Solutions for Assignment # 3

December 1, 2025

1. Write an explicit formula for this bijection.

   SOLUTION:

   The area in the picture is $\kappa \times \kappa$ affine transformation $A = \{(\alpha, \beta) \in \kappa \times \kappa \mid \beta \leq \alpha\}$. And there is a nature bijection

$$
\begin{aligned}
f : \kappa \times \kappa &\to A \\
(\alpha, \beta) &\mapsto (\beta + \alpha, \beta)
\end{aligned}
$$

   The picture give us a well-ordering of A:

$$
\begin{aligned}
g : A &\to \kappa \\
(\alpha, \beta) &\mapsto \sum_{\gamma < \alpha} (\gamma + 1) + \beta
\end{aligned}
$$

   Where $\sum_{\gamma<\alpha}(\gamma + 1)$ is the sum of all ordinals $< \alpha$, which can be defined recursively: For all ordinal number α,

   (a) $\sum_{\gamma<0}(\gamma + 1) = 0$.
   (b) $\sum_{\gamma<\alpha+1}(\gamma + 1) = \sum_{\gamma<\alpha}(\gamma + 1) + \alpha + 1$.
   (c) $\sum_{\gamma<\alpha} \gamma = \lim_{\xi\to\alpha} \sum_{\gamma<\xi} \gamma$, for limit $\alpha > 0$.

   And one can easily verify that g is a bijection. So the bijection from $\kappa \times \kappa \to \kappa$ should be

$$
\begin{aligned}
g \circ f : \kappa \times \kappa &\to \kappa \\
(\alpha, \beta) &\mapsto \sum_{\gamma < \beta + \alpha} (\gamma + 1) + \beta
\end{aligned}
$$

2. **Instructions: State the value of the following expressions.**

   (i) $\aleph_0 + \aleph_0$
   (ii) $\aleph_0 \cdot \aleph_0$
   (iii) $\mathfrak{c} + \mathfrak{c}$
   (iv) $\mathfrak{c} \cdot \mathfrak{c}$
   (v) $2^{\aleph_0}$
   (vi) $\mathfrak{c}^{\aleph_0}$ **Where** $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$**.**

   SOLUTION:

   (i) $\aleph_0$
   (ii) $\aleph_0$
   (iii) $\mathfrak{c}$
   (iv) $\mathfrak{c}$
   (v) $\mathfrak{c}$

(vi) $\mathfrak{c}$

3. Assuming the **Generalized Continuum Hypothesis (GCH), which states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α, evaluate the following expressions.**

   (i) $2^{\mathfrak{c}}$
   (ii) $\aleph_1^{\aleph_0}$
   (iii) $\mathfrak{c}^{\aleph_1}$

   SOLUTION:

   (a) $\aleph_2$
   (b) $\aleph_1$
   (c) $\aleph_2$

4. Determine the relationship between the two cardinal number $(\leq, \geq, =)$. You may use the fact that for infinite cardinals κ and λ, if $\kappa \leq \lambda$, then $\kappa + \lambda = \lambda$ and $\kappa \cdot \lambda$.

   (i) $\aleph_0 + \mathfrak{c}$______$\mathfrak{c}$
   (ii) $\aleph_0 \cdot \mathfrak{c}$______$\mathfrak{c}$
   (iii) $2^{\mathfrak{c}}$______$2^{\aleph_0} + \mathfrak{c}$
   (iv) $(\aleph_0 + \mathfrak{c})^{\aleph_0}$______$\mathfrak{c}$

   SOLUTION:

   (i) $=$
   (ii) $=$
   (iii) $\geq$
   (iv) $=$

5. Prove the following statements.

   (a) If $x \cap y = \varnothing$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.
   (b) If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.

   SOLUTION:

   (a) Intuitively speaking, $x \cup y \preccurlyeq y$ means if via an injective function one can get a copy of (a disjoint union of) x and y inside y. Iterating this process ω many times, one can get ω many disjoint copies of x inside y, i.e. $\omega \times x \preccurlyeq y$.
   Formally, let $f : x \cup y \to y$ be an injection. Define the function

$$\begin{aligned} g : \omega \times x &\to y \\ (n, a) &\mapsto f^{n+1}(a) \end{aligned}$$

   where f^{n+1} denotes the n times composition of f. One can check that g is an injection from $\omega \times x$ to y.

   (b) Intuitively speaking, $\omega \times x \preccurlyeq y$ means we can insert ω many x into y , then we can move the n-th copy of x in y to $n+1$-th, thus leave a place for one x.
   Formally, from the definition of $\omega \times x \preccurlyeq y$, there is an injection $f : \omega \times x \to y$. Define the function

$$\begin{aligned} g : x \cup y &\to y \\ a \mapsto & f(0, a), \text{ if } a \in x \\ & f(n+1, b), \text{ if } \exists (n, b) \in \omega \times x, a = f(n, b) \\ & a, \text{ otherwize} \end{aligned}$$

   Then one can check that g is a bijection from $x \cup y$ to y.

Ex.3.1 (a) A subset of a finite set is finite.

(b) The union of a finite set of finite sets is finite.

(c) The power set of a finite set is finite.

(d) The image of a finite set(under a mapping) is finite.

SOLUTION:

(a) By definition, X is finite iff $|X| < \omega$. Suppose $Y \subset X$, then it follows that $Y \preccurlyeq X$. Thus Y can be well-ordered and $|Y| \leq |X| < \omega$. So Y is finite.

(b) Let $X = \{x_1, x_2 \ldots x_n\}(n \in \mathbb{N})$, where $|x_i| < \omega$ for $i = 1, 2, \ldots, n$. Let

$$A = \bigcup_{i=1}^{n} (\{i\} \times x_i)$$

Then A can be well-ordered and $|A| = |x_1| + |x_2| + \cdots + |x_n| < \omega$. For any $x \in \bigcup X$, let $f(x) =$ (the least i s.t. $x \in x_i, x) \in A$. It is clear that f is an injection from $\bigcup X$ to A, which implies that $\bigcup X$ is finite.

(c) There is a bijection f from X to some $n < \omega$. Define F by $F(E) = f(E)$ for any $E \subset X$. F is a bijection from $\mathscr{P}(X)$ onto $\mathscr{P}(n)$. Since $|\mathscr{P}(n)| = 2^n < \omega$, $\mathscr{P}(X)$ is finite.

(d) Denote the finite set by X and the mapping f. g is the bijection from X onto $n < \omega$. Then $f \circ g^{-1}$ is a mapping from n to $\operatorname{ran}(f)$. For any $y \in \operatorname{ran}(f)$, let $F(y) = m \in n$, where m is the least element s.t $f \circ g^{-1}(m) = y$. F is a 1-1 mapping from $\operatorname{ran}(f)$ into m. So $\operatorname{ran}(f)$ is finite.

Ex.3.2 (a) A subset of a countable set is at most countable.

(b) The union of a finite set of countable sets is countable.

(c) The image of a countable set (under a mapping) is at most countable.

SOLUTION:

(a) By definition, X is countable iff $|X| = \omega$. Suppose $Y \subset X$, then it follows that $Y \preccurlyeq X$. Thus Y can be well-ordered and $|Y| \leq |X| \leq \omega$. So Y is at most countable.

(b) Let $X = \{x_1, x_2 \ldots x_n\}(n \in \mathbb{N})$, where $|x_i| = \omega$ (the bijection is f_i) for $i = 1, 2, \ldots, n$. Let

$$A = \bigcup_{i=1}^{n} (\{i\} \times x_i)$$

Let $F(i, y) = n \cdot f_i(y) + i - 1$. Then F is a bijection from A onto ω. So A can be well-ordered $A \approx \omega$. For any $x \in \bigcup X$, let $f(x) =$ (the least i s.t. $x \in x_i, x) \in A$. It is clear that f is an injection from $\bigcup X$ into A, which implies that $|\bigcup X| \leq \omega$. On the other hand, $|\bigcup X| \geq \omega$ since $\bigcup X$ has a countable subset x_1. So $\bigcup X$ is countable.

(c) Denote the finite set by X and the mapping f. g is the bijection from X onto ω. Then $f \circ g^{-1}$ is a mapping from ω to $\operatorname{ran}(f)$. For any $y \in \operatorname{ran}(f)$, let $F(y) = m \in \omega$, where m is the least element s.t $f \circ g^{-1}(m) = y$. F is a 1-1 mapping from $\operatorname{ran}(f)$ into ω. So $\operatorname{ran}(f)$ is at most countable.

Ex.3.3 $\mathbb{N} \times \mathbb{N}$ is countable. [$f(m, n) = 2^m(2n + 1) - 1$.]

SOLUTION: Define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $f(m, n) = 2^m(2n + 1) - 1$. For any $s \in \mathbb{N}$, let

$$m = \max\{k \in \mathbb{N} \mid 2^k | (s + 1)\}, n = \frac{(s + 1)2^{-m} - 1}{2} \in \mathbb{N}$$

Then $f(m, n) = s$. On the other hand, suppose $2^{m_1}(2n_1 + 1) - 1 = 2^{m_2}(2n_2 + 1) - 1 (m_1 \leq m_2)$, then $(2n_1 + 1) = 2^{m_2 - m_1}(2n_2 + 1)$, which implies $2^{m_2 - m_1} = 1$, i.e $m_1 = m_2$. Then we have $n_1 = n_2$. So f is a bijection. It follows that $\mathbb{N} \times \mathbb{N}$ is countable.

7 Prove that $\kappa^\kappa \leq 2^{\kappa\cdot\kappa}$.

Solution: For any mapping f from κ to κ , f is a subset of $\kappa \times \kappa$, i.e. $f \in \mathscr{P}(\kappa \times \kappa)$. From this we have ${}^\kappa\kappa \subset \mathscr{P}(\kappa \times \kappa)$. It follows that ${}^\kappa\kappa \preccurlyeq \mathscr{P}(\kappa \times \kappa)$, i.e. $\kappa^\kappa \leq 2^{\kappa\cdot\kappa}$.

8 If $A \preccurlyeq B$, then $A \preccurlyeq^* B$.

Solution: If $A = \varnothing$, then $A \preccurlyeq^* B$. Otherwise there is a $x_0 \in A$. Denote the one-to-one mapping for A into B by f. Let

$$g = \{(y, x) \mid (x, y) \in f\} \cup \{(y, x_0) \mid y \in B \setminus \operatorname{ran}(f)\}$$

Then g is a function. First, $\operatorname{dom}(g) = \operatorname{ran}(f) \cup (B \setminus \operatorname{ran}(f)) = B$, i.e. for any $y \in B$, there exists an x such that $(y, x) \in g$. Second, suppose $(y, x_1), (y, x_2) \in g$. If $y \notin ran(f)$, then $x_1 = x_0 = x_2$. If $y \in \operatorname{ran}(f)$, then $(x_1, y), (x_2, y) \in f$. Since f is 1-1, we have $x_1 = x_2$.

Furthermore, g is a surjection, since $\operatorname{ran}(g) = \operatorname{dom}(f) \cup \{x_0\} = \operatorname{dom}(f) = A$.

9 If $A \preccurlyeq^* B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.

Solution: If $A = \varnothing$, then $\mathscr{P}(A) = \{\varnothing\} \subseteq \mathscr{P}(B)$. Otherwise let $f : B \to A$ be a surjection. For any $x \subset A$, let $F(x) = f^{-1}[x] \subset B$. Then F is a function from $\mathscr{P}(A)$ into $\mathscr{P}(B)$. F is injection, since if $x_1 \neq x_2$, say $t \in x_1 \setminus x_2$, then $\varnothing \neq f^{-1}[\{t\}] \subseteq f^{-1}[x_1] \setminus f^{-1}[x_2]$ (f being onto ensures that $f^{-1}[\{t\}] \neq \varnothing$), thus $F(x_1) \neq F(x_2)$.

10 Let X be a set. If there is an injective function $f : X \longrightarrow X$ s.t $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

(Dedekind infinite is infinite)

Solution: *Method I.* Suppose NOT, X is finite. There is a bijection from X to some $n \in \mathbb{N}$, denote it by g. Then $F = g \circ f \circ g^{-1}$ is an injection from n into n, s.t $\operatorname{ran}(F) \subsetneq n$. But this contradicts to the fact that every $n \in \omega$ has no proper subsets of the same cardinality.

We show by induction that no $n \in \omega$ has proper subsets of the same cardinality. We go from n to $n+1$. Let $h : n+1 \to n+1$ be a non-surjective injection and $h' = h \restriction n$. We modify h' to get an $f : n \to n$. Consider n, if $n \notin \operatorname{ran}(h')$, let $f = h'$; if $n \in \operatorname{ran}(h')$, then let $f(h^{-1}(n)) = h(n)$ and for $i \in n \setminus h^{-1}(n)$, $f(i) = h'(i)$. In either case $f : n \to n$ is a non-surjective injection.

Method II. (AC). We shall construct an injection $g : \omega \to X$. Let $X_0 = X$ and $X_{n+1} = f[X_n]$ for all $n \in \omega$. Since $\operatorname{ran}(f) \subsetneq X$, by induction, one can see that for each $n \in \omega$, $X_{n+1} - X_n \neq \varnothing$, hence can select $g(n) \in X_{n+1} - X_n$ for each $n \in \omega$. These $g(n)$'s are clearly distinct, hence $g : \omega \to X$ is an injection. This shows that X is infinite.

1. Calculate the result of the cardinal operation $\aleph_0 + \aleph_2$.

   (a) $\aleph_0$
   (b) $\aleph_2$
   (c) $\aleph_3$
   (d) $2\aleph_2$

   Solution: (b)

2. Calculate the confinality of the ordinal $\alpha = \omega^4 + 1$.

   (a) ω
   (b) 1
   (c) 7
   (d) 4

   Solution: (b)

3. Which of the following cardinal numbers is singular?

   (a) $\aleph_1$

(b) $\aleph_0$
(c) $\aleph_\omega$
(d) $\aleph_5$

SOLUTION: (c)

4. Caculate the cofinality of the ordinal product $\alpha = \omega_1 \cdot \omega^2$

(a) ω^2
(b) ω_1
(c) 1
(d) ω

SOLUTION: (d)

5. What is the value of the cardinal exponentiation $2^{\aleph_1}$ under the **Generalized Continuum Hypothesis (GCH)?**

(a) $\aleph_1$
(b) $\aleph_0$
(c) $\aleph_2$
(d) $\mathfrak{c}$

SOLUTION: (c)

6. If κ is a **regular infinite cardinal, what is the cofinality of the sum $\kappa + \kappa$(cardinal sum)?**

(a) 2
(b) $\aleph_0$
(c) κ
(d) κ^+

SOLUTION: (c)

7. Caculate the cofinality of the ordinal $\beta = \omega^\omega + 1$.

(a) ω
(b) 1
(c) ω^ω
(d) 2

SOLUTION: (b)

8. Let κ be a regular cardinal. What is the value of κ^λ when $\lambda < \kappa$?

(a) κ
(b) 2^λ
(c) λ
(d) $\kappa^{\mathrm{cf}(\kappa)}$

SOLUTION: (a) if $\forall \delta < \kappa(\delta^\lambda \leq \kappa)$, (b) if $2^\lambda \geq \kappa$

9. What is the cofinality of the limit cardinal $\aleph_{\omega^2}$?

(a) $\aleph_{\omega^2}$
(b) $\aleph_0$

(c) $\aleph_2$

(d) ω^2

<u>Solution</u>: (b)

10. Which statement about the cofinality $\mathrm{cf}(\kappa)$ of the infinite cardinal κ is always true?

(a) $\mathrm{cf}(\kappa) = \aleph_0$

(b) $\mathrm{cf}(\kappa) = \kappa$

(c) $\mathrm{cf}(\kappa)$ is a regular cardinal.

(d) $\mathrm{cf}(\kappa) < \kappa$

<u>Solution</u>: (c)