Elementary Set Theory

Xianghui Shi

School of Mathematical Sciences
Beijing Normal University

Fall 2025

Ramsey theory

Finite Ramsey Theorem

Theorem 1 (Finite Ramsey Theorem)

For any $n, k, m \in \mathbb{N}$, there is an l such that $l \to (m)^n_k$, i.e. for any k-coloring (function) $f : [l]^n \to k$, there is an $H \subseteq l$ of size m such that $|f''[H]^n| = 1$.[1]

[1] Such H is called f-*homogeneous*

Remark

- Such H is called a homogeneous set (for f).
- The notation $l \to (m)^n_k$ is called Erdős arrow.
  k is omitted if $k = 2$.
- A variation for k colors: $l \to (m_1, \ldots, m_k)^n$.
- For $n = 2$, the least such l is denoted as $R(m_1, \ldots, m_k)$, called the Ramsey number for $(m_1, \ldots, m_k)$.
- $R(3,3) = 6$, $R(4,4) = 18$, $R(4,5) = 25$, $R(3,3,3) = 17$
- $k = 2$, $R(r,s) \leq R(r-1,s) + R(r,s-1)$
- $k > 2$, $R(n_1, \ldots, n_k) \leq R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k))$.
- $[1 + o(1)]\frac{\sqrt{2}s}{e}2^{\frac{s}{2}} \leq R(s,s) \leq s^{-(c \log s)/(\log \log s)}4^s$

Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[5]	36–41	49–61	59[10]–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149[10]–442
6						**102–165**	115[10]–298	134[10]–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090
9									**565–6588**	581–12677
10										**798–23556**

From `https://en.wikipedia.org/wiki/Ramsey%27s_theorem#Ramsey_numbers`

Infinite Ramsey Theorem

Theorem 2 (Infinite Ramsey Theorem)

$\omega \to (\omega)^n_k$, *for any* $n, k \in \omega$.

PROOF.

Suffices to prove for $k = 2$. Prove by induction on n. Fix a coloring $c : [\omega]^{n+1} \to \{0, 1\}$. Define $\langle A_n, a_n : n < \omega \rangle$ as follows:

- $A_0 = \omega$ and $a_n = \min A_n$,
- $A_{n+1} = A_{n,i_n}$, where for $i < 2$,

$$\{a_n\} \times [A_{n,i}]^n = c^{-1}(\{i\}) \cap \{a_n\} \times [A_n]^n$$

  and i_n is least such that $|A_{n,i_n}| = \omega$.

$c^* : a_n \mapsto i_n$, for $n < \omega$, is a 2-coloring of $B = \{a_n \mid n < \omega\}$. By the case $n = 1$, there is an infinite c^*-homogeneous $H \subset B$. This H is also c-homogeneous. □

IRT implies FRT

Theorem 3

Infinite *Ramsey Theorem* $\implies$ ***Finite*** *Ramsey Theorem.*

PROOF.

Use Compactness, prove by contradiction. Take $k = n = 2$

- Suppose m is such that $l \to (m)^2_2$ fails at (l, m) for any $l < \omega$.
- In the language of graph, for each l, there is a graph (model) G such that φ_l holds in G:

  $$\varphi_l \equiv \neg(\exists x_0 \cdots x_{m-1})\left[\bigwedge_{i<j<m} \neg R(x_i, x_j) \vee \bigwedge_{i<j<l} R(x_i, x_j)\right]$$

- The set $\{\varphi_l \mid l < \omega\}$, by Compactness, is realizable by some infinite G^*.

This G^* witnesses that $\omega \not\to (m)^2_2$, contradicting to $\omega \to (\omega)^2_2$. □

A stronger form of FRT

Theorem 4 (Paris-Harrington, 1977)

For any $n, k, m \in \mathbb{N}$, there is an l such that for any k-coloring (function) $f : [l]^n \to k$, there is an $H \subseteq l$ such that $|f''[H]^n| = 1$ and $|H| \geq \max\{m, \min H\}$.

Remark

Paris-Harrington Theorem (PH) is a statement that can be expressed in the 1^{st}-order language of arithmetic. It is

- provable in the 2^{nd}-order arithmetic, but
- unprovable in the 1^{st}-order (Peano) arithmetic.

IRT $\implies$ PH $\implies$[2] FRT.

[2] by an argument similar to that of IRT.

More infinite Ramsey theorems

Theorem 5

1. $\beth_n^+ \to (\omega_1)^{n+1}_{\omega_0}$. (Erdős-Rado)
2. $2^\kappa \not\to (\kappa^+)^2$. (Sierpiński)
3. $2^\kappa \not\to (3)^2_\kappa$.
4. $\kappa \to (\kappa, \omega_0)^2$. (Erdős-Dushnik-Miller)

Remarks

1. This implies that $(2^\omega)^+ \to (\omega_1)^2$.
2. Consider $({}^\kappa 2, <_{\mathrm{lex}})$. $\{f, g\} \mapsto 0$ if $f(\delta) < g(\delta)$, where $\delta =$ least ξ such that $f(\xi) \neq g(\xi)$. This implies that there is no κ^+-ascending or κ^+-descending sequence.
3. Color $[{}^\kappa 2]^2$ by $\{A, B\} \mapsto \lambda_{A,B}$, the ordertype of $\delta_{A,B}$. $\lambda_{A,B} = \lambda_{B,C} = \lambda_{A,C}$ is impossible!
4. An weaker variant.

Theorem 6

If $\kappa > \omega$ and $\kappa \to (\kappa)^2$, then κ is strongly inaccessible.

PROOF.

κ is regular. Suppose not.

- Let $\kappa = \bigcup_{i<\lambda} X_i$, where $\lambda < \kappa$ and each $|X_i| < \kappa$.
- Define $f : [\kappa]^2 \to \{0,1\}$ as follows:

$$f(\alpha, \beta) = \begin{cases} 1, & \alpha, \beta \text{ are in the same } X_i; \\ 0, & \text{otherwise.} \end{cases}$$

  There is no f-homogeneous set.

κ is a strong limit: if $\lambda < \kappa$ is such that $\kappa \leq 2^\lambda$, then $\kappa \to (\kappa)^2$ implies $2^\lambda \to (\lambda^+)^2$, contradicting to Theorem 5-2. □

Erdős arrows and Large cardinals

Definition 7

Let κ be an uncountable cardinal.

- κ is weakly compact if $\kappa \to (\kappa)^2$.
- κ is α-Erdős if $\kappa \to (\alpha)^{<\omega}$.
- κ is a Ramsey cardinal if $\kappa \to (\kappa)^{<\omega}$.

Erdős arrows and Large cardinals

Definition 7

Let κ be an uncountable cardinal.

- κ is weakly compact if $\kappa \to (\kappa)^2$.
- κ is α-Erdős if $\kappa \to (\alpha)^{<\omega}$.
- κ is a Ramsey cardinal if $\kappa \to (\kappa)^{<\omega}$.

ω is weakly compact, but not Ramsey.

Erdős arrows and Large cardinals

Definition 7

Let κ be an uncountable cardinal.

- κ is weakly compact if $\kappa \to (\kappa)^2$.
- κ is α-Erdős if $\kappa \to (\alpha)^{<\omega}$.
- κ is a Ramsey cardinal if $\kappa \to (\kappa)^{<\omega}$.

ω is weakly compact, but not Ramsey.

Hint: consider the coloring $f : [\omega]^{<\omega} \to 2$ defined by $f(\{m_1, \ldots, m_n\}) = 0$ if $m_1 < n$, and 1 otherwise.

Infinite exponents

Theorem 8

1. $\omega \nrightarrow (\omega)^{<\omega}$.
2. *Assume* AC*, then* $\kappa \nrightarrow (\omega)^{\omega}$*, for any* $\kappa \geq \omega$*.*
3. *Assume* AD*, then* $\omega_1 \rightarrow (\omega_1)^{\omega_1}$*.* *(D. Martin)*

A Ramsey-type theorem with structures

Theorem 9 (Hindman)

Given any finite coloring $c : \mathbb{N} \to k$*, some* $k < \omega$*, there exists an infinite* $A \subseteq \mathbb{N}$ *such that* c *is constant on the set*

$$A^* = \{\sum F \mid F \subset A \text{ is finite}\}.$$

Theorem 10 (Pigeonhole Principle for tree)

Given any finite coloring $c : T = 2^{<\omega} \to k$*, some* $k < \omega$*, there exists a strong subtree* $S \subset T$ *such that* c *is constant on* S*.*