Elementary Set Theory

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An Ultra-brief Introduction of Forcing

Forcing

- Gödel (1940) constructed L to prove that
$$\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZF} + \mathsf{AC} + \mathsf{CH}).$$
- Paul Cohen (1963) invented forcing to prove that
$$\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZF} + \mathsf{AC} + \neg\mathsf{CH}).$$
- The idea is to extend a countable transitive model M (ground model) by adjoining a new set G (generic set) in order to obtain a larger model $M[G]$ (generic extension).
- The generic set is approximated by forcing conditions in the ground model, and the properties of the generic extension can be described entirely within the ground model using the language of forcing.
- The existence of the generic set can be viewed as a generalization of **Baire Category Theorem**.

Definitions

Consider a partial order $(\mathbb{P}, \leq)$.

- We call $\mathbb{P}$ a **notion of forcing**, members of $\mathbb{P}$ are called **forcing conditions**.
- Say p **stronger** than q if $p \leq q$.
- p, q are **compatible** if $\exists r \in \mathbb{P}\,(r \leq p \wedge r \leq q)$.
- A set $W \subseteq \mathbb{P}$ is an **antichain** if elements of W are pairwise **in**compatible.
- A set $D \subseteq \mathbb{P}$ is **dense** if $(\forall p \in \mathbb{P})(\exists q \in D)\,[q \leq p]$

Let M be a countable transitive model (ctm). Suppose $M \models$ ZFC and $(\mathbb{P}, \leq) \in M$.

Definition 1

- A nonempty set $F \subseteq \mathbb{P}$ is a **filter** on $\mathbb{P}$ if
  1. if $p \leq q$ and $p \in F$, then $q \in F$.
  2. if $p, q \in F$ then $\exists r \in F$ s.t. $r \leq p$ and $r \leq q$
- A set of conditions $G \subset \mathbb{P}$ is **generic over** M[1] if
  1. G is a filter on P;
  2. if $D \subset \mathbb{P}$ is dense[2] in P and $D \in M$, then $G \cap D \neq \varnothing$.

The smallest transitive model containing $M \bigcup \{G\}$, denoted as $M[G]$ is called a **generic extension** of M.

[1] We also say that G is $(M, \mathbb{P})$-generic, M-generic, or P-generic (over M), or just generic.

[2] "dense" can be replaced by: open dense, maximal antichain, predense

Example 2

$(\mathbb{P}, \leq) = ({}^{<\omega}2, \sqsupseteq)$, i.e. members of $\mathbb{P}$ are finite 0-1 sequences, and

$p \leq q$ if p extends q, i.e. $q = p \restriction \mathrm{dom}(q)$.

Let

$$\mathcal{D}^M = \{D \subseteq \mathbb{P} \mid D \in M \text{ and } D \text{ is dense open}\}.$$

Let $G \subset \mathbb{P}$ be a generic filter. Then $G \cap D \neq \varnothing$ for every $D \in \mathcal{D}^M$. Let $x_G = \bigcup G = \bigcup\{p \mid p \in G\}$.

- Each $D_n = \{p \in \mathbb{P} \mid \mathrm{dom}(p) \geq n + 1\} \in \mathcal{D}^M$.

  Thus $\mathrm{dom}\, x_G = \omega$.
- For each $r \in M \cap {}^{\omega}2$, the following set is in $\mathcal{D}^M$.

  $$D_r = \{p \in \mathbb{P} \mid p(n) = 1 - r(n) \text{ for some } n \in \mathrm{dom}(p)\}.$$

  Therefore, $x_G \notin M \cap {}^{\omega}2$.

Thus $({}^{\omega}2)^{M[G]} \neq ({}^{\omega}2)^M$.

Example 3

$(\mathbb{P}, \leq) = ({}^{<\omega}\omega_1, \sqsupseteq)$, i.e.

- members of $\mathbb{P}$ are finite sequences of countable ordinals,
- $p \leq q$ if p extends q, i.e. $q = p \restriction \mathrm{dom}(q)$.

Let $\mathcal{D}^M$ and $G \subset \mathbb{P}$ as before. Let $f_G = \bigcup G$.

- Each $D_n = \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\} \in \mathcal{D}^M$.
  So $\mathrm{dom}\, f_G = \omega$.
- For every $\alpha < \omega_1^M$, $D_\alpha = \{p \in \mathbb{P} \mid \alpha \in \mathrm{ran}(p)\} \in \mathcal{D}^M$.
  Therefore, $\mathrm{ran}(f_G) = \omega_1^M$.

Thus in $M[G]$, ω_1^M is countable.

The Generic Model Theorem

Theorem 4

Let M be a countable transitive model of ZFC*, and $(\mathbb{P}, \leq)$ a notion of forcing in M. If $G \subset \mathbb{P}$ is a generic filter over $\mathbb{P}$, then there exists a transitive model $M[G]$ such that*

1. $M[G]$ *is a model of* ZFC*;*
2. $M \subseteq M[G]$ *and* $G \in M[G]$*;*
3. $\mathrm{Ord}^{M[G]} = \mathrm{Ord}^{M}$*;*
4. *if N is a transitive model of* ZF *such that (2) holds for N, then $M[G] \subseteq N$.*

The Forcing Language

- Every set in $M[G]$ is definable (in V) with parameters from $M \cup \{G\}$.
- Each element of $M[G]$ will have a **name** in M describing how it is constructed from $M \cup \{G\}$. $(M^{\mathbb{P}})$
- **The forcing language** contains a name for each element of $M[G]$, including a constant symbol $\dot{G}$ for the generic filter.
- **The forcing relation** $p \Vdash \sigma$ (read as "p *forces* σ") is a generalization of satisfaction/deduction:

$$\sigma \to \sigma' \wedge p \Vdash \sigma \quad \Longrightarrow \quad p \Vdash \sigma'$$

The Forcing Theorem

Theorem 5

Let $(\mathbb{P}, \leq)$ *be a notion of forcing in* M. *If* σ *is a sentence of the forcing language, then for every* $G \subset \mathbb{P}$ *generic over* M,

$$M[G] \models \sigma \quad \Longleftrightarrow \quad (\exists p \in G)\,(p \Vdash \sigma)$$

These are the fundamental theory of forcing method, please read the two textbooks by Thomas Jech and Kenneth Kunen (both titled *Set Theory*) for more thorough discussion.

Independence of CH

Theorem 6 (Cohen, 1963)

Assume $\mathrm{Con}(\mathsf{ZFC})$*, it is consistent that* $\mathsf{ZFC} + 2^{\aleph_0} > \aleph_1$*.*

Work with a ctm $M \models \mathsf{ZFC}$ and use the following $(\mathbb{P}, \leq)$:

- $p \in \mathbb{P}$ iff p is a finite function from $\omega_2^M \times \omega$ to $\{0, 1\}$.
- $p \leq q$ iff $p \supseteq q$.

If G is $(M, \mathbb{P})$-generic., let $f_G = \bigcup G$. Then

CLAIM. f_G *is a function with* $\mathrm{dom}(f_G) = \omega_2^M \times \omega$*, and* $\{f_G(\alpha, \cdot) \mid \alpha < \omega_2^M\}$ *are distinct* ω*-sequences.*

KEY: the following are open dense subsets of $\mathbb{P}$ in M.

- For each $\alpha < \omega_2^M$ and each $n < \omega$,

$$D_{\alpha,n} = \{p \in \mathbb{P} \mid (\alpha, n) \in \mathrm{dom}(p)\}.$$

- For every $\alpha, \beta < \omega_2^M$,

$$D_{\alpha,\beta} = \{p \in \mathbb{P} \mid \exists n \, [p(\alpha, n) \neq p(\beta, n)]\}.$$

If $\omega_2^{M[G]} = \omega_2^M$, then $M[G] \models 2^{\aleph_0} \geq \aleph_2$.

Definition 7

A forcing notion $\mathbb{P}$ satisfies the **countable chain condition** (c.c.c.) if every antichain in $\mathbb{P}$ is countable.

Theorem 8

If $\mathbb{P}$ satisfies c.c.c., then M and $M[G]$ have the same cardinals and cofinalities.[3]

PROOF.

Suffices to show that M-regular cardinals remain regular in $M[G]$.

- Let $\lambda < \kappa$ and $\dot{f} \in M$ be a name for some $f : \lambda \to \kappa$ in $M[G]$. Show $\dot{f}^G$, the interpretation of $\dot{f}$ in $M[G]$, remains bounded in κ (in $M[G]$), i.e. if some $p \in G$ forces that "$\dot{f}$ is a function", then there is a $p' \in G$ forces that "$\dot{f}$ is bounded".
- Let $p \in \mathbb{P}$ and assume $p \Vdash \dot{f} : \lambda \to \kappa$. By c.c.c., for $\alpha < \lambda$,

  $$A_\alpha = \{\beta < \kappa \mid \exists q \leq p \ (q \Vdash \dot{f}(\alpha) = \beta)\} \text{ is countable.}$$

  κ is regular, so $\bigcup A_\alpha$ is bounded in κ by some $\gamma < \kappa$. It follows that for each $\alpha < \lambda$, $p \Vdash \dot{f}(\alpha) < \gamma$. □

[3] I.e. $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

At last, we show that our forcing notion satisfies c.c.c.

Lemma 9

Cohen's forcing notion $\mathbb{P}$ satisfies c.c.c.

Sketch of proof.

- Let $X \subseteq \mathbb{P}$ be uncountable, show it cannot be an antichain, i.e. it must contain compatible conditions.
- There is an uncountable $Y \subseteq X$ forming a so-called **Δ-system**: there is a finite $r \in \mathbb{P}$ s.t. for any $p_1, p_2 \in Y$,

  $$(p_1 - r) \cap (p_2 - r) = \varnothing.$$

  This provides us an uncountable collection of mutually compatible conditions. □

This completes the proof of Cohen's Theorem. ⊠

More applications of forcing

Soon after Cohen invented the forcing technique, Donald Martin isolated the combinatoric ingredient of Cohen's argument. In fact, our presentation of Cohen's proof is based on this framework.

Martin's Axiom (MA)

Suppose $\mathbb{P}$ is a c.c.c forcing notion and $\mathscr{D}$ is a family of $< 2^\omega$ many dense (open) subsets of $\mathbb{P}$. Then there is a filter $G \subset \mathbb{P}$ such that $G \cap D \neq \varnothing$ for every $D \in \mathscr{D}$.

Such G is called a $\mathscr{D}$*-generic filter*.

Theorem 10

Assume MA. *Then*

1. *If* $\omega < \kappa < 2^\omega$, *then* $2^\kappa = 2^\omega$.
   As a corollary, 2^ω *is a regular cardinal.*
2. $\mathbb{R}$ *is not the union of* $< 2^\omega$ *many meager sets.*
3. *Every tower on* ω *has size* 2^ω.
4. *Every dominating family in* ${}^\omega\omega$ *has size* 2^ω.

(2)-(4) are typical statements in the study of a branch of set theory called "cardinal invariants", which studies cardinals associated to various structural properties of sets of reals.

- (1) is proved at the end.
- (2): $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.[4]

[4] $\mathrm{cov}(\mathcal{I})$ is the smallest size of a subfamily of $\mathcal{I}$ that covers $\mathbb{R}$.

- Suppose $x, y \in [\omega]^\omega$, define $x \subseteq^* y$ iff $x - y$ is finite.
- A $\subseteq^*$-descending sequence $\mathcal{T} \subset [\omega]^\omega$ is a tower on ω it has no $\subseteq^*$-lower bound, i.e. there is no $A \in [\omega]^\omega$ s.t. $A \subseteq^* X$ for every $X \in \mathcal{T}$.
- $\mathfrak{t}$ denotes the minimal cardinality of a tower.

- Suppose $x, y \in {}^\omega\omega$, define $x <^* y$ iff
$$\{n \in \omega \mid x(n) \geq y(n)\} \text{ is finite.}$$
- We say a set $\mathcal{D} \subseteq {}^\omega\omega$ is a dominating family on ω if $\forall f \in {}^\omega\omega$, $\exists g \in \mathcal{D}$ s.t. $f <^* g$.
- $\mathfrak{d} =$ the minimal cardinality of a dominating family.

(2) is the Baire Category Theorem for κ many dense open sets, for every $\kappa < 2^\omega$. We prove (3) and (4).

Theorem 11 (ZFC)

1. *Suppose* $\mathcal{A} \subset [\omega]^\omega$, $|\mathcal{A}| \leq \omega$ *and for every finite* $\mathcal{A}_0 \subset \mathcal{A}$, $\left|\bigcap \mathcal{A}_0\right| = \omega$. *Then there is some* $B \in [\omega]^\omega$ *s.t.* $B \subseteq^* A$ *for every* $A \in \mathcal{A}$.
2. *Every countable set* $\mathcal{A} \subset {}^\omega\omega$ *is dominated, i.e. there exists an* $f \in {}^\omega\omega$ *such that* $g <^* f$ *for every* $g \in \mathcal{A}$.

These says $\mathfrak{t} > \omega$ and $\mathfrak{d} > \omega$. MA implies that $\mathfrak{t} = \mathfrak{d} = 2^\omega$.

$\mathfrak{t} = 2^\omega$

We show that there is no tower of size $< 2^\omega$. Let $\mathcal{T}$ be a $\subseteq^*$-descending sequence of elements in $[\omega]^\omega$ with $|\mathcal{T}| < 2^\omega$. Define $(\mathbb{P}_\mathcal{T}, \leq)$ as follows

- $p \in \mathbb{P}_\mathcal{T}$ iff $p = (a, F)$, where $a \subset \omega$, $F \subset \mathcal{T}$ are both finite.
- $(a, F) \leq (a', F')$ iff $a \supset a'$, $F \supset F'$ and $a - a' \subset \bigcap F'$.

$(\mathbb{P}_\mathcal{T}, \leq)$ satisfies c.c.c. The conclusion follows from the genericity over the following dense sets:

- $D_T = \{(a, F) \mid T \in F\}$, for each $T \in \mathcal{T}$.
- $D_n = \{(a, F) \mid |a| \geq n\}$, for each $n \in \omega$.

$\mathfrak{d} = 2^\omega$

Show that there is no dominating family of size $< 2^\omega$.
Let $\mathcal{D}$ be a $<^*$-increasing sequence of elements in ${}^\omega\omega$ with $|\mathcal{D}| < 2^\omega$. Define $(\mathbb{P}_\mathcal{D}, \leq)$ as follows

- $p \in \mathbb{P}_\mathcal{D}$ iff $p = (s, F)$, where $s \subset \omega$, $F \subset \mathcal{D}$ are both finite.
- $(s, F) \leq (s', F')$ iff $s \sqsupset s'$, $F \supset F'$ and $\forall f \in F'$, $\forall n \in \mathrm{dom}(s) - \mathrm{dom}(s')$, $s(n) > f(n)$.

$(\mathbb{P}_\mathcal{D}, \leq)$ satisfies c.c.c. The conclusion follows from the genericity over the following dense sets:

- $D_f = \{(s, F) \mid f \in F\}$, for each $f \in \mathcal{D}$.
- $D_n = \{(s, F) \mid n \in \mathrm{dom}(s)\}$, for each $n \in \omega$.

$2^\kappa = 2^\omega$

A family $\mathcal{A} \subseteq [\omega]^\omega$ is an **almost disjoint family** if $\forall x, y \in \mathcal{A}$, $x \cap y$ is finite.

Lemma 12

There is an almost disjoint family of size 2^ω.

PROOF.

Fix a bijection $\pi : \omega \to {}^{<\omega}2$. Every $f \in {}^\omega\omega$ gives an infinite subset of ω: For each $f \in [\omega]^\omega$, let

$$\pi^*(f) = \{\pi^{-1}(f \restriction n) \mid n < \omega\} \subseteq \omega.$$

Then the family $\mathcal{A} = \{\pi^*(f) \mid f \in [\omega]^\omega\}$ works. □

Fix a cardinal $\kappa < 2^\omega$. Use MA to show that any $X \subset \kappa$ can be "coded" by an infinite subset of ω.

Definition 13

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family, and $\mathcal{C} \subsetneq \mathcal{A}$. The partial order $(\mathbb{P}, \leq)$[5] as follows

- $p \in \mathbb{P}$ iff $p = (a, E)$, where $a \subset \omega$, $E \subset \mathcal{C}$ are both finite.
- $(a, E) \leq (a', E')$ iff $a \supset a'$, $E \supset E'$ and
$$(a - a') \cap (\bigcup E') = \varnothing.$$[6]

$(\mathbb{P}, \leq)$ satisfies c.c.c. The following sets are dense:

- $D_A = \{(a, E) \mid A \in E\}$, for each $A \in \mathcal{C}$.

[5]Write $\mathbb{P}$ as $\mathbb{P}_{\mathcal{A},\mathcal{C}}$ if necessary.

[6]Idea: $a \setminus a'$ consists of only new elements that do not occur in E'.

- $D_n = \{(a, E) \mid |a| \geq n\}$, for each $n \in \omega$.
- $D_{X,n} = \{(a, E) \mid |a \cap X| \geq n\}$, for each $X \in \mathcal{A} - \mathcal{C}$.

Lemma 14 (Solovay's Lemma)

Assume MA, *Let* $\mathcal{A}, \mathcal{C}$ *be as above. If* $|\mathcal{A}| < 2^\omega$, *then there is a set* $X_\mathcal{C} \subset \omega$ *so that for each* $A \in \mathcal{A}$,

$$A \cap X_\mathcal{C} \text{ is finite} \quad \Leftrightarrow \quad A \in \mathcal{C}.$$

PROOF OF (1).

Suppose $\omega < \kappa < 2^\omega$, and $\mathcal{A}$ an almost disjoint family of size κ. $\mathcal{A}$ has 2^κ many sub-families $\mathcal{C}$, each "coded" by a set $X_\mathcal{C}$, according to the lemma. Thus $2^\omega \geq 2^\kappa$.

If CH holds, $2^\omega = \omega_1$; if CH fails, since $2^{\mathrm{cf}(2^\omega)} > 2^\omega$, $\mathrm{cf}(2^\omega)$ can not be $< 2^\omega$. Therefore 2^ω is regular under MA. □