Elementary Set Theory

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Additional Topic

GAMES ON REALS[1]

[1]Cf. *The Higher Infinite*, by A. Kanamori, §27

Infinite Games

For $A \subseteq {}^{\omega}\omega$, $G(A)$ denotes the following two-person game:

I	x_0		x_2		$\cdots$
II		x_1		x_3	$\cdots$

where each $x_i \in \omega$.

- Each choice is a **move** of the game.
- The result $x = \langle x_i : i < \omega \rangle \in {}^{\omega}\omega$ is a **play** of the game.
- A is called the **payoff** for the game $G(A)$.
- Rule: I wins if $x \in A$, otherwise II wins.

For $s \in {}^{<\omega}\omega$, let $G_s(A)$ be $G(A)$ restricted to O_s, i.e.

- I wins if $s^\frown x \in O_s \cap A$, and II wins if $s^\frown x \in O_s - A$.

- A **strategy** for I is a function

$$\sigma : \bigcup_n {}^{2n}\omega \to \omega$$

that tells him what to play next given the previous moves.

Given II's moves $y = \langle y_n = x_{2n+1} : n < \omega \rangle \in {}^{\omega}\omega$, σ produces a play $\sigma * y \in {}^{\omega}\omega$.

I	$u_0 = \sigma(0)$		$u_1 = \sigma(u_0{}^\frown y_0)$		$\sigma(u_0{}^\frown y_0{}^\frown u_1{}^\frown y_1)$	
II		y_0		y_1		$\cdots$

- σ is a **winning strategy** (w.s.) for I iff

$$\{\sigma * y \mid y \in {}^{\omega}\omega\} \subseteq A,$$

i.e. no matter what moves II makes, plays according to σ always yield members of A.

Analogously,

- a strategy for II is a function $\tau : \bigcup_n {}^{2n+1}\omega \to \omega$.
- τ is a winning strategy for II iff

$$\{z * \tau \mid z \in {}^{\omega}\omega\} \cap A = \varnothing,$$

  where $z * \tau$ is the result of applying τ to a move sequence z played by I.

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  where $z * \tau$ is the result of applying τ to a move sequence z played by I.

$G(A)$ is **determined** iff a player has a winning strategy.

Note that the players cannot both have winning strategies.

A is **determined** iff $G(A)$ is determined.

Determined Sets

Theorem 1

1. *If* $|A| < \mathfrak{c} = |{}^{\omega}\omega|$, *then I cannot have a winning strategy. Similarly, II cannot have a w.s., if* $|{}^{\omega}\omega - A| < \mathfrak{c}$.
2. (*Gale-Stewart*). *If* $A \subseteq {}^{\omega}\omega$ *is either open or closed then* $G(A)$ *is determined.*
3. (*Gale-Stewart*). AC *implies that there is a set of reals which is not determined.*

Proof

1. Each w.s. induces an injective function from ${}^{\omega}\omega$ to ${}^{\omega}\omega$.
2. Let $A_s = A \cap O_s$. Consider $s \in {}^{2n}\omega$. Note that
   - *If I has no w.s. in $G_s(A)$ then*
     $$\forall i \exists j\, (\text{I has no w.s. in } G_{s^\frown\langle ij\rangle}(A)).$$
     In this case, let $\tau(s^\frown\langle i\rangle) =$ least j as above.
   - *If I has no w.s. in $G_s(A)$ then $|O_s \cap A^c| > 1$.*
     So every play produced by τ is a limit point of A^c.

   Suppose A is open and I has no w.s, then every play produced by τ are in A^c, thus II has a w.s.

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   Suppose A is open and I has no w.s, then every play produced by τ are in A^c, thus II has a w.s.
3. By AC, there are $\mathfrak{c}$ many strategies. And for each strategy σ (or τ), the corresponding set R_σ (or R_τ) of plays
   $$\{\sigma * x \mid x \in {}^{\omega}\omega\} \text{ (for I)} \quad \text{or} \quad \{x * \tau \mid x \in {}^{\omega}\omega\} \text{ (for II)}$$
   has size $\mathfrak{c}$. Choose a play (without repetition) from each R_σ (or R_τ). This gives two disjoint sets, one from R_σ's, the other from R_τ's. They are non-determined.

Regularity Properties

Donald A. Martin (1975) showed that

Every Borel set is determined.

Mycielski and Steinhaus (1962) proposed the following axiom, now known as the **Axiom of Determinacy** (AD):

Every set of reals is determined.

Theorem 2

Assume AD. *Then every set of reals is Lebesgue measurable, has the property of Baire, and has the perfect set property.*

Mazur Game

- Let $G(A, X)$ denote the game on ${}^{\omega}X$.
  Then $G(A) = G(A, \omega)$.
- $G(A, X)$ for an X with $|X| = \omega$ and $A \subset {}^{\omega}X$ is "equivalent to" a $G(A^*)$ for some $A^* \subset {}^{\omega}\omega$.

The game for the property of Baire is the Mazur game $G_{\mathcal{M}}(A)$ formulated as follows:

$$\begin{array}{c|ccccc} \mathrm{I} & s_0 & & s_2 & & \cdots \\ \hline \mathrm{II} & & s_1 & & s_3 & \cdots \end{array}$$

where $s_i \in {}^{<\omega}\omega - \{\varnothing\}$. Let $x = s_0{}^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown \cdots$, then I wins if $x \in A$, and II wins otherwise.

Proposition 3 (Mazur, Banach)

For $A \subseteq {}^{\omega}\omega$,

1. *A is meager iff II has a w.s. in $G_{\mathcal{M}}(A)$.*
2. *$O_s - A$ is meager for some $s \in {}^{<\omega}\omega$ iff I has a w.s. in $G_{\mathcal{M}}(A)$.*

Corollary 4

For $A \subseteq {}^{\omega}\omega$, let $C_A = \bigcup\{O_s \mid O_s - A \text{ is meager}\}$. If $G_{\mathcal{M}}(A - C_A)$ is determined then A has the property of Baire.

Proof of Proposition 3

1. "$\Rightarrow$". Suppose $\{C_i \mid i < \omega\}$ are (decreasing) dense open sets such that $A \cap (\bigcap_i C_i) = \varnothing$. Suppose $p = s_0{}^\frown \cdots {}^\frown s_{n-1}$ is an n-round play. For each $s \in {}^{<\omega}\omega$, let $\tau(p{}^\frown s)$ be a $t \in {}^{<\omega}\omega$ such that $O_{p^\frown s^\frown t} \subseteq C_n$.

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   "$\Leftarrow$". Let τ be a w.s. for II. For each play p, let

   $D_p = \{x \in {}^{\omega}\omega \mid p \not\subset x\} \cup (\bigcup\{O_{p^\frown s^\frown t} \mid t = \tau(p^\frown s),\ s \in {}^{<\omega}\omega\})$.

   Then $A \cap \bigcap_p D_p = \varnothing$. Since τ is winning for II, D_p is open dense.

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   "$\Leftarrow$". Given σ, winning for I. $\sigma(\varnothing)$ is a such s.

Proof of Corollary 4

1. Note that $C_A - A$ is meager. If II wins, then $A - C_A$ is meager, and then $A \,\Delta\, C_A = (C_A - A) \cup (A - C_A)$ is meager, therefore A has the property of Baire.

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2. If I wins. For some $s \in {}^{<\omega}\omega$, $O_s - (A - C_A)$ is meager.

   $$O_s - (A - C_A) \supseteq O_s - A.$$

   Thus $O_s - A$ is meager, and hence $O_s \subseteq C_A$. Then $O_s \cap (A - C_A) = (O_s \cap A) - (O_s \cap O_A) = \varnothing$, therefore $O_s - (A - C_A) = O_s$. This contradicts to the fact that O_s is not meager.

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   So I can not win!

An embedding

We shall present the other two games as games over ${}^{\omega}2$. The following embedding $\pi : {}^{\omega}\omega \to {}^{\omega}2$ can transfer the results back to the Baire space ${}^{\omega}\omega$.

$$\pi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown s_{x(2)}{}^\frown \cdots$$

where $s_{x(k)} = \underbrace{1 \cdots 1}_{x(k)} 0$ for even k, and $\underbrace{0 \cdots 0}_{x(k)} 1$ for odd k.

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It's easy to check that ${}^{\omega}2 - \mathrm{ran}(\pi)$ is countable, and for $\varphi \in \{\text{BP, PSP, LM}\}$, for every set $X \subseteq {}^{\omega}2$,

$$\varphi(X) \text{ is true in } {}^{\omega}2 \text{ iff } \varphi(\pi^{-1}(X)) \text{ is true in } {}^{\omega}\omega.$$

Davis game

Davis game $G_{\mathcal{C}}(A)$ is formulated as follows:

$$\begin{array}{c|cccccc} \text{I} & s_0 & & s_2 & & \cdots \\ \hline \text{II} & & k_1 & & k_3 & \cdots \end{array}$$

where $s_i \in {}^{<\omega}2 - \{\varnothing\}$, $k_i \in \{0, 1\}$. Let

$$x = s_0{}^\frown\langle k_1\rangle{}^\frown s_2{}^\frown\langle k_3\rangle{}^\frown \cdots .$$

I wins if $x \in A$, otherwise II wins.

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I wins if $x \in A$, otherwise II wins.

Proposition 5 (Davis)

For any $A \subseteq {}^\omega 2$,

1. *A is countable iff II has a w.s. in $G_{\mathcal{C}}(A)$.*
2. *A contains a perfect subset iff I has a w.s. in $G_{\mathcal{C}}(A)$.*

Proof of Proposition 5

1. "$\Rightarrow$" is easy. Argue for "$\Leftarrow$". Let τ be a w.s. for II. Let $R_\tau = \{y * \tau \mid y \in {}^\omega 2\}$, i.e. all the plays produced by τ. Then $A \cap R_\tau = \varnothing$. Thus for each $x \in A$, there is a play $p_x = \langle s_0, k_0, \ldots, s_n, k_n \rangle$ such that
$$p_x^* = s_0{}^\frown \langle k_0 \rangle^\frown \cdots s_n{}^\frown \langle k_n \rangle \sqsubset x,$$
and no matter what I plays with along x, he is defeated by τ, i.e. for every $i \geq |p^*|$, $x(i) = 1 - \tau(x \restriction i)$.

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and no matter what I plays with along x, he is defeated by τ, i.e. for every $i \geq |p^*|$, $x(i) = 1 - \tau(x \restriction i)$.
2. Let $T \subseteq {}^{<\omega}2$ be a perfect tree such that $[T] \subseteq A$. Suppose p is an n-round play, let $\sigma(p)$ to be the next splitting node extending p^*. Then σ is a w.s. for I.

Harrington Game

For $A \subset {}^{\omega}2$ and $\varepsilon \in \mathbb{R}^+$, $G_{\mathcal{N}}(A, \varepsilon)$ is

I	i_0		i_1		$\cdots$
II		$\bar{s}_0$		$\bar{s}_1$	$\cdots$

where $i_k \in \{0, 1\}$, $\bar{s}_k \in [{}^{<\omega}2 - \{\varnothing\}]^{<\omega}$ with the additional requirement

$$\mu(N_{\bar{s}_k}) < \varepsilon / 2^{2(n+1)}, \quad N_{\bar{s}_k} = \bigcup_j O_{\bar{s}_k(j)}.$$

Let $x = \langle i_0 i_1 \cdots \rangle$. I wins iff $x \in A - \bigcup_k N_{\bar{s}_k}$, otherwise II wins.

(Here $\mu(O_s) = 1/2^{\mathrm{dom}(s)}$ for each $s \in {}^{<\omega}2$.)

Proposition 6

In $G_{\mathcal{N}}(A, \varepsilon)$, $A \subset {}^{\omega}2$ and $\varepsilon \in \mathbb{R}^+$,

1. *If I has a w.s. then there is a Lebesgue measurable $B \subseteq A$ such that $\mu(B) > 0$.*
2. *If II has a w.s. then there is an open set $O \supseteq A$ s.t. $\mu(O) < \varepsilon$.*

Proposition 6

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2. *If II has a w.s. then there is an open set $O \supseteq A$ s.t. $\mu(O) < \varepsilon$.*

Corollary 7

For $A \subseteq {}^{\omega}\omega$, let $Q_A \supseteq A$ be Lebesgue measurable and with $\mu(Q_A)$ minimal. Then if $G_{\mathcal{N}}(Q_A - A, \varepsilon)$ is determined for every $\varepsilon > 0$, then A is Lebesgue measurable.

Proof of Corollary 7

By choice of Q_A, II must have a winning strategy in $G_{\mathcal{N}}(Q_A - A, 1/n)$ for each $n < \omega$. Hence

$$Q_A - A \subseteq \bigcap_n C_n,$$

where $\mu(C_n) < 1/n$, for each n. Therefore $\mu(Q_A - A) = 0$ and A is Lebesgue measurable with $\mu(A) = \mu(Q_A)$.