

Elementary Set Theory

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Course Information

Webpage

- ▶ <http://math0.bnu.edu.cn/~shi/teaching/fall2025/ST/>

Textbook

- ▶ Thomas Jech. *Set Theory: The Third Millennium Edition, Revised and Expanded, Part I*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

Suggested Readings

- [1] Judith Roitman. *Introduction to Modern Set Theory*. John Wiley & Sons, 1990.
- [2] Karel Hrbacek, Thomas Jech. *Introduction to set theory*. M. Dekker, New York, 1978.
- [3] Wikipedia (en). <http://en.wikipedia.org>

Advanced Readings

- [1] Kenneth Kunen, *Set theory, An introduction to independence proofs*. Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Company, 1980.
- [2] Kenneth Kunen. *Set theory*, volume 34 of Studies in Logic (London). College Publications, London, 2011.
- [3] Akihiro Kanamori, *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.
- [4] Ralf Schindler. *Set theory*. Universitext. Springer, Cham, 2014. Exploring independence and truth.

Coming up next

Introduction

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- ▶ Mathematical concepts are defined in terms of
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- ▶ Axiom system for set theory: formulate a few simple
 axioms about these primitive notions in an attempt to
 capture the basic “obviously true” set-theoretic principles.
- ▶ From such axioms, all mathematics may be derived.
- ▶ Unfortunately, usual axioms can not settle all problems.
 (Continuum Hypothesis)

History

Before axiomatization

- ▶ Naïve set theory had been used everywhere in mathematics since the very beginning of mathematics, dating back to the age of Aristotle.

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² Reading assignment: Paradox (wikipedia).

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- ▶ Modern set theory: infinity.
Antient Greeks: the idea of infinity.
Antient China: 不积跬步，无以至千里，不积细流，无以成江海。¹

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Antient China: 不积跬步，无以至千里，不积细流，无以成江海。¹
- ▶ Cantor's work between 1874 and 1884 mark the birth of modern set theory.
- ▶ 1895, 1899 and 1902: three paradoxes,² especially **Russell paradox**, were discovered and shaken the foundations of mathematics.
– the 3rd crisis in foundations of mathematics

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- ▶ Zermelo (1908) was the first to attempt an axiomatization of set theory. Then followed by Fraenkel, von Neumann, Bernays and Gödel etc. (**ZF, ZFC**)
- ▶ (late 1930s) limitations of naïve axiomatic approaches.
(**Gödel's 1st & 2nd Incompleteness**)

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- ▶ Large cardinals, inner models and determinacy axioms are the central topics in modern pure set theory.
- ▶ W. Hugh Woodin (late 1990s) proposed a stronger form of logic, Ω -logic, for set theory transcending forcing, and recently, an ultimate universe for large cardinal and inner model program, meanwhile, settling CH.

Areas of study³

- ▶ Combinatorial set theory
- ▶ Descriptive set theory
- ▶ Fuzzy set theory
- ▶ Inner model theory
- ▶ Large cardinals
- ▶ Determinacy
- ▶ Forcing
- ▶ Cardinal invariants
- ▶ Set-theoretic topology

³See https://en.wikipedia.org/wiki/Set_theory#Areas_of_study

Naive and Rigorous

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- ▶ In **naïve set theory**, sets are introduced and understood using what is taken to be the self-evident concept of sets as **collections of objects** considered as a whole.
- ▶ In **axiomatic set theory**, the concepts of sets and set membership are defined indirectly by first **postulating certain axioms which specify their properties**. In this conception, sets and set membership are primitive concepts like point and line in Euclidean geometry, and are not themselves directly defined.

Paradoxes in Naïve Set Theory

Need for axiomatic approach

In Naive Set Theory, one's first intuition might be that we can form any sets we want, but this view leads to inconsistencies.

Define

$$Z = \{x \mid x \notin x\}.$$

Then

$$Z \in Z \leftrightarrow Z \notin Z.$$

This is the famous **Russell's paradox**.

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- ▶ **Zermelo**¹⁹⁰⁸-**Fraenkel**¹⁹²² set theory is the most commonly used system of set-theoretic axioms, based on Zermelo set theory and further developed by Abraham Fraenkel and Thoralf Skolem.

- ▶ **Von Neumann**^{1920s}-**Bernays**¹⁹³⁷-**Gödel**¹⁹⁴⁰ set theory is an axiom system for set theory designed to yield the same results as Zermelo-Fraenkel set theory, together with the axiom of choice (AC), but with only a finite number of axioms, that is without axiom schemata.

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- ▶ **New Foundations** (W. Quine, 1937) and **positive** set theory are among the alternative set theories which have been proposed.

People in Set Theory, I



Georg Cantor
1845 - 1918



Ernst Zermelo
1871 - 1953



Kurt Gödel
1906 - 1978

People in Set Theory, II



Adolf Fraenkel
1891 - 1965



Thoralf Skolem
1887 - 1963



Bertrand Russell
1872 - 1970

People in Set Theory, III



Saharon Shelah

1945 -



Alexander S. Kechris

1946 -



W. Hugh Woodin

1955 -



Stevo Todorčević

1955 -

Assignment

- ▶ Read https://en.wikipedia.org/wiki/Set_theory.
- ▶ Follow links of your interest in that article.