Elementary Set Theory

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Structures of Numbers

- We've defined $(\omega, +_\omega, \cdot_\omega, <_\omega)$, via either operations on ordinals or operations on cardinals.
- $(\mathbb{Z}, +_\mathbb{Z}, \cdot_\mathbb{Z}, <_\mathbb{Z})$ is defined as: $\mathbb{Z} = \omega \times \omega / \approx_1$, where

$$\begin{aligned}
(a,b) \approx_1 (c,d) &\Leftrightarrow a +_\omega d = c +_\omega b \\
(a,b) +_\mathbb{Z} (c,d) &= (a +_\omega c,\ b +_\omega d) \\
(a,b) \cdot_\mathbb{Z} (c,d) &= (ac +_\omega bd,\ ad +_\omega bc) \\
(a,b) <_\mathbb{Z} (c,d) &\Leftrightarrow a +_\omega d <_\omega c +_\omega b
\end{aligned}$$

- $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \approx_2$, where

$$(p,q) \approx_2 (r,s) \quad \Leftrightarrow \quad p \cdot_\mathbb{Z} s = q \cdot_\mathbb{Z} r$$

  The reader should try to define $+_\mathbb{Q}$, $\cdot_\mathbb{Q}$ and $<_\mathbb{Q}$.
- $\mathbb{R}$ is the set of Dedekind cuts, and $\mathbb{C} = \mathbb{R} \times \mathbb{R}$.

Homework 4.1

1. Define $+_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions are independent of the choice of representatives.
2. Exercises in Ch4: 1-5.

The Cardinality of the Continuum, I

Theorem 1 (Cantor)

The set of all real numbers is uncountable, i.e. $\omega \prec \mathbb{R}$.

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PROOF.

Suppose $\mathbb{R} = \{c_k \mid k < \omega\}$. Construct a $r \in \mathbb{R}$ s.t. $r \neq c_k$ for all k. Use the theorem of Nested Closed Intervals. Start with an interval I_0 such that $c_0 \notin I_0$. For each $n < \omega$, choose an interval I_{n+1} inductively such that

- $I_{n+1} \subset I_n$, $|I_{n+1}| \leq |I_n|/3$ and
- $c_i \notin I_{n+1}$, for $i \leq n+1$.

This produces a nested sequence $\langle I_n : n < \omega \rangle$. Let r be such that $\{r\} = \bigcap_n I_n$. This r works as desired. □

The Cardinality of the Continuum, II

Next, we find the precise size of $\mathbb{R}$.

- Since $\mathbb{R}$ is defined by Dedekind cut over $\mathbb{Q}$,

$$|\mathbb{R}| \leq |\mathscr{P}(\mathbb{Q})| = 2^\omega.$$

- Cantor set is the set

$$C = \{\sum_n f(n)/3^n \mid f : \omega \to \{0,2\}\}$$

  $|C| = 2^\omega$, hence $|\mathbb{R}| \geq 2^\omega$.

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By Cantor-Bernstein, $|\mathbb{R}| = 2^\omega$.

In fact, there is a natural bijection between $\mathscr{P}(\omega)$ and $\mathbb{R}$ via "continuous fractions"（**连分数**）.

Continuous Fractions

We define $f : (\omega - \{0\})^{\leq\omega} \to [0,1]$ as follows. $f(\varnothing) = 0$ and

$$f(\langle a_n \rangle) = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}$$

f^{-1}: Given $r_0 \in (0,1]$, $a_0 = \left[\frac{1}{r_0}\right]$ and
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Some beautiful continuous fractions

$$\frac{\sqrt{5}+1}{2} = [1; 1, 1, 1, 1, \cdots]$$
$$\sqrt{2} = [1; 2, 2, 2, 2, \cdots]$$
$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \cdots]$$

Generalized continuous fractions [see wikipedia]

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}} = \cfrac{1}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}}$$

The Ordering of $\mathbb{R}$

The Order-type of $\mathbb{Q}$

- A linear ordering $(P, <)$ is **dense** if for all $a < b$ there exists c s.t. $a < c < b$. P is **unbounded** if it has neither a least nor a greatest element.
- A set $D \subset P$ is a **dense subset** if for all $a < b \in P$, there exists $d \in D$ s.t. $a < d < b$. D is a **bounded above** in P if there exist $e \in P$ s.t. for all $a \in D$, $a < e$. (or simply $D < e$)
- A linear ordering $(P, <)$ is **complete** if every nonempty bounded subset of P has a least upper and a largest lower bound.

Theorem 2 (Cantor)

Any two countable unbounded dense linear orderings are isomorphic.

The Ordering of $\mathbb{R}$

The Continuum, uniqueness

Theorem 3 (Cantor-Dedekind)

$(\mathbb{R}, <)$ *is the unique complete linear ordering that has a countable unbounded dense subset.*

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PROOF.

Prove the uniqueness only.

- Let C, C' be two such linear orderings, and $P \subset C$, $P' \subset C'$ be the two countable unbounded dense subset repectively.
- Let $f : P \to P'$ be an isomorphism. Then $F : C \to C'$ is defined as: for $x \in C$,
$$F(x) = \sup\{f(t) \mid t \in P \wedge t \leq x\}$$
- Verify that F is an isomorphism. □

The Ordering of $\mathbb{R}$

The Continuum, existence

The linear ordering $(\mathbb{R}, <)$ is often called **continuum** and the size of $\mathbb{R}$ is often denoted as $\mathfrak{c}$.

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Theorem 4

Let $(P, <)$ be a dense unbounded linear ordering. Then there is a complete unbounded linear ordering $(C, \prec)$ s.t.

1. *$P \subset C$ and $<, \prec$ agree on P.*
2. *P is dense in C.*

- If P is countable, then $C \cong \mathbb{R}$.
- If P is not countable, C is not necessarily unique.

The Ordering of $\mathbb{R}$

Dedekind cut

- A **D-cut** is a pair of disjoint set of rationals (A, B) s.t. $A \cup B = \mathbb{Q}$, $A <_{\mathbb{Q}} B$ and A has no maximal elements.
- $(A, B) < (C, D)$ iff $A \subsetneq C$.
  $(A, B) + (C, D) = (A +_{\mathbb{Q}} C, B +_{\mathbb{Q}} D)$. Thus
  $-(A, B) = (-(B \setminus \{\min B\}), -(A \cup \{\min B\}))$.
  For $(A, B), (C, D) >_{\mathbb{Q}} 0_{\mathbb{Q}}$,
  $$(A, B) \cdot (C, D) = (\mathbb{Q} - BD, BD)$$
- $(\mathbb{R}, +, \cdot)$ forms a field, and
  - If $x < y$ then $x + z < y + z$.
  - If $x < y$ and $z > 0$ then $x \cdot z < y \cdot z$.
- As $(\mathbb{R}, <)$ is complete, $(\mathbb{R}, +, \cdot, <, 0, 1)$ is a complete ordered field. In fact, every complete ordered field is isomorphic to $(\mathbb{R}, +, \cdot, <, 0, 1)$.

Trees

Definitions

- A **tree** is a partially ordered set $(T, <_T)$ s.t. for every $t \in T$, the set $(\cdot, t)_T = \{s \in T \mid s <_T t\}$ is well-ordered.
- The **height** of t in T, $\mathrm{ht}_T(t) = \mathrm{ordertype}(\cdot, t)_T$.
- The **α-th level** of T, $\mathrm{Lev}_\alpha T = \{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$.
- The **height** of T, $\mathrm{ht}\, T = \min\{\alpha \mid \mathrm{Lev}_\alpha T = \varnothing\}$.
- A **chain** of T is a $<_T$-well-ordered subset of T. (α-chain).
- A **path** P is a chain which is also an initial part of T, i.e., $(\cdot, t) \subset T$ for every $t \in P$. (α-path).

- A **branch** of T is a maximal chain/path of T. (α-branch). The set of all branches of T is denoted as B_T.
- A **cofinal branch** is a branch that intersects each level of T. The set of all cofinal branches of T is denoted as $[T]$.
- $A \subset T$ is an **antichain** if members of A are pairwise **incompatible**, i.e. $\forall s, t \in A\,(s \neq t \implies s \perp_T t)$, where

$$s \perp_T t \quad \Leftrightarrow \quad s \not\leq_T t \wedge t \not\leq_T s.$$

- If T is a tree and $s, t \in T$, define $\delta_{st} = (\cdot, s)_T \cap (\cdot, t)_T$ and $n_{st} = \mathrm{ordertype}(\delta_{st})$.

Order a tree linearly

- ${}^{<\omega}2$ and ${}^{<\omega}\omega$ carry natural tree orderings:

$$f \sqsubseteq g \quad \Leftrightarrow \quad f \text{ is an initial segment of } g.$$

- The general **X-ary** tree $({}^{<\alpha}X, \sqsubseteq)$ is defined similarly for any set X and $\alpha \in \mathrm{Ord}$.
- Only consider subtrees $T \subseteq {}^{<\omega}2$ or ${}^{<\omega}\omega$ that are downward closed under $\sqsubseteq$.

Let $<_X$ be a linear ordering of X, then the **lexicographical ordering** $\prec$ of T is defined by $s \prec t$ iff

1. $s \sqsubseteq t$ or
2. $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge s(n_{st}) <_X t(n_{st})$.

Order a tree linearly

Proposition 5

Let T be a tree.

1. *If $s, t, u \in T$, then $R_{stu} = \{\delta_{st}, \delta_{tu}, \delta_{su}\}$ has ≤ 2 elements, and $p, q \in R_{stu} \implies p \subseteq q \vee q \subseteq p$.*
2. *$\prec$ is a linear ordering of T which extends $\sqsubset$.*
3. *For every $t \in T$, $T^t = \{s \in T \mid t \sqsubset s\}$ is an interval in $(T, \prec)$.*

PROOF.

1. Key: R_{stu} is linearly ordered by $\sqsubset$.
2. By the definition of $\prec$
3. Suppose $s, s' \in T^t$, $\sigma \in T$ and $s \prec \sigma \prec s'$. $\delta_{s,s'} \sqsupseteq \delta_{s,\sigma} \cap \delta_{s',\sigma} = \min\{\delta_{s,\sigma}, \delta_{s',\sigma}\}$. If not equal, then $s(n_{s,s'}) = \sigma(n_{s,s'}) = s'(n_{s,s'})$, but $s(n_{s,s'} < s'(n_{s,s'})$. □

Extend the lex-ordering to cofinal branches

The ordering $\prec$ on T can be further extended lexicographically to the set B_T of all branches of T.

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Proposition 6

Let T, B_T be as above.

1. *$\prec$ is a linear ordering of $T \cup B_T$.*
2. *For every $t \in T$, $B_t = \{f \in T \cup B_T \mid t \in f\}$ is an interval in $(T \cup B_T, \prec)$.*

The lexicographical ordering of ${}^{<\omega}\mathbb{Z}$ is an unbounded dense linear order ($\cong \mathbb{Q}$), its extension to $T \cup B_T$ is complete ($\cong \mathbb{R}$).

The Standard Topology of $\mathbb{R}$

- The standard topology over $\mathbb{R}$ is induced by the standard linear ordering: using intervals of the form
$$(a, b), \quad a \leq b \in \mathbb{Q},$$
as basis.
- For the Baire space $\mathcal{N} = {}^{\omega}\omega$, the basis consists of sets of the form: $s \in {}^{<\omega}\omega$,
$$O_s = \{f \in \mathcal{N} \mid s \sqsubset f\}.$$
- The continuum is the unique linear ordering that is dense, unbounded, complete and **separable**.

Separability & c.c.c.

- (Suslin's Problem). Is it still true if "separable" is replaced by **"countable chain condition" (c.c.c)**, i.e., there is no uncountable pairwise-disjoint collection of open intervals?
- A linear ordering is a **Suslin line** if it is dense, unbounded, complete, c.c.c but not separable.
- $\mathbb{R}$ is separable and has c.c.c. In general,
  If X is separable, then X has c.c.c.

- The product of two separable spaces is separable. However, separability is not preserved under arbitrary products (for $\geq (2^\omega)^+$ factors).

- $\neg\mathrm{SH}$ implies that the product of two c.c.c spaces is not necessarily c.c.c. (see Homework 4.2 problem $\sharp 3$). Strangely, if c.c.c is preserved by products with two factors, then it is preserved by arbitrary products.

Suslin Hypothesis

Definition 7

A **Suslin tree** is a tree of size ω_1 that has neither uncountable chain nor uncountable antichain.

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Theorem 8 (Kurepa, 1936)

There is an ω_1-Suslin tree iff there is a Suslin line.

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Suslin's Hypothesis (SH)

There are no Suslin lines/trees.

SH turns out to be independent of ZFC.

Homework 4.2

1. Prove Proposition 5.
2. Prove Proposition 6.
3. If X is a Suslin line, then X^2 is not c.c.c.

   Hint: construct $\{U_\alpha = (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \mid \alpha < \omega_1\}$, s.t.

   $a.$ $a_\alpha < b_\alpha < c_\alpha$

   $b.$ $(a_\alpha, b_\alpha) \neq \varnothing$ and $(b_\alpha, c_\alpha) \neq \varnothing$

   $c.$ for every $\xi < \alpha$, $b_\xi \notin (a_\alpha, c_\alpha)$.
4. Exercises in Ch4: 8-13, 15, 18

Plan

We shall discuss three properties of sets of reals:

- The Perfect Set Property
- The Property of Baire
- Lebesgue Measurability

Size of Closed Sets

Consider $\mathbb{R}$ together with its standard topology and metric.

$$d(a, b) = |a - b|.$$

Some simple counting:

- There are $\mathfrak{c}$ many open sets.
- There are $\mathfrak{c}$ many closed sets.
- Every nonempty open set has size $\mathfrak{c}$.

What about the size of closed sets? The answer is what Cantor considered as an evidence to his famous Hypothesis (CH).

Theorem

Every closed set either is countable or has size $\mathfrak{c}$.

Perfect Sets

Definition 9

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Theorem 10

Every perfect set has cardinality $\mathfrak{c}$.

More generally, every perfect set in a separable complete metric space contains a copy of Cantor set.

Cantor-Bendixson

Theorem 11 (Cantor-Bendixson)

If $F \subseteq \mathbb{R}$ *is an uncountable closed set, then* $F = P \cup C$*, where* P *is perfect and* $|C| \leq \omega$*.*

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PROOF.

For every $A \subset \mathbb{R}$, define the derivative of A as

$$A' = \{r \in A \mid r \text{ is a limit point of } A\}.$$

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Iterate the process

$$A_0 = A,\ A_{\alpha+1} = (A_\alpha)' \text{ and } A_\lambda = \bigcap_{\alpha<\lambda} A_\alpha \text{ for limit } \lambda,$$

till it stops, say at τ. Let $P = A_\tau$ and $C = A - A_\tau$.

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QUESTION: Why are P perfect and C countable? □

Perfect sets in $\mathcal{N} = {}^\omega\omega$

Proposition 12

1. *$F \subseteq \mathcal{N}$ is closed iff $F = [T]$ for some tree $T \subseteq {}^{<\omega}\omega$ with $\mathrm{ht}(T) = \omega$.*
2. *If f is an isolated point of a closed set $F \subseteq \mathcal{N}$, then there is $n \in \omega$ s.t. $\forall g \in F\,(f \neq g \to f{\restriction}n \neq g{\restriction}n)$.*
3. *A closed set $F \subseteq \mathcal{N}$ is perfect iff*

$$T_F = \{f{\restriction}n \mid f \in F,\ n < \omega\}$$

is a perfect tree.

Definition 13

A tree $T \subseteq {}^{<\omega}\omega$ is **perfect** iff for every $t \in T$, there exist $s_1, s_2 \in T$ s.t. $t \sqsubset s_1$ and $t \sqsubset s_2$, but s_1, s_2 are incomparable.

Cantor-Bendixson for $\mathcal{N}$

PROOF.

For each $T \subseteq {}^{<\omega}\omega$, define

$$T' = \{t \in T \mid \exists s_1, s_2 \in T\,(t \sqsubset s_1 \wedge t \sqsubset s_2 \wedge s_1 \perp_T s_2)\}$$

Iterate the process

$$T_0 = T,\ T_{\alpha+1} = (T_\alpha)' \text{ and } T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha \text{ for limit } \lambda,$$

till it stops, say at τ. Then $\tau < \omega_1$, and

- Each $[T_\alpha] - [T_{\alpha+1}]$ is countable, as T_α is countable. $(\alpha < \tau)$
- If $T_\tau \neq \varnothing$, then it is perfect.
- $[\bigcap T_\alpha] = \bigcap [T_\alpha]$.

  Hence $[T] - [T_\tau] = \bigcup_{\alpha<\tau}([T_\alpha] - [T_{\alpha+1}])$ is countable □

σ-Algebra

Definition 14

Fix a set X.

- An **algebra of sets** is a collection $\mathcal{S} \subseteq \mathscr{P}(X)$ s.t.
  - (i) $X \in \mathcal{S}$.
  - (ii) $U, V \in \mathcal{S} \implies U \cup V \in \mathcal{S}$.
  - (iii) $U \in S \implies X - U \in \mathcal{S}$.
- A collection $\mathcal{I} \subseteq \mathscr{P}(X)$ forms an **ideal** if
  - (i) $X \notin \mathcal{I}$.
  - (ii) $U, V \in \mathcal{I} \implies U \cup V \in \mathcal{I}$.
  - (iii) $U \in \mathcal{I} \wedge V \subset U \implies V \in \mathcal{I}$.
- A **σ-algebra (σ-ideal)** is an algebra (ideal) closed under countable union, i.e.
  - (iv) $\{U_n \mid n \in \omega\} \subseteq \mathcal{S} \implies \bigcup_n U_n \in \mathcal{S}$.

σ-Algebra, II

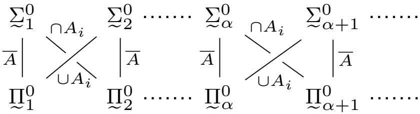
- $\mathscr{P}(X)$ is a σ-algebra.
- For every $\mathcal{A} \subseteq \mathscr{P}(X)$, there is a smallest ($\sigma$-)algebra containing $\mathcal{A}$, which is the intersection of all (σ-)algebras containing $\mathcal{A}$.
- A set $A \subseteq \mathbb{R}$ is **Borel** if it belongs to the smallest σ-algebra that contains all open sets.
- The Lebesgue measurable sets form a σ-algebra.
- The sets having the Baire property form a σ-algebra.

Borel Hierarchy, I

Definition 15

For each $\alpha < \omega_1$,

$$\utilde{\Sigma}^0_1 = \text{the collection of all open sets;}$$
$$\utilde{\Pi}^0_1 = \text{the collection of all closed sets;}$$
$$\utilde{\Sigma}^0_\alpha = \{\textstyle\bigcup_n A_n \mid \text{each } A_n \in \utilde{\Pi}^0_\beta, \text{ some } \beta < \alpha\}$$
$$\utilde{\Pi}^0_\alpha = \{\overline{A} \mid A \in \utilde{\Sigma}^0_\alpha\} \qquad \text{(where } \overline{A} = \mathbb{R} \setminus A\text{)}$$
$$\utilde{\Delta}^0_\alpha = \utilde{\Sigma}^0_\alpha \cap \utilde{\Pi}^0_\alpha$$



Borel Hierarchy, II

- $\underset{\sim}{\Delta}_1^1$ = the collection of all Borel sets.
- The construction ends at $\alpha = \omega_1$.
- The elements of each $\underset{\sim}{\Sigma}_\alpha^0$ (or $\underset{\sim}{\Pi}_\alpha^0$) are Borel sets.
- For $\alpha < \beta < \omega_1$,

$$\begin{array}{ll} \underset{\sim}{\Sigma}_\alpha^0 \subset \underset{\sim}{\Sigma}_\beta^0, & \underset{\sim}{\Sigma}_\alpha^0 \subset \underset{\sim}{\Pi}_\beta^0, \\ \underset{\sim}{\Pi}_\alpha^0 \subset \underset{\sim}{\Pi}_\beta^0, & \underset{\sim}{\Pi}_\alpha^0 \subset \underset{\sim}{\Sigma}_\beta^0 \end{array}$$

- All inclusions above are strict.
- The collection $\bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}_\alpha^0 = \bigcup_{\alpha<\omega_1} \underset{\sim}{\Pi}_\alpha^0$ is a σ-algebra.

- Every irrational number has a unique representation by an infinite continued fraction.

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}$$

  where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}^+$ for $i \geq 1$.

- Let A be the set of all irrational numbers that correspond to sequences $\langle a_i : i < \omega \rangle$ with the following property:

  > *there exists an infinite subsequence* $\langle a_{k_i} : i < \omega \rangle$ *such that for every* $i < \omega$, a_{k_i} *is a factor of* $a_{k_{i+1}}$.

- A is not Borel. (Assume AC_ω)
- A is constructed in ZF, however, it cannot be proven to be non-Borel in ZF alone.

Meager Sets

Definition 16

- A set $X \subseteq \mathbb{R}$ is **nowhere dense** if its closure has empty interior.
- X is of **the first category (or meager)** if it is the union of countably many nowhere dense sets. A non-meager set is called a set of **the second category**. A set is **comeager** if its complement is meager.
- A set $A \subseteq \mathbb{R}$ has **the property of Baire** if there exists an open set G such that $G \,\Delta\, A$ is meager.

The collection of meager sets is a σ-ideal, and the collection of sets that have the property of Baire is a σ-algebra.

The Baire Category Theorem

Theorem 17 (The Baire Category Theorem)

If $\{D_n \mid n < \omega\}$ are dense open subsets of $\mathbb{R}$ (or $\mathcal{N}$), then the intersection $D = \bigcap_n D_n$ is dense in $\mathbb{R}$.

Equivalent versions:

- Replace “dense” by “comeager”.
- Every open set (in particular $\mathbb{R}$) is not meager.

Baire Category for $\mathcal{N}$

Note that

- Sets of the form O_s, $s \in {}^{<\omega}\omega$, form a basis.
- Every open dense subset of $\mathcal{N}$ corresponds to a maximal antichain in the sequential tree ${}^{<\omega}\omega$.

Theorem 18 (The Baire Category Theorem for $\mathcal{N}$)

Let $\mathcal{A} = \{A_n \mid n < \omega\}$ be a family of maximal antichains of the sequential tree $T = {}^{<\omega}\omega$. Then for every $s \in T$, there is a cofinal branch $f \in {}^{\omega}\omega$ such that

- $s \sqsubset f$, *and*
- *for each n, there is exactly one $t_n \in A_n$ such that $t_n \sqsubset f$.*

Lebesgue Measurable

The standard definition of Lebesgue measure uses the **outer measure**:

$$\mu^*(A) = \inf\{\sum \mathrm{lh}(I_i) \mid A \subset \bigcup I_i\},$$

where $\{I_i \mid i < \omega\}$ refers to a sequence of open intervals.

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Definition 19

- A set A is **Lebesgue measurable** if there exist an F_σ-set F and a G_δ-set G such that $F \subset A \subset G$ and $\mu^*(G - F) = 0$.

  When A is measurable, write $\mu(A)$ instead of $\mu^*(A)$.
- A set A is **null** if $\mu^*(A) = 0$.

In addition to the properties mentioned before, we add a few more:

- μ is σ-additive: If $\{A_n \mid n < \omega\}$ are pairwise disjoint and measurable, then
$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$
- μ is σ-finite: If A is measurable, then there exist measurable sets A_n $(n < \omega)$ such that
$$A = \bigcup_n A_n \text{ and } \mu(A_n) < \infty \text{ for each } n.$$
- Every null set is measurable. The null sets form a σ-ideal and contain all singletons.

(Lebesgue) Measure on $\mathcal{N}$

The above theory of Lebesgue measure on $\mathbb{R}$ can be carried over to $(\mathcal{N}, \mu)$, where μ is the extension of the product measure ν on open sets in $\mathcal{N}$ induced by the probability measure on ω such that

$$\nu(\{n\}) = 1/2^{n+1}, \text{ for every } n.$$

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$$\nu(\{n\}) = 1/2^{n+1}, \text{ for every } n.$$

Thus for every nonempty sequence $s \in {}^{<\omega}\omega$,

$$\begin{aligned} \mu(O_s) &= \textstyle\prod_{n \in \mathrm{dom}(s)} \nu(\{s(n)\}) \\ &= \textstyle\prod_{n \in \mathrm{dom}(s)} 1/2^{s(n)+1}. \end{aligned}$$

Homework 4.3

1. Prove Proposition 12.
2. Show that $\bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}^0_\alpha =$ the collection of all Borel sets.
3. Show that the collection of Lebesgue measurable sets (of reals) form a σ-algebra.
4. Show that the collection of sets (of reals) having the property of Baire forms a σ-algebra.

Solovay model

Theorem (Solovay 1970)

Assume the existence of an (a strongly) inaccessible cardinal is consistent with ZFC. *Then there is an inner model of* ZF + *Dependent Choice*[1] *such that every set of reals*

- *is Legesgue measurable,*
- *has the perfect set property, and*
- *has the Baire property.*

[1] $\mathrm{HOD}_{\mathrm{Ord}^\omega}$ or $L(\mathbb{R})$, computed in the generic extension $V[G]$ by Levy's poset $\mathrm{Coll}(\omega, {<}\kappa)$, which collapses all cardinals below the least inaccessible κ to ω.

The inaccessible cardinal

Theorem

1. *(Shelah 1984) The inaccessible cardinal is not necessary for the Baire property.*
2. *(Specker 1957, Solovay 1970) The existence of an inaccessible cardinal is equivalent to the statement that every set of reals has the perfect set property.*
3. *(Shelah 1984) If every $\undertilde{\Sigma}^1_3$ set of reals is Lebesgue measurable then $(\aleph_1)^V$ is inaccessible in L. So the inaccessible is also necessary.*

   Moreover, Shelah also construct a model (without using an inaccessible cardinal) in which every $\undertilde{\Delta}^1_3$ set of reals is Lebesgue measurable.