# Elementary Set Theory

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# Axioms of Set Theory

**Need to know:**

- 9 axioms, expressed by the formulas of Set Theory.
- Applications of axioms: defining new sets, deriving contradictions, etc.
- *Concepts*: set/class, partition/equivalence relation

# Ordinal Numbers

**Need to know:**

- *Concepts*: partial/linear/well ordering, order type, ordinal, successor/limit ordinal, addition/multiplication/exponentiation of ordinals, etc.
- *Techniques*: transfinite recursion (for definition), transfinite induction, argument with least element.

# Cardinal Numbers

**Need to know:**

- *Concepts*: $|X| \leq |Y|$, $|X| = |Y|$, cardinal, cardinal addition/multiplication/exponentiation, cofinality
- *Techniques*:
  - Cantor's diagonalization argument
  - verify properties of cardinal arithmetic
  - do transfinite counting
- *Theorems*: Cantor-Bernstein, Theorem 3.8 (Cantor), 3.31 (König)

# Real Numbers

**Need to know:**

- *Concepts*: open/closed/perfect subsets of $\mathbb{R}$ (and $\omega^\omega$)
- *Techniques*:
  - Tree representation of Baire space, $\mathcal{N}$.
- *Theorems*:
  - Theorem 4.3 (Cantor-Dedekind)[1]
  - Cantor-Bendixson,
  - Baire Category Theorem

[1] $\mathbb{R}$ is the unique complete dense unbounded separable linear order.

# The Axiom of Choice

**Need to know:**

- *Concepts*:
  - Statements of AC, WO, ZL, MP, $AC_\omega$, DC;
- *Theorems*:
  - Implications among AC, WO, ZL, MP, $AC_\omega$, DC;
  - König Theorem ($\Sigma_i \kappa_i < \prod_i \lambda_i$) and its consequences:
    - $2^\kappa > \kappa$,
    - $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, (simple version of König)
    - $\mathrm{cf}(2^\kappa) > \kappa$.

# Infinite sum and infinite product

- If $\lambda \geq \omega$ and $\kappa_i > 0$, for each $i < \lambda$, then
$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$
- Suppose $\lambda \geq \omega$ and $\langle \kappa_i \mid i < \lambda \rangle$ is a nondecreasing sequence of cardinals $> 0$. Then
$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

# Exercises

- There are arbitrarily large singular cardinals.
- There are arbitrarily large singular cardinals $\aleph_\alpha$ such that $\aleph_\alpha = \alpha$.
- About cofinality
  - $\mathrm{cf}(\alpha + \beta) = \mathrm{cf}(\beta)$.
  - $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$, $\alpha$ is a limit ordinal.
  - $\mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$.
- Cardinal exponentiations under $\mathrm{GCH}$: for any $\kappa, \lambda \geq \omega$,
  $\kappa^\lambda = \kappa$, if $\lambda < \mathrm{cf}(\kappa)$;
  $\kappa^\lambda = \kappa^+$, if $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$; and
  $\kappa^\lambda = \lambda^+$, if $\kappa < \lambda$.

# Cardinality

- If a linearly ordered set $P$ has a countable dense subset, then $|P| \leq 2^{\aleph_0}$.
- The cardinality of the set of all null sets.
- The set of all 1-1 function from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

# A set or a proper class?

- $\mathrm{Ord} =_{\mathsf{def}} \{\alpha \mid \alpha \text{ is an ordinal}\}$
- $\mathrm{Card} =_{\mathsf{def}} \{\alpha \mid \alpha \text{ is a cardinal}\}$
- $\{X \mid X \text{ is a wellordered set}\}$
- $\{X \subseteq \mathbb{R} \mid X \text{ is wellordered}\}$
- $\{P \mid P \text{ is a partially ordered set and } |P| < \aleph_\omega\}$
- the range of $\aleph$-function : $\aleph(i) =$ the $i$-th cardinal in ordinals.
- Assume $\kappa \in \mathrm{Ord}$ is a strongly inaccessible cardinal. The collection of wellordered sets in $V_\kappa$.

# Yes, No, or ...?

Assume ZFC, determine the truth of the following statement, if you can.

- Every dense subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$
- $\mathrm{cf}(2^{\aleph_\omega}) = \aleph_\omega$
- Every real in $(0, 1)$ is uniquely represented by a $\{0, 1\}$-sequence of length $\omega$
- Every real in $(0, 1)$ is uniquely represented by a continuous fraction
- Every wellordering has no nontrivial automorphism.

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F, F, F, T, T

# Yes, No, or ...?

## Cont'd

- Every linear order $L$ with the following property is a well order: if $f : L \to L$ is ordering preserving, then $f(x) \geq x$ for every $x \in L$.
- There are more than $\aleph_1$ many reals.
- There is no Suslin line, i.e. a linear ordering that is dense, unbounded, complete and has the countable chain condition but is not nonseparable.
- There is a unique complete ordered field.
- For any set $A$, $\bigcap A \subset a$, for all $a \in A$.
- For any set $A$, $\bigcup A \supset a$, for all $a \in A$.

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F, I (for Indepedent), I, T, F, T

# Counting

Compute the cardinalities of the following sets.

- $\{F \mid F = (A, +, \cdot, 0, 1, <)$ is a complete ordered field$\}$
- The collection of comeager subsets of the Baire space $\mathcal{N}$
- The collection of all Lebesgue measure zero sets of reals.
- The collection of all Borel sets that are of Lebesgue measure zero.
- The collection of all meager sets of reals
  [Hint: The Cantor set is nowhere dense.]

# Cofinality

- $\mathrm{cf}(\aleph_\omega)$
- $\mathrm{cf}(\aleph_{\omega+\omega^2+3})$
- Given continuous increasing ordinal function $f$, for $A \subset$ Ord with no maximal element,
$$\mathrm{cf}(f(\sup A)) = \mathrm{cf}(A).$$
- For instance, the cofinality of $(\omega_1)^{\omega^\omega}$ (as ordinal exponentiation) is $\omega$.
$$(\omega_1)^{\omega^\omega} = \sup_{n<\omega} (\omega_1)^{\omega^n}$$
- But as cardinal exponentiation,
$$\mathrm{cf}((\aleph_1)^{\aleph_0^{\aleph_0}}) > \aleph_0^{\aleph_0}.$$

# Miscellaneous

- Every comeager set of reals is dense.
- Every comeager set of reals contains a perfect subset.
- The Cantor set $\mathbb{C}$ is nowhere dense and null.
  - $\mathbb{C}$ is closed. $\mathbb{C}$ contains no intervals, so its interior is empty.
  - Given $\varepsilon > 0$, let $n < \omega$ be such that $(\frac{2}{3})^n < \varepsilon$, then the collection $\bigcup\{O_s \mid s \in {}^{<\omega}\omega \wedge |s| = n\}$ is an open set containing $\mathbb{C}$.

# Cardinal arithmetic

## Exercise

Show that $\prod_{m,n<\omega}(mn+1) = 2^{\aleph_0}$.

PROOF.

$$
\begin{aligned}
2^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} &= \prod_{n,m<\omega} 2 \\
&\leq \prod_{n,m<\omega} (mn+1) \\
&\leq \Big(\sup_{m,n}(mn+1)\Big)^{\aleph_0\cdot\aleph_0} \\
&= \aleph_0^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0}.
\end{aligned}
$$

□

# Hausdorff formula

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$

# Hausdorff formula

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- For $n < \omega$, $\aleph_{\alpha+n+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \prod_{i \leq n} \aleph_{\alpha+i+1} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+n+1}$.
- For limit ordinal $\alpha$,

$$\begin{aligned}
{\aleph_{\alpha+\omega}}^{\aleph_\beta} = {\aleph_{\alpha+\omega}}^{\aleph_0 \cdot \aleph_\beta} = \left({\aleph_{\alpha+\omega}}^{\aleph_0}\right)^{\aleph_\beta} = \left(\prod_{n<\omega} \aleph_{\alpha+n+1}\right)^{\aleph_\beta} \\
= \prod_{n<\omega} {\aleph_{\alpha+n+1}}^{\aleph_\beta} = \prod_{n<\omega} \left(\aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+n+1}\right) \\
= \aleph_\alpha^{\aleph_\beta \cdot \aleph_0} \cdot \left(\prod_{n<\omega} \aleph_{\alpha+n+1}\right) = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+\omega}.
\end{aligned}$$

# Cofinality of a fixed-point ordinal

### Exercise

Suppose $f : \mathrm{Ord} \to \mathrm{Ord}$ is increasing and continuous. Show

1. The set of fixed-points for $f$ is unbounded in $\mathrm{Ord}$.
2. The cofinality of the least fixed-point for $f$ is $\omega$.[2]

[2] For any regular cardinal $\kappa$, there are fixed-points for $f$ of cofinality $\kappa$.

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PROOF.

1. Let $\alpha_0$ be any ordinal. For $n < \omega$, define $\alpha_{n+1} = f(\alpha_n)$. Then $\alpha^* = \sup_{n<\omega} \alpha_n$ is a fixed-point for $f$.
2. This is the case $\alpha_0 = 0$.

□

[2] For any regular cardinal $\kappa$, there are fixed-points for $f$ of cofinality $\kappa$.

# Cofinality of power sets

**Exercise**

Show that $\mathrm{cf}(2^\kappa) > \kappa$, for all infinite cardinal $\kappa$.

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<u>Proof</u>.

Suppose not, $\mathrm{cf}(2^\kappa) \leq \kappa$. Then

$$2^\kappa < (2^\kappa)^{\mathrm{cf}(2^\kappa)} \leq (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa.$$

Contradiction! □

# Cofinality of Products of Cardinals

## Exercise

Let $\langle \kappa_i \mid i < \mu \rangle$ be a non-decreasing sequence of infinite cardinals. Define $\lambda = \prod_{i<\mu} \kappa_i$. Prove that $\mathrm{cf}(\lambda) > \mu$.

PROOF.

The point is that $\lambda = (\sup_{i<\mu} \kappa_i)^\mu$. Let $\kappa = \sup_{i<\mu} \kappa_i$. Suppose $\mathrm{cf}(\lambda) \leq \mu$. Then by König Theorem,

$$\lambda = \kappa^\mu < (\kappa^\mu)^{\mathrm{cf}(\kappa^\mu)} \leq \kappa^{\mu \cdot \mu} = \kappa^\mu = \lambda.$$

Contradiction! □

Good luck!