Solutions for Assignment # 4.1

November 27, 2025

1. Define $+_Q$, $\cdot_Q$ and $<_Q$ and verify that your definitions doesn't depend on the choice of representatives.

   <u>SOLUTION</u>: Define $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}_+ /_\approx$, where $(p, q) \approx (r, s)$ iff $p \cdot_\mathbb{Z} s = q \cdot_\mathbb{Z} r$. For rest of the solution, we write (a, b) for equivalent class $[(a, b)]_\approx$, and $+, \cdot, <$ for $+_\mathbb{Z}, \cdot_\mathbb{Z}, <_\mathbb{Z}$.

   - Define $(a, b) +_\mathbb{Q} (c, d) = (a \cdot d + b \cdot c, b \cdot d)$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

$$\begin{aligned}(a_1 \cdot d_1 + b_1 \cdot c_1, b_1 \cdot d_1) &\approx ((a_1 \cdot d_1 + b_1 \cdot c_1) \cdot b_2 \cdot c_2, (b_1 \cdot d_1) \cdot b_2 \cdot c_2) \\ &= ((a_2 \cdot d_2 + b_2 \cdot c_2) \cdot b_1 \cdot c_1, (b_2 \cdot d_2) \cdot b_1 \cdot c_1) \\ &\approx (a_2 \cdot d_2 + b_2 \cdot c_2, b_2 \cdot d_2).\end{aligned}$$

   - Define $(a, b) \cdot_\mathbb{Q} (c, d) = (a \cdot c, b \cdot d)$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

$$\begin{aligned}(a_1 \cdot c_1, b_2 \cdot d_2) &\approx (a_1 \cdot c_1 \cdot b_2 \cdot d_2, c_1 \cdot d_1 \cdot b_2 \cdot d_2) \\ &= (a_2 \cdot c_2 \cdot b_1 \cdot d_1, b_2 \cdot d_2 \cdot b_1 \cdot d_1) \\ &\approx (a_2 \cdot c_2, b_2 \cdot d_2).\end{aligned}$$

   - Define $(a, b) <_\mathbb{Q} (c, d)$ iff $a \cdot d < b \cdot c$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

$$\begin{aligned}a_1 \cdot d_1 < b_1 \cdot c_1 &\Leftrightarrow a_1 \cdot b_2 \cdot d_1 \cdot d_2 < c_1 \cdot d_2 \cdot b_1 \cdot b_2 \\ &\Leftrightarrow a_2 \cdot b_1 \cdot d_1 \cdot d_2 < c_2 \cdot d_1 \cdot b_1 \cdot b_2 \\ &\Leftrightarrow a_2 \cdot d_2 < b_2 \cdot c_2.\end{aligned}$$

2. *Ch4: 1.* The set of all continue functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality c (while the set of all functions has cardinality 2^c).

   <u>SOLUTION</u>: Denote the set of all continue functions by $c(\mathbb{R})$.

   For any $a \in \mathbb{R}$, let $g_a(x) = a(\forall x \in \mathbb{R})$. Obviously, $g_a \in c(\mathbb{R})$. Define

$$\begin{array}{rccc} G: & \mathbb{R} & \longrightarrow & c(\mathbb{R}) \\ & a & \longmapsto & g_a \end{array}$$

   G is an injection, so $|c(\mathbb{R})| \geq |\mathbb{R}| = c$.

   $\mathbb{Q}$ is countable, denoted by $\{r_i \mid i < \omega\}$. Let

$$\begin{array}{rccc} F: & c(\mathbb{R}) & \longrightarrow & {}^\omega\mathbb{R} \\ & f & \longmapsto & \langle f(r_0), f(r_1), \dots \rangle \end{array}$$

   F is injection. For any $f \neq g \in c(\mathbb{R})$, there exists $x \in \mathbb{R}$, s.t $f(x) \neq g(x)$. Since f and g are continue, there exists an interval I, s.t $f(y) \neq g(y)(\forall y \in I)$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, so $f(r_i) \neq g(r_i)$ for some $i < \omega$, i.e $F(f) \neq F(g)$. From this, we can get

$$|c(\mathbb{R})| \leq |{}^\omega\mathbb{R}| = |\mathbb{R}|^\omega = (2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega = c$$

   Hence, $|c(\mathbb{R})| = c$.

   The set of all functions on $\mathbb{R}$ is ${}^\mathbb{R}\mathbb{R}$. $|{}^\mathbb{R}\mathbb{R}| = (2^\omega)^{2^\omega} = 2^{\omega \cdot 2^\omega} = 2^{2^\omega} = 2^c$.

3. *Ch4: 2.* There are at least c countable order-types of linearly ordered sets.

SOLUTION: For every sequence $a = \langle a_n : n \in \mathbb{N}\rangle$ of natural numbers, let

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \dots$$

where ξ is the order-type of the integers.

If $a \neq b$, then $\tau_a \neq \tau_b$. Suppose not, let $\varphi : \tau_a \to \tau_b$ be the isomorphism and i be the least s.t $a_i \neq b_i$. Since φ is order preserving, it must be that $\varphi(a_0 + \xi + a_1 + \xi + \dots + a_{i-1} + \xi) = b_0 + \xi + b_1 + \xi + \dots + b_{i-1} + \xi$ (by induction). Without lose of generality, suppose $a_i < b_i$. Let $\eta_i = a_0 + \xi + a_1 + \xi + \dots + a_{i-1} + \xi$, and we identify η_i with $\varphi(\eta_i)$. Then $\varphi[\eta_i + a_i]$ is an initial part of $\eta_i + b_i$. Suppose c is the k-th element of b_i. Let c^* denote the element such that

$$\text{ordertype}(\{d \in \tau_b \mid d <_b c^*\}) = \eta_i + k.$$

Note that $\varphi^{-1}(a_i^*) > c$, for any $c \in \eta_i + a_i$. Thus $\varphi^{-1}(a_i^*) \in (\eta_i + a_i, +\infty)$. Then for any $c \in (\eta_i + a_i, \varphi^{-1}(a_i^*)]$, there is a $d \in (\eta_i + a_i, c)$. But this is not true on the τ_b side – there is a least element in $(\eta_i + a_i, a_i^*]$. This contradicts to that φ is isomorphism.

The map $a \to \tau_a$ is a injection from ${}^{\omega}\mathbb{N}$ into the set of order-types of linearly ordered sets. Hence

$$|\{\text{order-types of linearly ordered sets}\}| \geq |{}^{\omega}\mathbb{N}| = \omega^\omega = 2^\omega = \mathfrak{c}.$$

4. *Ch4: 3.* The set of all algebraic reals is countable.

SOLUTION: Since every algebraic real is one element of the finite roots of some polynomials in $\mathbb{Z}[x]$, and $|\mathbb{Z}[x]| = |\mathbb{Z}^{<\omega}|$ is countable, there are only countably many algebraic reals.

5. *Ch4: 4.* If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$.

SOLUTION: Let S^* be a countable subset of $\mathbb{R}^2$. Let $PS^* = \{x \in \mathbb{R} \mid \exists y((x, y) \in S^*)\}$. Since $PS^* \subset \mathbb{R}$ is countable, there exists an $x_0 \notin PS^*$. Then $(\{x_0\} \times \mathbb{R}) \subset ((\mathbb{R} - PS^*) \times \mathbb{R}) \subset (\mathbb{R}^2 - S^*)$. So $|\mathbb{R}^2 - S^*| \geq \mathfrak{c}$. But $|\mathbb{R}^2 - S^*| \leq \mathfrak{c}$ since it is a subset of $\mathbb{R}^2$. We can get that $|\mathbb{R}^2 - S^*| = \mathfrak{c}$.

Now, since $\phi : \mathbb{R} \approx \mathbb{R}^2$, there exists a countable sunset $S^* = \phi(S)$ of $\mathbb{R}^2$. So $(\mathbb{R} - S) \approx (\mathbb{R}^2 - S^*)$. Hence $|\mathbb{R} - S| = \mathfrak{c}$.

6. *Ch4: 5.*
   (a) The set of all irrational numbers has cardinality $\mathfrak{c}$.
   (b) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

SOLUTION: This is the direct conclusion of previous two exercises.

7. Prove proposition 5. Let T be a tree.
   (a) If $s, t, u \in T$, then $R_{stu} = \{\delta_{st}, \delta_{tu}, \delta_{su}\}$ has ≤ 2 elements, and $p, q \in R_{stu} \to p \subset q \vee q \subset p$.
   (b) $\prec$ is a linear ordering of T which extends $\sqsubseteq$.
   (c) For every $t \in T$, $T^t = \{s \in T \mid t \sqsubseteq s\}$ is an interval in $(T, \prec)$.

SOLUTION:

(a) It is to easy to prove that δ_{st} is an initial segment of $(\cdot, s)_T$. Fix $x \in \delta_{st}$, for any $x' < x$, $x' < s$ and $x' < t$ ($<$ is a partial order), i.e $x' \in \delta_{st}$.
Without losing generality, suppose $n_{st} \leq n_{tu} \leq n_{su}$. Since δ_{st}, δ_{tu} are both initial segments of well-ordered set $(\cdot, t)_T$. So $\delta_{st} \subset \delta_{tu}$. Similar, we have $\delta_{st} \subset \delta_{tu} \subset \delta_{su}$. Then $\delta_{st} = \delta_{st} \cap \delta_{su} = (\cdot, s)_T \cap (\cdot, t)_T \cap (\cdot, u)_T = \delta_{tu} \cap \delta_{su} = \delta_{tu}$. Hence $R_{stu} = \{\delta_{tu}, \delta_{su}\}$ and $\delta_{tu} \subset \delta_{su}$.

(b) i. (irreflective) Suppose $s \prec t$. if $s \sqsubseteq t$, obviously $t \not\prec s$. Otherwise, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge s(n_{st}) <_X t(n_{st})$, $t \not\prec s$ since X is linear ordered.

ii. (transitive) Suppose $s \prec t \wedge t \prec u$. It is easier when $s \sqsubseteq t$ or $t \sqsubseteq u$. Now prove the other case. If $n_{st} < n_{tu}$, then $n_{su} = n_{st}$(see the proof of (a)), $s(n_{su}) = s(n_{st}) <_X t(n_{st}) = u(n_{st}) = u(n_{su})$. If $n_{tu} < n_{st}$, then $n_{su} = n_{tu}$, similar, $s(n_{su}) <_X u(n_{su})$. If $n_{tu} = n_{st}$, then $n_{tu} = n_{st} = n_{su}$, since X is linear ordered, $s(n_{su}) <_X u(n_{su})$

iii. (trichotomous) Suppose $s \neq t$. There are exactly four cases $s \sqsubseteq t$, $t \sqsubseteq s$, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge s(n_{st}) <_X t(n_{st})$, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge t(n_{st}) <_X s(n_{st})$

(c) It suffices to prove that $s_1 \prec s \prec s_2$, $t \sqsubseteq s_1 \wedge t \sqsubseteq s_2$ implies $t \sqsubseteq s$. By definition, $t \sqsubseteq s \Rightarrow t \in (\cdot, s)_T$. So we have $t \in \delta_{s_1 s_2}$. Suppose $t \notin (\cdot, s)_T$. Then $\delta_{s_1 s} \neq \delta_{s_1 s_2} \wedge \delta_{s_2 s} \neq \delta_{s_1 s_2}$. By (a), $\delta_{s_1 s} = \delta_{s_2 s} \subsetneq \delta_{s_1 s_2}$. So $s_2(n_{s_2 s}) = s_1(n_{s_1 s}) < s(n_{s_1 s}) = s(n_{s_2 s})$ (contradiction!).

8. Prove Proposition 6. Let T, B_T be as above.

(a) $\prec$ is a linear ordering of $T \cup B_T$.

(b) For every $t \in T$, $B_t = \{f \in T \cup B_T \mid t \in f\}$ is an interval in $(T \cup B_T, \prec)$.

Solution: Consider $(T \cup B_T, \sqsubseteq^*)$, where $\sqsubseteq^* = \sqsubseteq \cup \{(t, f) \mid t \in T \wedge f \in B_T \wedge t \in f\}$. If $t \in T$, $(\cdot, t)_{T \cup B_T} = (\cdot, t)_T$ is well-ordered. If $t \in B_T$, $(\cdot, t)_{T \cup B_T} = \{s \in T \mid s \in t\} = t$ is also well-ordered. From this we can say $(T \cup B_T, \sqsubseteq^*)$ is a tree.

(a) By 2(b), we can define $\prec^*$ as a linear ordering of the tree $T \cup B_T$ which extends $\sqsubseteq^*$. Notice that for $f, g \in T \cup B_T, f \sqsubseteq g \Rightarrow f(n_{fg}) = \varnothing \leq_X g n_{fg}$. So $\prec = \prec^*$ is a linear ordering of $T \cup B_T$.

(b) By 2(c), $(T \cup B_T)^t = \{s \in T \cup B_T \mid t \sqsubseteq^* s\}$ is an interval in $(T \cup B_T, \prec)$. So $B_t = (T \cup B_T)^t$ is an interval in $(T \cup B_T, \prec)$.

9. If X is a Suslin line, then X^2 is not c.c.c.

Solution: We make the interval by recursion on α.

First, pick $a_0 < b_0 < c_0$ and make the first interval $(a_0, b_0) \times (b_0, c_0)$.

For $\alpha < \omega_1$, having constructed $(a_\beta, b_\beta) \times (b_\beta, c_\beta)$ for any $\beta < \alpha$, since $\{b_\beta \mid \beta < \alpha\} \in X$ is countable, it can't be dense in X. So there is $a_\alpha < c_\alpha$ s.t $\{b_\beta \mid \beta < \alpha\} \cap (a_\alpha < c_\alpha) = \varnothing$. We can choose $b_\alpha \in (a_\alpha, c_\alpha)$ since X is dense. It is obvious that $(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \cap (a_\beta, b_\beta) \times (b_\beta, c_\beta) = \varnothing$ for any $\beta < \alpha$.

Thus, we get $M = \{(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \mid \alpha < \omega_1\}$ is a pairwise-disjoint collection of open intervals of X^2, but $|M| = \omega_1 > \omega$.

10. *Ch4: 8.* If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \varnothing$, then $|P \cap (a, b)| = c$.

Solution: Obviously, $|P \cap (a, b)| \leq c$ since it is a subset of $\mathbb{R}$.

Choose $x_0 \in P \cap (a, b)$, if $[x_0, b) \subset P$ or $(a, x_0] \subset P$, we can easily get $|P \cap (a, b)| \geq c$. Now, suppose that $[x_0, b) \not\subseteq P$ and $(a, x_0] \not\subseteq P$. Pick $a_1 \in (a, x_0] - P$ and $b_1 \in [x_0, b) - P$, Then $P \cap [a_1, b_1]$ is a perfect. Since $P \cap [a_1, b_1]$ is a nonempty closed set, it suffices to prove the set has no isolated point. But $P \cap [a_1, b_1] = P \cap (a_1, b_1)$. For any $x \in P \cap (a_1, b_1)$ and any open neighborhood I of x, $(P \cap (a_1, b_1)) \cap I - \{x\} = P \cap ((a_1, b_1) \cap I) - \{x\} \neq \varnothing$, since x is a limit point of P. This means that $P \cap [a_1, b_1]$ is a perfect set. Thus $|P \cap (a, b)| \geq |P \cap [a_1, b_1]| = c$.

11. *Ch4: 9.* If $P_2 \not\subseteq P_1$ are perfect sets, then $|P_2 - P_1| = c$.

Solution: Pick $x \in P_2 - P_1$. Since P_1 is closed, there exists $\delta > 0$ such that $B(x, \delta) \subset P_1^c$. By exercise 4, $|P_2 \cap B(x, \delta)| = c$. Then $P_2 \cap B(x, \delta) \subset P_2 - P_1 \subset \mathbb{R}$, so $|P_2 - P_1| = c$.

12. *Ch4: 10.* If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

If P is perfect then $P = P^*$.

Solution: For any $x \in P$ and $\delta > 0$, $B(x, \delta) \cap P \neq \varnothing$. Using the conclusion of exercise 4, $|B(x, \delta) \cap P| = \mathfrak{c} > \aleph_0$, i.e $x \in P^*$.

On the other hand, $P' = \{x \in \mathbb{R} \mid \forall\delta > 0(B(x,\delta) \cap P - \{x\} \neq \varnothing)\}$. Clearly $P^* \subset P'$. But because P is closed, we have $P' \subset P$. Thus $P^* \subset P$.

So $P = P^*$ holds.

13. *Ch4: 11.* If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.

SOLUTION: We have prove that $P = P^*$ in the above exercise, so it is sufficient to show that $P^* \subset F^*$. For any $x \in P^*$, by definition, $\forall\delta > 0$, $|B(x,\delta) \cap P| > \aleph_0$. Then $|B(x,\delta) \cap F| \geq |B(x,\delta) \cap P| > \aleph_0$, which implies that $x \in F^*$.

14. *Ch4: 12.* If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$, thus $F^* = P$.

SOLUTION: For any $x \in F^*$ and $\delta > 0$, $|B(x,\delta) \cap F| > \aleph_0$. By theorem 4.6, $F = P \cup S$, where P is a perfect set and S is at most countable.

$$B(x,\delta) \cap F = (B(x,\delta) \cap P) \cup (B(x,\delta) \cap S)$$

$|B(x,\delta) \cap P|$ must be larger than $\aleph_0$. Otherwise, both of $B(x,\delta) \cap P$ and $B(x,\delta) \cap S$ are at most countable, which leads to that $B(x,\delta) \cap F$ is at most countable (contradiction). So we get $x \in P^* = P$. Thus $F^* \subset P$.

On the other hand, we have proved $P \subset F^*$ in exercise 7. So $F^* = P$.

15. *Ch4: 13.* If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.

SOLUTION: In the above exercise, we have proved that $F = F^* \cup (F - F^*)$ is a partition required. Here we prove the uniqueness. Suppose $F = P_1 \cup S_1 = P_2 \cup S_2$ are two partitions. If $P_1 \neq P_2$, without lost of generality, assume $P_2 \nsubseteq P_1$. By exercise 5, $|P_2 - P_1| = c$. But $|P_2 - P_1| = |S_1 - S_2| \leq \aleph_0$, which is a contradiction. So the partition is unique.

16. *Ch4: 15.* If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

SOLUTION: For each $\alpha < \omega_1$,

$$\begin{aligned}
\Sigma_1^0 &= \text{the collection of all open sets} \\
\Pi_1^0 &= \text{the collection of all closed sets} \\
\Sigma_\alpha^0 &= \{\bigcup B_n \mid \text{ each } B_n \in \Pi_\beta^0 \text{ some } \beta < \alpha\} \\
\Pi_\alpha^0 &= \{A^c \mid A \in \Sigma_\alpha^0\}
\end{aligned}$$

Then

$$\bigcup_{\alpha<\omega_1} \Sigma_\alpha^0 = \bigcup_{\alpha<\omega_1} \Pi_\alpha^0 = \mathscr{B}$$

Now we prove by induction that each Σ_α^0 is closed under inverse image by continuous function.

Σ_1^0 holds the property by the definition of continuous function.

For any $A \in \Sigma_\alpha^0$,

$$A = \bigcup B_n = \bigcup A_n^c$$

where each $B_n \in \Pi_{\beta_n}^0$, $A_n \in \Sigma_{\beta_n}^0$ for some $\beta_n < \alpha$. $f_{-1}(A) = f_{-1}(\bigcup A_n^c) = \bigcup f_{-1}(A_n^c) = \bigcup f_{-1}(A_n)^c$. By induction, each $f_{-1}(A_n) \in \Sigma_{\beta_n}^0$. Thus $f_{-1}(A) \in \Sigma_\alpha^0$.

So $\mathscr{B}$ is closed under inverse image by continuous function, i.e if B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

17. *Ch4: 18.* The tree T_F has no maximal note, i.e, $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential tree without maximal nodes.

Solution:

$$T_F = \{s \in \text{Seq} : s \subset f \text{ for some } f \in F\}$$

For any $s \in T$, there exists an f in F such that $s \subset f$, i.e $s = f \restriction n$ for some $n < \omega$. Let $t = f \restriction (n+1)$, thus we have $s \in t$ and $s \subset t$.

It suffices to prove that the map $F \mapsto T_F$ is one-to-one. It is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f \restriction n \in T_F$ for all $n \in \mathbb{N}$, then for each n, there is some $g \in F$ such that $g \restriction n = f \restriction n$; and since F is closed, it follows that $f \in F$. Let F_1, F_2 be closed sets in $\mathcal{N}$. If $F_1 \neq F_2$, say $g \in F_1 - F_2$, then for all n, $g \restriction n \in T_{F_1}$ (by definition of T_{F_1}), but $g \restriction m \notin T_{F_2}$ for some m (since F_2 is closed), $T_{F_1} \neq T_{F_2}$.