the corresponding element of *Form*.

If M is a set and E is a binary relation on M and if $a_1, \ldots, a_n$ are elements of M, then

$$\varphi^{M,E}(a_1, \ldots, a_n) \leftrightarrow (M, E) \vDash \ulcorner\varphi\urcorner[a_1, \ldots, a_n] \tag{12.16}$$

as can easily be verified. Thus in the case when M is a set and φ a particular (metamathematical) formula, we shall not make a distinction between the two meanings of the symbol $\vDash$. We note however that the left-hand side of (12.16) (relativization) is *not* defined for $\varphi \in \mathit{Form}$, and the right-hand side (satisfaction) is *not* defined if M is a proper class.

Below we sketch a proof of a theorem of Tarski, closely related to Gödel's Second Incompleteness Theorem. The theorem states that there is no set-theoretical property $T(x)$ such that if σ is a sentence that $T(\ulcorner\sigma\urcorner)$ holds if and only if σ holds.

Let us arithmetize the syntax and consider some fixed effective enumeration of all expressions by natural numbers (*Gödel numbering*). In particular, if σ is a sentence, then $\#\sigma$ is the Gödel number of σ, a natural number. We say that $T(x)$ is a *truth definition* if:

(12.17) (i) $\forall x\, (T(x) \to x \in \omega)$;
(ii) if σ is a sentence, then $\sigma \leftrightarrow T(\#\sigma)$.

Theorem 12.7 (Tarski). *A truth definition does not exist.*

Proof. Let us assume that there is a formula $T(x)$ satisfying (12.17). Let

$$\varphi_0, \quad \varphi_1, \quad \varphi_2, \quad \ldots$$

be an enumeration of all formulas with one free variable. Let $\psi(x)$ be the formula

$$x \in \omega \wedge \neg T(\#(\varphi_x(x))).$$

There is a natural number k such that ψ is φ_k. Let σ be the sentence $\psi(k)$. Then we have

$$\sigma \leftrightarrow \psi(k) \leftrightarrow \neg T(\#(\varphi_k(k))) \leftrightarrow \neg T(\#\sigma)$$

which contradicts (12.17). □

Relative Consistency

By Gödel's Second Incompleteness Theorem it is impossible to show the consistency of ZF (or related theories) by means limited to ZF alone.

Once we assume that ZF (or ZFC) is consistent, we may ask whether the theory remains consistent if we add an additional axiom A.

Let T be a mathematical theory (in our case, T is either ZF or ZFC), and let A be an additional axiom. We say that T + A is *consistent relative to* T (or that A is *consistent with* T) if the following implication holds:

if T is consistent, then so is T + A.

If both A and ¬A are consistent with T, we say that A is *independent of* T.

The question whether A is consistent with T is equivalent to the question whether the negation of A is provable in T (provided T is consistent); this is because T + A is consistent if and only if ¬A is not provable in T.

The way to show that an axiom A is consistent with ZF (ZFC) is to use models. For assume that we have a model M (possibly a proper class) of ZF such that $M \vDash$ A. (More precisely, the relativizations σ^M hold for all axioms σ of ZF, as well as A^M.) Then A is consistent with ZF: If it were not, then ¬A would be provable in ZF, and since M is a model of ZF, M would satisfy ¬A. However, $(\neg\mathrm{A})^M$ contradicts A^M.

Transitive Models and Δ_0 Formulas

If M is a transitive class then the model $(M, \in)$ is called a *transitive model*. We note that transitive models satisfy the Axiom of Extensionality (see Exercise 12.4) and that every well-founded extensional model is isomorphic to a transitive model (Theorem 6.15).

Definition 12.8. A formula of set theory is a Δ_0-formula if

(i) it has no quantifiers, or
(ii) it is $\varphi\wedge\psi$, $\varphi\vee\psi$, $\neg\varphi$, $\varphi\to\psi$ or $\varphi\leftrightarrow\psi$ where φ and ψ are Δ_0-formulas, or
(iii) it is $(\exists x\in y)\,\varphi$ or $(\forall x\in y)\,\varphi$ where φ is a Δ_0-formula.

Lemma 12.9. *If M is a transitive class and φ is a Δ_0-formula, then for all $x_1,\ldots,x_n$,*

(12.18) $$\varphi^M(x_1,\ldots,x_n)\leftrightarrow\varphi(x_1,\ldots,x_n).$$

If (12.18) holds, we say that the formula φ is *absolute* for the transitive model M.

Proof. If φ is an atomic formula, then (12.18) holds. If (12.18) holds for φ and ψ, then it holds for $\neg\varphi$, $\varphi\wedge\psi$, $\varphi\vee\psi$, $\varphi\to\psi$, and $\varphi\leftrightarrow\psi$.

Let φ be the formula $(\exists u\in x)\,\psi(u,x,\ldots)$ and assume that (12.18) is true for ψ. We show that (12.18) is true for φ (the proof for $\forall u\in x$ is similar).

If φ^M holds then we have $(\exists u\,(u\in x\wedge\psi))^M$, i.e., $(\exists u\in M)(u\in x\wedge\psi^M)$. Since $\psi^M\leftrightarrow\psi$, it follows that $(\exists u\in x)\,\psi$. Conversely, if $(\exists u\in x)\,\psi$, then since M is transitive, u belongs to M, and since $\psi(u,x,\ldots)\leftrightarrow\psi^M(u,x,\ldots)$, we have $\exists u\,(u\in M\wedge u\in x\wedge\psi^M)$ and so $((\exists u\in x)\,\psi)^M$. □

Lemma 12.10. *The following expressions can be written as Δ_0-formulas and thus are absolute for all transitive models.*

(i) $x=\{u,v\}$, $x=(u,v)$, *x is empty*, $x\subset y$, *x is transitive*, *x is an ordinal*, *x is a limit ordinal*, *x is a natural number*, $x=\omega$.
(ii) $Z=X\times Y$, $Z=X-Y$, $Z=X\cap Y$, $Z=\bigcup X$, $Z=\operatorname{dom}X$, $Z=\operatorname{ran}X$.
(iii) *X is a relation*, *f is a function*, $y=f(x)$, $g=f{\restriction}X$.

Proof.

(i) $x=\{u,v\}\leftrightarrow u\in x\wedge v\in x\wedge(\forall w\in x)(w=u\vee w=v)$.
$x=(u,v)\leftrightarrow(\exists w\in x)(\exists z\in x)(w=\{u\}\wedge z=\{u,v\})$
$\wedge\,(\forall w\in x)(w=\{u\}\vee w=\{u,v\})$.
x is empty $\leftrightarrow(\forall u\in x)\,u\neq u$.
$x\subset y\leftrightarrow(\forall u\in x)\,u\in y$.
x is transitive $\leftrightarrow(\forall u\in x)\,u\subset x$.
x is an ordinal $\leftrightarrow x$ is transitive $\wedge\,(\forall u\in x)(\forall v\in x)(u\in v\vee v\in u\vee u=v)$
$\wedge\,(\forall u\in x)(\forall v\in x)(\forall w\in x)(u\in v\in w\to u\in w)$.
x is a limit ordinal $\leftrightarrow x$ is an ordinal $\wedge\,(\forall u\in x)(\exists v\in x)\,u\in v$.
x is a natural number $\leftrightarrow x$ is an ordinal $\wedge$ (x is not a limit $\vee\,x=0$)
$\wedge\,(\forall u\in x)(u=0\vee u$ is not a limit).
$x=\omega\leftrightarrow x$ is a limit ordinal $\wedge\,x\neq0\wedge(\forall u\in x)\,x$ is a natural number.
(ii) $Z=X\times Y\leftrightarrow(\forall z\in Z)(\exists x\in X)(\exists y\in Y)\,z=(x,y)$
$\wedge\,(\forall x\in X)(\forall y\in Y)(\exists z\in Z)\,z=(x,y)$.
$Z=X-Y\leftrightarrow(\forall z\in Z)(z\in X\wedge z\notin Y)\wedge(\forall z\in X)(z\notin Y\to z\in Z)$.
$Z=X\cap Y$ ... similar.
$Z=\bigcup X\leftrightarrow(\forall z\in Z)(\exists x\in X)\,z\in x\wedge(\forall x\in X)(\forall z\in x)\,z\in Z$.
$Z=\operatorname{dom}(X)\leftrightarrow(\forall z\in Z)\,z\in\operatorname{dom}X\wedge(\forall z\in\operatorname{dom}X)\,z\in Z$,

and we show that:

(12.19) (a) $z\in\operatorname{dom}X$ is a Δ_0-formula;
(b) if φ is Δ_0, then $(\forall z\in\operatorname{dom}X)\,\varphi$ is Δ_0.

(a) $z\in\operatorname{dom}X\leftrightarrow(\exists x\in X)(\exists u\in X)(\exists v\in u)\,x=(z,v)$.
(b) $(\forall z\in\operatorname{dom}X)\,\varphi\leftrightarrow(\forall x\in X)(\forall u\in x)(\forall z,v\in u)(x=(z,v)\to\varphi)$.
An assertion similar to (12.19) holds for $\operatorname{ran}(X)$, and for $\exists$.

(iii) X is a relation $\leftrightarrow (\forall x \in X)(\exists u \in \operatorname{dom} X)(\exists v \in \operatorname{ran} X)\, x = (u, v)$.
f is a function $\leftrightarrow f$ is a relation $\wedge$

$$(\forall x \in \operatorname{dom} f)(\forall y, z \in \operatorname{ran} f)((x, y) \in f \wedge (x, z) \in f \to y = z)$$

where

$$(x, y) \in f \leftrightarrow (\exists u \in f)\, u = (x, y).$$

$$g = f{\restriction}X \leftrightarrow g \text{ is a function} \wedge g \subset f \wedge (\forall x \in \operatorname{dom} g)\, x \in X \wedge (\forall x \in X)(x \in \operatorname{dom} f \to x \in \operatorname{dom} g). \quad \square$$

It should be emphasized that cardinal concepts are generally not absolute. In particular, the following expressions are known not to be absolute:

$$Y = P(X), \quad |Y| = |X|, \quad \alpha \text{ is a cardinal}, \quad \beta = \operatorname{cf}(\alpha), \quad \alpha \text{ is regular}.$$

Compare with Exercise 12.6.

Consistency of the Axiom of Regularity

As an application of the theory of transitive models we show that the Axiom of Regularity is consistent with the other axioms of ZF. In this section only we work in the theory ZF minus Regularity, i.e., axioms 1.1–1.7.

The cumulative hierarchy V_α is defined as in Chapter 6, and we denote (in the present section only) V not the universal class but the class $\bigcup_{\alpha \in Ord} V_\alpha$. We shall show that V is a transitive model of ZF. Thus the Axiom of Regularity is consistent relative to the theory 1.1–1.7.

Theorem 12.11. *In* ZF *minus Regularity, σ^V holds for every axiom σ of* ZF.

Proof. We use absoluteness of Δ_0-formulas and the fact that for every set x, if $x \subset V$, then $x \in V$.

Extensionality. The formula

$$((\forall u \in X)\, u \in Y \wedge (\forall u \in Y)\, u \in X) \to X = Y$$

is Δ_0.

Pairing. Given $a, b \in V$, let $c = \{a, b\}$. The set c is in V and since "$c = \{a, b\}$" is Δ_0 (see Lemma 12.10), the Pairing Axiom holds in V.

Separation. Let φ be a formula; we shall show that

$$V \vDash \forall X\, \forall p\, \exists Y\, \forall u\, (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

Given $X, p \in V$, we let $Y = \{u \in X : \varphi^V(u, p)\}$. Since $Y \subset X$ and $X \in V$, we have $Y \in V$, and

$$V \vDash \forall u\, (y \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

Union. Given $X \in V$, let $Y = \bigcup X$. The set Y is in V and since "$Y = \bigcup X$" is Δ_0, the Axiom of Union holds in V.

Power Set. Given $X \in V$, let $Y = P(X)$. The set Y is in V, and we claim that $V \vDash \forall u\, \varphi(u)$ where $\varphi(u)$ is the formula $u \in Y \leftrightarrow u \subset X$. Since $\varphi(u)$ is Δ_0 and because $\varphi(u)$ holds for all u, we have $\varphi^V(u)$ for all $u \in V$, as claimed.

Infinity. We want to show that

$$V \vDash \exists S\, (\emptyset \in S \wedge (\forall x \in S)\, x \cup \{x\} \in S). \tag{12.20}$$

The formula in (12.20) contains defined notions, $\{\ \}$, $\cup$, and $\emptyset$; and strictly speaking, we should first eliminate these symbols and use a formula in which they are replaced by their definitions, using only $\in$ and $=$. However, we have already proved that both pairing and union are the same in the universe as in V, and similarly one shows that $X \in V$ is empty if and only if (X is empty)^V. In other words,

$$\{a, b\}^V = \{a, b\}, \qquad \bigcup\nolimits^V X = \bigcup X, \qquad \emptyset^V = \emptyset$$

where $\{a, b\}^V$, $\bigcup^V$, and $\emptyset^V$ denote pairing, union, and the empty set in the model V.

Since $\omega \in V$, we easily verify that (12.20) holds when $S = \omega$.

Replacement. Let φ be a formula; we shall show that

$$V \vDash \forall x\, \forall y\, \forall z\, (\varphi(x, y, p) \wedge \varphi(x, z, p) \to y = z) \to \forall X\, \exists Y\, \forall y\, (y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p)).$$

Given $p \in V$, assume that $V \vDash \forall x\, \forall y\, \forall z\, (\dots)$. Thus

$$F = \{(x, y) \in V : \varphi^V(x, y, p)\}$$

is a function, and we let $Y = F(X)$. Since $Y \subset V$, we have $Y \in V$, and one verifies that for every $y \in V$,

$$V \vDash y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p).$$

Regularity. We want to show that $V \vDash \forall S\, \varphi(S)$, where φ is the formula

$$S \neq \emptyset \to (\exists x \in S)\, S \cap x = \emptyset.$$

If $S \in V$ is nonempty, then let $x \in S$ be of least rank; then $S \cap x = \emptyset$. Hence $\varphi(S)$ is true for any S; moreover, $(S \cap x)^V = S \cap x$, and φ is Δ_0. Thus $V \vDash \forall S\, \varphi(S)$. $\square$

Inaccessibility of Inaccessible Cardinals

Theorem 12.12. *The existence of inaccessible cardinals is not provable in* ZFC. *Moreover, it cannot be shown that the existence of inaccessible cardinals is consistent with* ZFC.

We shall prove the first assertion and invoke Gödel's Second Incompleteness Theorem to obtain the second part.

First we prove (in ZFC):

Lemma 12.13. *If κ is an inaccessible cardinal, then V_κ is a model of* ZFC.

Proof. The proof of all axioms of ZFC except Replacement is as in the proof of consistency of the Axiom of Regularity (see Exercises 12.7 and 12.8). To show that $V_\kappa \models$ Replacement, it is enough to show:

(12.21) If F is a function from some $X \in V_\kappa$ into V_κ, then $F \in V_\kappa$.

Since κ is inaccessible, we have $|V_\kappa| = \kappa$ and $|X| < \kappa$ for every $X \in V_\kappa$. If F is a function from $X \in V_\kappa$ into V_κ, then $|F(X)| \le |X| < \kappa$ and (since κ is regular) $F(X) \subset V_\alpha$ for some $\alpha < \kappa$. It follows that $F \in V_\kappa$. □

Proof of Theorem 12.12. If κ is an inaccessible cardinal, then not only is V_κ a model of ZFC, but in addition

$$(\alpha \text{ is an ordinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an ordinal.}$$
$$(\alpha \text{ is a cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a cardinal.}$$
$$(\alpha \text{ is a regular cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a regular cardinal.}$$
$$(\alpha \text{ is an inaccessible cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an inaccessible cardinal.}$$

We leave the details to the reader.

In particular, if κ is inaccessible cardinal, then

$$V_\kappa \models \text{there is no inaccessible cardinal.}$$

Thus we have a model of ZFC + "there is no inaccessible cardinal" (if there is no inaccessible cardinal, we take the universe as the model). Hence it cannot be proved in ZFC that inaccessible cardinals exist.

To prove the second part, assume that it can be shown that the existence of inaccessible cardinals is consistent with ZFC; in other words, we assume

$$\text{if ZFC is consistent, then so is ZFC} + \text{I}$$

where I is the statement "there is an inaccessible cardinal."

We naturally assume that ZFC is consistent. Since I is consistent with ZFC, we conclude that ZFC + I is consistent. It is provable in ZFC + I that there is a model of ZFC (Lemma 12.13). Thus the sentence "ZFC is consistent" is provable in ZFC + I. However, we have assumed that "I is consistent with ZFC" is provable, and so "ZFC + I is consistent" is provable in ZFC + I. This contradicts Gödel's Second Incompleteness Theorem. □

The wording of the second part of Theorem 12.12 (and its proof) is somewhat vague; "it cannot be shown" means: It cannot be shown by methods formalizable in ZFC.

Reflection Principle

The theorem that we prove below is the analog of the Löwenheim-Skolem Theorem. While that theorem states that every model has a small elementary submodel, the Reflection Principle provides, for any *finite* number of formulas, a set M that is like an "elementary submodel" of the universe, with respect to the given formulas. The theorem is proved without the use of the Axiom of Choice, but using the Axiom of Choice, one can obtain countable model.

Theorem 12.14 (Reflection Principle).

(i) *Let $\varphi(x_1, \ldots, x_n)$ be a formula. For each M_0 there exists a set $M \supset M_0$ such that*

(12.22) $$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)$$

for every $x_1, \ldots, x_n \in M$. (We say that M reflects φ.)

(ii) *Moreover, there is a transitive $M \supset M_0$ that reflects φ; moreover, there is a limit ordinal α such that $M_0 \subset V_\alpha$ and V_α reflects φ.*

(iii) *Assuming the Axiom of Choice, there is an $M \supset M_0$ such that M reflects φ and $|M| \le |M_0| \cdot \aleph_0$. In particular, there is a countable M that reflects φ.*

Remarks. 1. We may require either that M be transitive or that $|M| \le |M_0| \cdot \aleph_0$ but not both.

2. The proof works for any finite number of formulas, not just one. Thus if $\varphi_1, \ldots, \varphi_n$ are formulas, then there exists a set M that reflects each of $\varphi_1, \ldots, \varphi_n$.

3. If σ is a true sentence, then the Reflection Principle yields a set M that is a model of σ; using the Axiom of Choice, one can get a countable transitive model of σ.

4. As a consequence of the Reflection Principle, and of Gödel's Second Incompleteness Theorem, it follows that the theory ZF is not finitely axiomatizable: Any finite number of theorems of ZF have a model (a set) by the Reflection Principle, while the existence of a model of ZF is not provable. (By the same argument, no consistent extension of ZF is finitely axiomatizable.)

The key step in the proof of Theorem 12.14 is the following lemma, which we prove first.

Lemma 12.15.

(i) *Let $\varphi(u_1, \ldots, u_n, x)$ be a formula. For each set M_0 there exists a set $M \supset M_0$ such that*

$$\text{(12.23)} \qquad \textit{if } \exists x\, \varphi(u_1, \ldots, u_n, x) \quad \textit{then } (\exists x \in M)\, \varphi(u_1, \ldots, u_n, x)$$

for every $u_1, \ldots, u_n \in M$. Assuming the Axiom of Choice, there is $M' \supset M_0$ such that (12.23) holds for M' and $|M'| \leq |M_0| \cdot \aleph_0$.

(ii) *If $\varphi_1, \ldots, \varphi_k$ are formulas, then for each M_0 there is an $M \supset M_0$ such that (12.23) holds for each $\varphi_1, \ldots, \varphi_k$.*

Proof. We shall give a detailed proof of (i). An obvious modification of the proof gives (ii); we leave that to the reader.

Note that the operation $H(u_1, \ldots, u_n)$ defined below plays the same role as Skolem functions in the Löwenheim-Skolem Theorem.

Let us recall the definition (6.4):

$$\text{(12.24)} \qquad \hat{C} = \{x \in C : (\forall z \in C)\ \operatorname{rank} x \leq \operatorname{rank} z\}.$$

For every $u_1, \ldots, u_n$, let

$$\text{(12.25)} \qquad H(u_1, \ldots, u_n) = \hat{C}$$

where

$$\text{(12.26)} \qquad C = \{x : \varphi(u_1, \ldots, u_n, x)\}.$$

Thus $H(u_1, \ldots, u_n)$ is a set with the property

$$\text{(12.27)} \quad \text{if } \exists x\, \varphi(u_1, \ldots, u_n, x), \quad \text{then } (\exists x \in H(u_1, \ldots, u_n))\, \varphi(u_1, \ldots, u_n, x).$$

We construct the set M by induction. We let $M = \bigcup_{i=0}^{\infty} M_i$ where for each $i \in \boldsymbol{N}$,

$$\text{(12.28)} \qquad M_{i+1} = M_i \cup \bigcup\{H(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M_i\}.$$

Now, if $u_1, \ldots, u_n \in M$, then there is an $i \in \boldsymbol{N}$ such that $u_1, \ldots, u_n \in M_i$ and if $\varphi(u_1, \ldots, u_n, x)$ holds for some x, then it holds for some $x \in M_{i+1}$, by (12.27) and (12.28).

Assuming the Axiom of Choice, let F be a choice function on $P(M)$. For every $u_1, \ldots, u_n \in M$, let $h(u_1, \ldots, u_n) = F(H(u_1, \ldots, u_n))$ (and let $h(u_1, \ldots, u_n)$ remain undefined if $H(u_1, \ldots, u_n)$ is empty). Let us define $M' = \bigcup_{i=0}^{\infty} M_i'$, where $M_0' = M_0$ and for each $i \in \boldsymbol{N}$,

$$M_{i+1}' = M_i' \cup \{h(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M_i'\}.$$

Condition (12.23) can be verified for M' in the same way as for M. Moreover, each M_i' has cardinality at most $|M_0| \cdot \aleph_0$, and so does M'. □

Proof of Theorem 12.14. Let $\varphi(x_1, \ldots, x_n)$ be a formula. We may assume that the universal quantifier does not occur in φ ($\forall x \ldots$ can be replaced by $\neg\exists x \neg \ldots$). Let $\varphi_1, \ldots, \varphi_k$ be all the subformulas of the formula φ.

Given a set M_0, there exists, by Lemma 12.15(ii), a set $M \supset M_0$, such that

$$\text{(12.29)} \qquad \exists x\, \varphi_j(u, \ldots, x) \to (\exists x \in M)\, \varphi_j(u, \ldots, x), \qquad j = 1, \ldots, k$$

for all $u, \ldots \in M$. We claim that M reflects each φ_j, $j = 1, \ldots, k$, and in particular M reflects φ. This is proved by induction on the complexity of φ_j.

It is easy to see that (every) M reflects atomic formulas, and that if M reflects formulas ψ and χ, then M reflects $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$, and $\psi \leftrightarrow \chi$. Thus assume that M reflects $\varphi_j(u_1, \ldots, u_m, x)$ and let us prove that M reflects $\exists x\, \varphi_j$.

If $u_1, \ldots, u_m \in M$, then

$$\begin{aligned} M \vDash \exists x\, \varphi_j(u_1, \ldots, u_m, x) &\leftrightarrow (\exists x \in M)\, \varphi_j^M(u_1, \ldots, u_m, x) \\ &\leftrightarrow (\exists x \in M)\, \varphi_j(u_1, \ldots, u_m, x) \\ &\leftrightarrow \exists x\, \varphi_j(u_1, \ldots, u_m, x). \end{aligned}$$

The last equivalence holds by (12.29).

This proves part (i) of the theorem. Part (iii) is proved by taking M of size $\leq |M_0| \cdot \aleph_0$. To prove (ii), one has to modify the proof of Lemma 12.15 so that the set M used in (12.29) is transitive (or $M = V_\alpha$). This is done as follows: In (12.28), we replace M_{i+1} by its transitive closure (or by the least $V_\gamma \supset M_{i+1}$). Then M is transitive (or $M = V_\alpha$). □

Exercises

12.1. Let U be a principal ultrafilter on S, such that $\{a\} \in U$. Show that the ultraproduct $\operatorname{Ult}_U\{\mathfrak{A}_x : x \in S\}$ is isomorphic to $\mathfrak{A}_a$.

12.2. If U is a principal ultrafilter, then the canonical embedding j is an isomorphism between $\mathfrak{A}$ and $\operatorname{Ult}_U \mathfrak{A}$.

12.3. Let κ be a measurable cardinal and let U be an ultrafilter on κ. Let $(A, <^*)$ be the ultrapower of $(\kappa, <)$ by U, and let $j : \kappa \to A$ be the canonical embedding.

(i) $(A, <^*)$ is a linear ordering.
(ii) If U is σ-complete then $(A, <^*)$ is a well-ordering; $(A, <^*)$ is isomorphic, and can be identified with, $(\gamma, <)$, where γ is an ordinal.
(iii) If U is κ-complete then $j(\alpha) = \alpha$ for all $\alpha < \kappa$
(iv) If d is the diagonal function, $[d] \geq \kappa$. The measure U is normal if and only if $[d] = \kappa$.
[Compare with Exercise 10.5.]

12.4. A class M is extensional if and only if σ^M holds where σ is the Axiom of Extensionality.

12.5. The following can be written as Δ_0-formulas: x is an ordered pair, x is a partial (linear) ordering of y, x and y are disjoint, $z = x \cup y$, $y = x \cup \{x\}$, x is an inductive set, f is a one-to-one function of X into (onto) Y, f is an increasing ordinal function, f is a normal function.

12.6. Let M be a transitive class.

(i) If $M \vDash |X| \le |Y|$, then $|X| \le |Y|$.
(ii) If $\alpha \in M$ and if α is a cardinal, then $M \vDash \alpha$ is a cardinal.

[$|X| \le |Y| \leftrightarrow \exists f\, \varphi(f, X, Y)$; α is a cardinal $\leftrightarrow \neg\exists f\, (\exists \beta \in \alpha)\, \psi(\alpha, \beta, f)$, where φ and ψ are Δ_0-formulas.]

12.7. If α is a limit ordinal, then V_α is a model of Extensionality, Pairing, Separation, Union, Power Set, and Regularity. If AC holds, then V_α is a model of AC.

12.8. If $\alpha > \omega$, then V_α is a model of Infinity.

12.9. V_ω, the set of all hereditarily finite sets, is a model of ZFC minus Infinity.

12.10. The existence of an infinite set is not provable in ZFC minus Infinity. Moreover, it cannot be shown that the existence of an infinite set is consistent with ZFC minus Infinity.

12.11. If κ is an inaccessible cardinal then $V_\kappa \vDash$ there is a countable model of ZFC.

[Since $\langle V_\kappa, \in\rangle$ is a model of ZFC, there is a countable model (by the Löwenheim-Skolem Theorem). Thus there is $E \subset \omega \times \omega$ such that $\mathfrak{A} = (\omega, E)$ is a model of ZFC. Verify that $V_\kappa \vDash$ ($\mathfrak{A}$ is a countable model of ZFC).]

12.12. If κ is an inaccessible cardinal, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle$ Moreover, the set $\{\alpha < \kappa : \langle V_\alpha, \in\rangle \prec \langle V_\kappa, \in\rangle\}$ is closed unbounded.

[Construct Skolem functions h for V_κ, and let $\alpha = \lim_n \alpha_n$, where $\alpha_{n+1} < \kappa$ is such that $h(V_{\alpha_n}) \subset V_{\alpha_{n+1}}$ for each h.]

For every infinite regular cardinal κ let H_κ be the set of all x such that $|\operatorname{TC}(x)| < \kappa$. The sets in H_ω are hereditarily finite sets. The sets in H_{ω_1} are *hereditarily countable* sets. Each H_κ is transitive and $H_\kappa \subset V_\kappa$.

12.13. If κ is a regular uncountable cardinal then H_κ is a model of ZFC minus the Power Set Axiom.

12.14. For every formula φ, there is a closed unbounded class C_φ of ordinals such that for each $\alpha \in C_\varphi$, V_α reflects φ.

[$C_{\varphi\wedge\psi} = C_\varphi \cap C_\psi$, $C_{\exists x\, \varphi} = C_\varphi \cap K_\varphi$, where K_φ is the closed unbounded class $\{\alpha \in \mathit{Ord} : \forall x_1, \ldots, x_n \in V_\alpha\, (\exists x\, \varphi(x, x_1, \ldots, x_n) \to (\exists x \in V_\alpha)\, \varphi(x, x_1, \ldots, x_n))\}$.]

12.15. Let M be a transitive class and let φ be a formula. For each $M_0 \subset M$ there exists a set $M_1 \supset M_0$ such that $M_1 \subset M$ and that $\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^{M_1}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in M_1$.

A transfinite sequence $\langle W_\alpha : \alpha \in \mathit{Ord}\rangle$ is called a *cumulative hierarchy* if $W_0 = \emptyset$ and

(12.30) (i) $W_\alpha \subset W_{\alpha+1} \subset P(W_\alpha)$,
(ii) if α is limit, then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.

Each W_α is transitive and $W_\alpha \subset V_\alpha$.

12.16. Let $\langle W_\alpha : \alpha \in \mathit{Ord}\rangle$ be a cumulative hierarchy, and let $W = \bigcup_{\alpha\in \mathit{Ord}} W_\alpha$. Let φ be a formula. Show that there are arbitrary large limit ordinals α such that $\varphi^W(x_1, \ldots, x_n) \leftrightarrow \varphi^{W_\alpha}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in W_\alpha$.

Historical Notes

For concepts of model theory, the history of the subject and for model-theoretical terminology, I refer the reader to Chang and Keisler's book [1973].

Reduced products were first investigated by Łoś in [1955], who also proved Theorem 12.3 on ultraproducts.

For Tarski's Theorem 12.7, see Tarski [1939].

The impossibility of a consistency proof of the existence of inaccessible cardinals follows from Gödel's Theorem [1931]. An argument that more or less establishes the consistency of the Axiom of Regularity appeared in Skolem's work in 1923 (see Skolem [1970], pp. 137–152).

The study of transitive models of set theory originated with Gödel's work on constructible sets. The Reflection Principle was introduced by Montague; see [1961] and Lévy [1960b].

Exercise 12.12: Montague and Vaught [1959].

Exercise 12.14: Galvin.