Solutions for Assignment # 3

October 28, 2025

1. Let $\alpha, \beta, \gamma \in \text{Ord}$ and let $\alpha < \beta$. Then

   (a) $\alpha + \gamma \leq \beta + \gamma$

   (b) $\alpha \cdot \gamma \leq \beta \cdot \gamma$

   (c) $\alpha^\gamma \leq \beta^\gamma$

   Given examples to show that $\leq$ cannot be replaced by $<$ in either inequality.

   SOLUTION: We prove the proposition by induction on γ. In order to use the conclusion later, we only suppose that $\alpha \leq \beta$.

   (a) i. $\gamma = 0$, it is obvious that $\alpha + 0 \leq \beta + 0$.

   ii. Suppose the inequality holds for γ, i.e. $\alpha + \gamma \leq \beta + \gamma$, then we have

$$\alpha + \gamma < (\beta + \gamma) + 1 = \beta + (\gamma + 1)$$

   Noting that

$$\alpha + (\gamma + 1) = (\alpha + \gamma) + 1 = \inf\{\xi \in \text{Ord} \mid \alpha + \gamma < \xi\}$$

   it follows that

$$\alpha + (\gamma + 1) \leq \beta + (\gamma + 1)$$

   iii. Suppose γ is a limit ordinal and the inequality holds for any ordinal less than γ. Then by definition,

$$\alpha + \gamma = \lim_{\xi \to \gamma} (\alpha + \xi) = \sup\{\alpha + \xi \mid \xi < \gamma\}$$

$$\beta + \gamma = \lim_{\xi \to \gamma} (\beta + \xi) = \sup\{\beta + \xi \mid \xi < \gamma\}$$

   By induction, for any $\xi < \gamma$, $\alpha + \xi \leq \beta + \xi \leq \beta + \gamma$, then

$$\alpha + \gamma \leq \beta + \gamma$$

   Example: $1 < 2$, but $1 + \omega = \omega = 2 + \omega$

   (b) i. $\gamma = 0$, it is obvious that $\alpha \cdot 0 = 0 = \beta \cdot 0$.

   ii. Suppose the inequality holds for γ, i.e. $\alpha \cdot \gamma \leq \beta \cdot \gamma$, then we have

$$\begin{array}{rcll} \alpha \cdot (\gamma + 1) & = & \alpha \cdot \gamma + \alpha & \text{(definition)} \\ & \leq & \alpha \cdot \gamma + \beta & \text{(Lemma 2.25)} \\ & \leq & \beta \cdot \gamma + \beta & \text{(induction+(a))} \\ & = & \beta \cdot (\gamma + 1) & \end{array}$$

   iii. Suppose γ is a limit ordinal and the inequality holds for any ordinal less than γ. Then by definition,

$$\alpha \cdot \gamma = \lim_{\xi \to \gamma} (\alpha \cdot \xi) = \sup\{\alpha \cdot \xi \mid \xi < \gamma\}$$

$$\beta \cdot \gamma = \lim_{\xi \to \gamma} (\beta \cdot \xi) = \sup\{\beta \cdot \xi \mid \xi < \gamma\}$$

   By induction, for any $\xi < \gamma$, $\alpha \cdot \xi \leq \beta \cdot \xi \leq \beta \cdot \gamma$, then

$$\alpha \cdot \gamma \leq \beta \cdot \gamma$$

   Example: $1 < 2$, but $1 \cdot \omega = \omega = 2 \cdot \omega$

(c) i. $\gamma = 0$, it is obvious that $\alpha^0 = 1 = \beta^0$.

ii. Suppose the inequality holds for γ, i.e. $\alpha^\gamma \le \beta^\gamma$, then we have

$$\begin{array}{rcll} \alpha^{\gamma+1} & = & \alpha^\gamma \cdot \alpha & \text{(definition)} \\ & \le & \alpha^\gamma \cdot \beta & \text{(Lemma 2.25, if } \alpha^\gamma = 0 \text{ or } \alpha = \beta, \text{ ``='' holds)} \\ & \le & \beta^\gamma \cdot \beta & \text{(induction+(b))} \\ & = & \beta^{\gamma+1} & \end{array}$$

iii. Suppose γ is a limit ordinal and the inequality holds for any ordinal less than γ. Then by definition,

$$\alpha^\gamma = \lim_{\xi \to \gamma} (\alpha^\xi) = \sup\{\alpha^\xi \mid \xi < \gamma\}$$

$$\beta^\gamma = \lim_{\xi \to \gamma} (\beta^\xi) = \sup\{\beta^\xi \mid \xi < \gamma\}$$

By induction, for any $\xi < \gamma$, $\alpha^\xi \le \beta^\xi \le \beta^\gamma$, then

$$\alpha^\gamma \le \beta^\gamma$$

Example: $2 < 3$, but $2^\omega = \omega = 3^\omega$

2. Show that the following rules do not hold for all. $\alpha, \beta, \gamma \in \text{Ord}$:

(a) If $\alpha + \gamma = \beta + \gamma$ then $\alpha = \beta$.

(b) If $\gamma > 0$ and $\alpha \cdot \gamma = \beta \cdot \gamma$ then $\alpha = \beta$.

(c) $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$

<u>Solution</u>:

(a) $1 + \omega = \omega = 2 + \omega$, but $1 < 2$.

(b) If $\omega > 0$ and $1 \cdot \omega = \omega = 2 \cdot \omega$, but $1 < 2$.

(c) $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega < \omega + 1 \le \omega + \omega = 1 \cdot \omega + 1 \cdot \omega$

3. Find a set $A \subset \mathbb{Q}$, such that$(A, <_{\mathbb{Q}}) \cong (\alpha, \in)$, where

(a) $\alpha = \omega + 1$

(b) $\alpha = \omega \cdot 2$

(c) $\alpha = \omega \cdot \omega$

(d) $\alpha = \omega^\omega$

<u>Solution</u>:

(a) $A = \{-1, -\frac{1}{2}, \cdots, -\frac{1}{2^n}, \cdots, 0\}$. The isomorphism $f : A \to \omega + 1$ is:

$$f(-\frac{1}{2^n}) = n, f(0) = \omega$$

(b) $A = \{-1, -\frac{1}{2}, \cdots, -\frac{1}{2^n}, \cdots, 0, \frac{1}{2}, \frac{3}{4}, \cdots, 1 - \frac{1}{2^n}, \cdots\}$. The isomorphism $f : A \to \omega + 2$ is:

$$f(-\frac{1}{2^n}) = n, f(1 - \frac{1}{2^n}) = \omega + n$$

(c) $A = \{m - \frac{1}{2^n} \mid m, n \in \mathbb{N}\}$. The isomorphism $f : A \to \omega \cdot \omega$ is:

$$f(m - \frac{1}{2^n}) = \omega \cdot m + n$$

(d) By Cantor's Normal form Theorem, for any ordinal $\alpha \in \omega^\omega$,

$$\alpha = k_n + \omega \cdot k_{n-1} + \cdots + \omega^n \cdot k_0 (k_0 \neq 0)$$

Let

$$g(\alpha) = n - 2^{-k_0} - 2^{-k_0-(k_1+1)} - \cdots - 2^{-k_0-(k_1+1)-\cdots-(k_n+1)}$$

Then g is an isomorphism from ω^ω to $g(\omega^\omega) \subset \mathbb{Q}$

4. An ordinal α is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some $\beta \in \text{Ord} \setminus \{0\}$

SOLUTION: Suppose α is a limit ordinal, then there exists a unique β and n, such that $\alpha = \omega \cdot \beta + n$, and $n < \omega$. If $n \neq 0$, it must be $m + 1$ for some $m < \omega$. But then $\alpha = (\omega \cdot \beta + m) + 1$, which contradicts to that α is limit.

(*Method I*). Suppose $\alpha = \alpha' + 1$ for some α', then there exists a unique β' and n', such that $\alpha' = \omega \cdot \beta' + n'$, and $n' < \omega$. Let $\beta = \beta', n = n' + 1$, we have $\alpha = \omega \cdot \beta + n$, where $n < \omega$. By the uniqueness of β and n, α can't be written as $\alpha = \omega \cdot \beta$ for some $\beta \in \text{Ord}$.

(*Method II*). There are two cases for β. (a) $\beta = \gamma + 1$ for some γ. Then $\alpha = \omega(\gamma+1) = \sup\{(\omega \cdot \gamma + n \mid n < \omega\}$ is a limit ordinal. (b) β is a limit ordinal. Then $\alpha = \sup\{\omega \cdot \gamma \mid \gamma < \beta\}$ is also a limit ordinal.

5. Find the first three $\alpha > 0$ s.t. $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

SOLUTION: The least α is 1. The only ordinal less than 1 is 0, which satisfies that $0 + 1 = 1$. On the other hand, 1 is the least ordinal > 0.

If we suppose $\alpha > 1$, the least ordinal is ω. For any $n < \omega$, $n + \omega = \lim_{m \to \omega}(n + m) = \omega$. On the other hand, for any $1 < m < \omega$, there exists an m' such that $m = m' + 1$ and $m' > 0$, thus $m' + m > m$.

Suppose $\alpha > \omega$, the least ordinal is ω^2. For any $\beta < \omega^2$, $\beta = \omega \cdot m + n$ and $m, n < \omega$. $\beta + \omega^2 = \omega \cdot m + n + \omega \cdot \omega = \omega^2$. On the other hand, for any $\omega < \beta = \omega \cdot m + n < \omega^2$, there exists β' such that $\beta = \beta' + \omega + n$ and $\beta' > 0$; thus $\beta' + \beta > \beta$.

6. Find the least ξ such that

(a) $\omega + \xi = \xi$

(b) $\omega \cdot \xi = \xi$, $\xi \neq 0$

(c) $\omega^\xi = \xi$

(Hint for (1): Consider a sequence $\langle \xi_n \rangle$ s.t. $\xi_{n+1} = \omega + \xi_n$.)

SOLUTION:

(a) Construct a sequence $\langle \xi_n \rangle$: $\xi_1 = \omega$, $\xi_{n+1} = \omega + \xi_n$. Then $\langle \xi_n \rangle$ is a set belongs to Ord. In fact, $\xi_n = \omega \cdot n$, let

$$\xi = \lim_{n \to \omega} \xi_n = \omega \cdot \omega$$

It is easy to verify that $\omega + \xi = \xi$. On the other hand, for any $\alpha < \xi$, $\alpha = \omega \cdot k_1 + k_2$, where $k_1, k_2 < \omega$. Then

$$\omega + \alpha = \omega + \omega \cdot k_1 + k_2 = \omega \cdot (k_1 + 1) + k_2 > \omega \cdot k_1 + k_2 = \alpha$$

(b) Construct a sequence $\langle \xi_n \rangle$: $\xi_1 = \omega$, $\xi_{n+1} = \omega \cdot \xi_n$. Then $\langle \xi_n \rangle$ is a set belongs to Ord. In fact, $\xi_n = \omega^n$, let

$$\xi = \lim_{n \to \omega} \xi_n = \omega^\omega$$

It is easy to verify that $\omega \cdot \xi = \xi$. On the other hand, for any $\alpha < \omega^\omega$, there exists an n such that

$$\omega^n \leq \alpha < \omega^{n+1}$$

Actually, $n = \sup\{m \in \omega \mid x \geq \omega^m\}$, where $\{m \in \omega \mid x \geq \omega^m\}$ is an initial segment of ω. Thus we have

$$\omega \cdot \alpha \geq \omega \cdot \omega^n = \omega^{n+1} > \alpha$$

(c) Construct a sequence $\langle \xi_n \rangle$: $\xi_1 = \omega$, $\xi_{n+1} = \omega^{\xi_n}$. Then $\langle \xi_n \rangle$ is a set belongs to Ord. Let

$$\xi = \lim_{n \to \omega} \xi_n$$

It is easy to verify that $\omega^\xi = \xi$. On the other hand, for any $\alpha < \xi$, there exists an n such that

$$\xi_n \leq \alpha < \xi_{n+1}$$

Actually, $n = \sup\{m \in \omega \mid x \geq \xi_m\}$, where $\{m \in \omega \mid x \geq \xi_m\}$ is an initial segment of ω. Thus we have

$$\omega^\alpha \geq \omega^{\xi_n} = \xi_{n+1} > \alpha.$$

Exercises in About V

By transfinite recursion, define

$$V_0 = \varnothing,$$
$$V_{n+1} = \mathcal{P}(V_n),$$
$$V_\omega = \bigcup_{n<\omega} V_n.$$

1. Every $x \in V_\omega$ is finite.

SOLUTION: Fix $x \in V_\omega$. There is an n such that $x \in V_n$.
Claim. For each n, V_n is transitive, and $|V_{n+1}| = 2^n$.

Proof of the Claim. We prove by induction on n. This is clearly true for $n = 0$. We proceed from n to $n+1$. Clearly $|V_{n+2}| = 2^{n+1}$ by induction and simple calculation. Let y be any element of V_{n+1}. Then $y \subseteq V_n$, by definition. Since V_n is transitive, $\forall z(z \in V_n \to z \subseteq V_n)$, we have $V_n \subseteq V_{n+1}$. Thus $y \subseteq V_{n+1}$. This shows that for every n, V_n is transitive, and $|V_{n+1}| = 2^n$. $\dashv$ (*Claim*)

By the claim, $x \subseteq V_n$, and $|x| \leq |V_n|$. Therefore x is finite.

2. V_ω is transitive.

SOLUTION: This follows from the above claim. Let $x \in V_\omega$, then $x \in V_n$ for some n. By the transitivity of V_n and the definition of V_ω, $x \subseteq V_n \subseteq V_\omega$.

3. V_ω is an inductive set.

SOLUTION: First $\varnothing \in V_1 \subseteq V_\omega$. Now fix $x \in V_\omega$ and an n such that $x \in V_n$. Then $x \cup \{x\} \subseteq V_n \cup V_{n+1} \subseteq V_{n+1}$. The last step follows from the claim that every V_n is transitive. Hence $x \cup \{x\} \in V_{n+2} \subseteq V_\omega$.

1. If $x, y \in V_\omega$ then $\{x, y\} \in V_\omega$.

SOLUTION: Suppose $x \in V_m$ and $y \in V_n$. We may assume that $m \leq n$. By the transitivity of V_n's, $x, y \in V_n$, and hence $\{x, y\} \in V_{n+1} \subseteq V_\omega$.

2. If $x \in V_\omega$, then $\bigcup x \in V_\omega$ and $\mathcal{P}(x) \in V_\omega$.

SOLUTION: Fix an n such that $x \in V_n$. Since V_n is transitive, $x \subseteq V_n$. Then $\bigcup x \subseteq \bigcup\{V_n \mid z \in x\} = V_n$ and $\mathcal{P}(x) \subseteq \mathcal{P}(V_n)$. These implies that $\bigcup x \in V_{n+1}$ and $\mathcal{P}(x) \in V_{n+2}$, therefore both in V_ω.

3. If $A \in V_\omega$ and f is a function on A such that $f(x) \in V_\omega$ for each $x \in A$, then $f[A] \in V_\omega$.

SOLUTION: If $A \in V_\omega$, by (61), A is finite. Then $f[A]$ is a finite subset of V_ω, so the conclusion follows from (64).

4. If x is a finite subset of V_ω, then $x \in V_\omega$.

SOLUTION: Suppose $x = \{a_i \mid i = 1, \ldots, n\}$. Let $C = \{k_i \mid a_i \in V_{k_i}\}$. C is finite set of numbers and has a largest number K. Since V_n's are all transitive, $x \subseteq V_K$, hence $x \in V_{K+1} \subseteq V_\omega$.