Solutions for Assignment # 1

October 20, 2025

1. Using only $\hat{\in}$ and $\hat{=}$ to express the following formulas:
   (a) $z \hat{=} ((x,y),(u,v))$
   (b) $\forall x[\neg(x \hat{=} \varnothing) \to (\exists y \hat{\in} x)(x \cap y \hat{=} \varnothing)]$
   (c) $\forall u[\forall x \exists y (x,y) \hat{\in} u \to \exists f \forall x (x, f(x)) \hat{\in} u]$

   <u>Solution</u>:

   (a) Note that $(x,y) = \{\{x\},\{x,y\}\}$. The formula $z \hat{=} (x,y)$ can be expressed as:

   $$\varphi(z,x,y) \equiv \forall u(u \hat{\in} z \leftrightarrow \forall v(v \hat{\in} u \leftrightarrow v \hat{=} x) \vee \forall v(v \hat{\in} u \leftrightarrow v \hat{=} x \vee v \hat{=} y))$$

   So $z \hat{=} ((x,y),(u,v))$ can be expressed as:

   $$\exists z_1 \exists z_2 (\varphi(z_1,x,y) \wedge \varphi(z_2,u,v) \wedge \varphi(z,z_1,z_2))$$

   (b) Note that $x \cap y \hat{=} \varnothing$ iff $\neg \exists z(z \hat{\in} y \wedge z \hat{\in} x)$, $\exists y \hat{\in} x$ means $\exists y(y \hat{\in} x)$. This formula can be expressed as:

   $$\forall x\,(\exists v\,(v \in x) \to \exists y\,(y \hat{\in} x \wedge \neg \exists u\,(u \in x \wedge u \in y))$$

   (c) Note that function f is a binary relation and $(\forall (x,y),(x,z) \hat{\in} f)(y \hat{=} z)$. "$f$ is a binary relation" can be expressed as:

   $$\varphi_1(f) \equiv \forall z(z \hat{\in} f \leftrightarrow \exists x \exists y \varphi(z,x,y))$$

   $\varphi(z,x,y)$ means $z \hat{=} (x,y)$ defined in (a).
   "f is a function" can be expressed as:

   $$\varphi_2(f) \equiv \varphi_1(f) \wedge \forall x \forall y \forall z \exists u \exists v(\varphi(x,y,u) \wedge \varphi(x,z,v) \wedge (u \hat{\in} f \wedge v \hat{\in} f \to y \hat{=} z))$$

   So the formula $\forall u[\forall x \exists y (x,y) \hat{\in} u \to \exists f \forall x (x,f(x)) \hat{\in} u]$ can be expressed as:

   $$\forall u(\forall x \exists y \exists z(\varphi(z,x,y) \wedge z \hat{\in} u) \to \exists f(\varphi_2(f) \wedge \forall x \exists y \exists z(\varphi(z,x,y) \wedge z \hat{\in} f \wedge z \hat{\in} u)))$$

2. Suppose that R, S are two relations. Show that R_{-1} and $S \circ R$ exist.

   <u>Solution</u>: Since $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are two sets, so are $\operatorname{ran}(R) \times \operatorname{dom}(R)$. By Comprehension Schema,

   $$R_{-1} = \{(u,v) \in \operatorname{ran}(R) \times \operatorname{dom}(R) \mid (v,u) \in R\}$$

   exists. $R_{-1} \subset \mathscr{P}(\mathscr{P}(\bigcup\bigcup R))$ with Comprehension also shows R_{-1} is a set.

   Since $\operatorname{dom}(R)$ and $\operatorname{ran}(S)$ are two sets, so does $\operatorname{dom}(R) \times \operatorname{ran}(S)$. By Comprehension Schema,

   $$S \circ R = \{(u,v) \in \operatorname{dom}(R) \times \operatorname{ran}(S) \mid \exists w((u,w) \in R \wedge (w,v) \in S)\}$$

   exists. $S \circ R \subset \mathscr{P}(\mathscr{P}(\bigcup\bigcup(R \cup S)))$ with Comprehension also shows $S \circ R$ is a set.

3. There is no set X such that $\mathscr{P}(X) \subseteq X$.

   <u>Solution</u>: Suppose NOT. There exists a set X s.t. $\mathscr{P}(X) \subseteq X$.

Method I We have $X \in X \in X \cdots$, since X is a subset of itself $X \in \mathscr{P}(X) \subseteq X$. But it contradicts Regularity/Well-foundedness axioms.

Method II Let $W = \{x \in X \mid x \notin x\}$. $W \subset X$, thus $W \in \mathscr{P}(X) \subseteq X$. But $W \in W \leftrightarrow W \notin W$. Contradiction!

Let $N = \bigcap\{X \mid X \text{ is inductive}\}$. N is the smallest inductive set. Let us use the following notation:

$$0 = \varnothing, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}$$

If $n \in N$, let $n + 1 = n \cup \{n\}$. And for $n, m \in N$,

$$n < m \leftrightarrow n \in m$$

A set T is *transitive* if $x \in T$ implies $x \subseteq T$.

4. If X is inductive, then the set

$$\{x \in X \mid x \subseteq X\}$$

is inductive. Hence N is transitive, and for each n, $n = \{m \in N \mid m < n\}$.

Solution: Let $E = \{x \in X \mid x \subseteq X\}$.

(a) It is clear that $\varnothing \in X$(X is inductive) and $\varnothing \subseteq X$ (trivial). So $\varnothing$ belongs to E.

(b) For all $x \in E$, $x \cup \{x\} \in X$ because x is an element of X and X is inductive. Since both x and $\{x\}$ are subsets of X, we have $x \cup \{x\} \subseteq X$. Hence $x \cup \{x\} \in E$.

According to (a) and (b), $\{x \in X \mid x \subseteq X\}$ is inductive.

Let $E_0 = \{x \in N \mid x \subseteq N\} \subseteq N$. But E_0 is inductive, so N is a subset of E_0. That means $\{x \in N \mid x \subseteq N\} = N$, thus N is transitive.

It is obvious that $\{m \in N \mid m < n\} = \{m \in N \mid m \in n\} \subseteq n$. On the other hand, since $n \in N$ and N is transitive, we have $n \subseteq N$. Then $m \in n \rightarrow m \in N$ which equals $n \subseteq \{m \in N \mid m < n\}$. Hence $n = \{m \in N \mid m < n\}$.

5. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive}\}$$

is inductive. Hence every $n \in N$ is transitive.

Solution: Let $E = \{x \in X \mid x \text{ is transitive}\}$.

(a) $\varnothing \in E$ since $\varnothing$ is transitive.

(b) For all $x \in X$, x is transitive. Our goal is to show that $x \cup \{x\}$ is transitive, too. For all $y \in x \cup \{x\}$, no matter wether $y \in x$ or $y \in \{x\}$, we have $y \subseteq x \cup \{x\}$. Thus $x \cup \{x\}$ is transitive.

According to (a) and (b), $\{x \in X \mid x \text{ is transitive}\}$ is inductive.

Since N is the smallest inductive set, $N \subseteq \{x \in N \mid x \text{ is transitive}\}$. Thus every element n of N is transitive.

6. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive and } x \notin x\}$$

is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in N$.

Solution: According to the conclusion above, it is sufficient to prove that $x \cup \{x\} \notin x \cup \{x\}$ if x is transitive and $x \notin x$. Suppose NOT, we have $x \cup \{x\} \in x$ or $x \cup \{x\} \in \{x\}$. Both of them lead to $x \cup \{x\} \subseteq x$ (x is transitive). But

$$x \cup \{x\} \subseteq x \rightarrow \{x\} \subseteq x \rightarrow x \in x$$

It is a contradiction.

Since N is the smallest inductive set, $N \subseteq \{x \in X \mid x \text{ is transitive and } x \notin x\}$. Thus $n \notin n$. $n+1 = n \cup \{n\}$ by definition. Since there is n s.t. $n \notin n$ but $n \in n + 1$, we have $n \neq n + 1$.

7. If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\!-\text{minimal element}\}$ is inductive. (t is $\in$-*minimal* in z if there is no $s \in z$ such that $s \in t$.)

SOLUTION: It is sufficient to show that every nonempty $z \subseteq x \cup \{x\}$ has an $\in$-minimal element if x belongs to the above set. For any nonempty $z \subseteq x \cup \{x\}$, Suppose $z = \{x\}$, x is the $\in$-minimal element in z (Otherwise $x \in x$, but then $\{x\}$, as a nonempty subset of x, has no $\in$-minimal element). Otherwise, $z \subsetneq x$, there would exist a $y \in z \backslash \{x\} \subseteq x$ is a $\in$-minimal element in $z \backslash \{x\}$. Meanwhile, $x \notin y$ (Otherwise $x \in y \in x \to x \in x$, since x is transitive). That implies y is an $\in$-minimal element in z.

8. Every nonempty $X \subseteq N$ has an $\in$-minimal element.

SOLUTION: Since N is the smallest inductive set, $N \subseteq \{x \in N \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\text{-minimal element}\}$. For all $X \subseteq N$, pick $n \in X$. If $X \cap n = \varnothing$, $(\forall m < n)(m \notin X)$. So n is an $\in$-minimal element. If $X \cap n \neq \varnothing$, $X \cap n \subseteq n$ has an $\in$-minimal element. It is an $\in$-minimal element in X.

9. If X is inductive then so is $\{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq \varnothing$ is $m+1$ for some m.

SOLUTION: Let $E = \{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$. Suppose an nonempty set $x \in E$. Then $x \cup \{x\} = y \cup \{y\}$ for $y = x$. Thus $x \cup \{x\} \in E$. So the above set is inductive.

Since $N \subseteq \{x \in N \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$, each $n \neq 0$ is $m+1$ for some m.

10. (Induction) Let A be a subset of N such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = N$.

SOLUTION: By definition, A is inductive. So N is a subset of A. But $A \subseteq N$ naturally. Hence $A = N$.

11. Show that the function f given in the proof of Theorem 11 is an isomorphism.

SOLUTION:

$$f = \{(x, y) \mid x \in U \wedge y \in V \wedge (U_x, (<_U)_x) \cong (V_y, (<_V)_y)\}$$

It suffices to show that $x_1 <_U x_2 \Leftrightarrow f(x_1) <_V f(x_2)$, since this implies that f is an injective function as well as order-preserving. We only need to consider the case $x_1 <_U x_2$ and $x_1 = x_2$.

Suppose $x_1 = x_2$. Then $U_{x_1} \cong V_{f(x_1)} \cong V_{f(x_2)}$, since there are no two distinct initial segments of $(V, <_V)$ are isomorphic, it must be that $f(x_1) = f(x_2)$. This implies that f is a (well-defined) function between its domain and range.

Suppose $x_1 <_U x_2$. Since U_{x_1} is an initial segment of U_{x_2}, $V_{f(x_1)}$ is isomorphic to an initial segment of $V_{f(x_2)}$, say V_y, $y < f(x_2)$. Since no well-ordering is isomorphic to its proper initial segments, it must be that $f(x_1) = y < f(x_2)$.

12. The relation "$(P, <) \cong (Q, <)$" is an equivalence relation (on the class of all partially ordered sets).

SOLUTION:

(a) (reflexive) For any partially ordered sets P, id is the natural automorphism.

(b) (symmetric) Suppose $f : P \to Q$ is an isomorphism, then so does $f^{-1} : Q \to P$, since

$$y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$$

(c) (transitive) For any partially ordered sets P, Q, R, Suppose $f : P \to Q$ and $g : Q \to R$ are isomorphisms, then $g \circ f$ is an isomorphism between P and R.

13. Let $\mathcal{A}$ denote the class of all well orderings. For any $a, b \in \mathcal{A}$, $a \prec b$ iff a is isomorphic to an initial segment of b. Show that $\prec$ is a well ordering on $\mathcal{A}/_{\cong}$, where $\cong$ is the equivalence relation given in Ex.2.

SOLUTION:

It is obvious that $a \prec b \Leftrightarrow [a] \prec [b]$, in which $[a]$ denotes the equivalence class containing a.

(a) (irreflexive) $a \not\prec a$ since any a is isomorphic to itself, therefore, can't be isomorphic to its own initial segment.

(b) (transitive) For any $a, b, c \in \mathcal{A}$, suppose $a \prec b$ and $b \prec c$. It follows that a is isomorphic to an initial segment of an initial segment of c, which is still an initial segment of c. Thus $a \prec c$.

(c) (trichotomous) For any $a, b \in \mathcal{A}$, by theorem 2. 3, we have $a \prec b$, $a = b$ (actually, $a \cong b$) or $a \succ b$

(d) (well-ordering) Suppose NOT, there is a nonempty subclass $P \subseteq \mathcal{A}$, such that P has no least element, i.e. there exists a infinite sequence $\{U^i\} \subseteq P$

$$U^0 \succ U^1 \succ U^2 \succ \dots$$

By definition, for every $i \in \mathbb{N}^*$, there exist a $x_i \in U^0$ such that $U^i \cong U^0_{x_i}$. It is obvious that $\{x_i\}$ is an infinite decreasing sequence of U^0 (Contradict to the fact that U^0 is well-ordering).

14. Prove Proposition 6.

(a) If $(W, <)$ is a well ordering and $U \subseteq W$, then $(U, < \cap (U \times U))$ is a well ordering.

(b) If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, $W_1 \cap W_2 = \emptyset$, then $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$ is a well ordering, where

$$\prec = <_1 \cup <_2 \cup \{(a, b) \mid a \in W_1 \wedge b \in W_2\}$$

(c) If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, then $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$ is a well ordering, where

$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \vee (b_1 = b_2 \wedge a_1 <_1 a_2)$$

Solution:

(a) Let $<_1$ denote $< \cap (U \times U)$. Notice the fact that for any $p, q \in U$, $p <_1 q \Leftrightarrow p < q$.

- (irreflexive) For any $p \in U$, $p \not<_1 p$ because $p \not< p$.
- (transitive) For any $p, q, r \in U$, $p <_1 q \wedge q <_1 r \Rightarrow p < q \wedge q < r \Rightarrow p < r \Rightarrow p <_1 r$
- (trichotomous) For any $p, q \in U$, $p < q \vee p = q \vee q < p \Rightarrow p <_1 q \vee p = q \vee q <_1 p$
- (well-ordered) For any nonempty subset $P \subseteq U$, it is also a nonempty subset of W. Since W is a well-ordering, P has a least element p. For all element $x \in P$, $p \leq x$, which implies that $p \leq_1 x$. So p is the least element in $(P, <_1)$.

(b)
- (irreflexive) For any $a \in W_1 \cup W_2$, $a \in W_1$ or $a \in W_2$. In either case, $a \not\prec a$.
- (transitive) Suppose $a, b, c \in W_1 \cup W_2$ are such that $a \prec b \prec c$. We show that $a \prec c$. Two cases:

CASE 1: $a, c \in W_i$, $i = 1$ or 2. Then $a \prec c$ follows from $a <_i b <_i c$ and the transitivity of $<_i$.

CASE 2: $a \in W_1$, $c \in W_2$. Then $a \prec c$ follows from the definition of $\prec$.

- (trichotomous) For any $a, b \in W_1 \cup W_2$, if $a \in W_1$, $b \in W_2$ or $a \in W_2$, $b \in W_1$, then a, b are comparable according to the definition of $\prec$; otherwise if $a, b \in W_i$ ($i = 1$ or 2), then a, b are comparable by the trichotomy of $<_i$.
- (well-ordered) Let P be a nonempty subset of $W_1 \cup W_2$. If $P \cap W_1 \neq \varnothing$, then $<_1$-min of $P \cap W_1$ gives the $\prec$-min element of P. Otherwise $P \subset W_2$, hence the $\prec$-min element of P is in fact its $<_2$-min.

(c) It is trivial to prove that $W_1 \otimes W_2$ is a linear order. So it suffices to show that for any nonempty subset $P \subseteq W_1 \times W_2$, P has a least element. Let

$$U = \{a \in W_1 \mid (a, b) \in P\}$$

U is a nonempty subset of W_1, therefore has a least element a_0. Let

$$V = \{b \in W_1 \mid (a_0, b) \in P\}$$

V is a nonempty subset of W_2, therefore has a least element b_0. Then (a_0, b_0) is a least element of P.

15. Show that the following are equivalent:

(a) T is transitive;

(b) $\bigcup T \subseteq T$;

(c) $T \subseteq \mathscr{P}(T)$.

SOLUTION:

$(a) \Rightarrow (b)$**.** For any $x \in \bigcup T$, let $y \in T$ be s.t. $x \in y$. Since T is transitive, $y \subseteq T$ thus $x \in T$. Hence $\bigcup T \subseteq T$.

$(b) \Rightarrow (c)$**.** For any $x \in T$. By (b), $x \subseteq \bigcup T \subseteq T$, thus x is an element of $\mathscr{P}(T)$. Hence $T \subseteq \mathscr{P}(T)$.

$(c) \Rightarrow (a)$**.** For any $x \in T$, we have $x \in \mathscr{P}(T)$, i.e. x is a subset of T. Hence T is transitive by definition.