Elementary Set Theory

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Coming up next

Cardinality

We use injective functions to compare the size of sets.

Definition 1

1. $X \approx Y$ iff there is a bijection from X to Y.
2. $X \preccurlyeq Y$ iff there is an injection from X to Y.[1]
3. $X \prec Y$ iff $X \preccurlyeq Y$ and $\neg(Y \preccurlyeq X)$.

[1] Note that empty function is injective.

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Easy to check:

Proposition 2

1. $\approx$ *is an equivalence relation.*
2. $\preccurlyeq$ *is transitive.*

[1] Note that empty function is injective.

Cantor-Bernstein

Next is a much deeper result

Theorem 3 (Cantor-Bernstein-Schröder)

Let X, Y be any two sets. Then

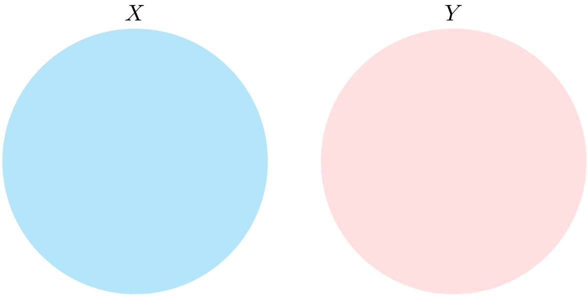
$$X \preccurlyeq Y \wedge Y \preccurlyeq X \implies X \approx Y.$$

A bit history

As it is often the case in mathematics, the name of this theorem does not truly reflect its history.

- The traditional name "Schröder-Bernstein" is based on two proofs published independently in 1898.
- Cantor is often added because he investigated it around 1870s, and first stated it as a theorem in 1895,
- while Schröder's name is often omitted because his proof turned out to be flawed
- and while the name of the mathematician who first proved it (Dedekind, 1887, 1897) is not connected with the theorem.

Proof of Cantor-Bernstein



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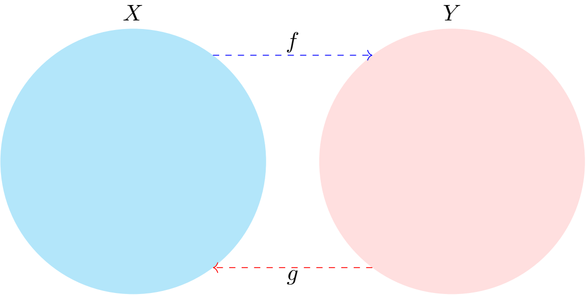

If f (or g) is onto, then we are done!
f (or g^{-1}) is a bijection.

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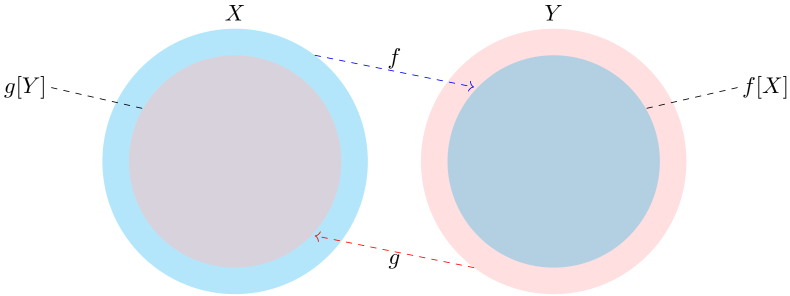

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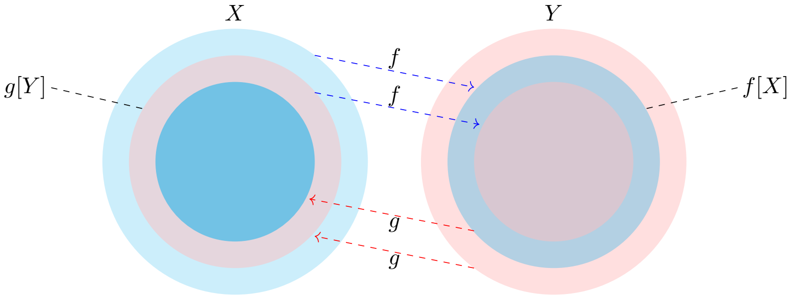

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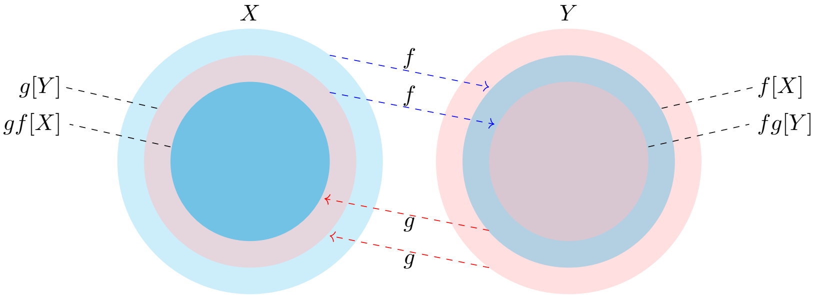

$$g[Y] - gf[X] \approx Y - f[X] \text{ via } g^{-1}$$

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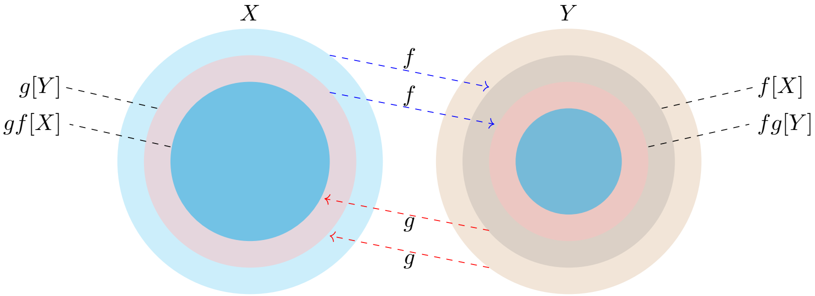

$g[Y] - gf[X] \approx Y - f[X]$ via g^{-1}
$X - g[Y] \approx f[X] - fg[Y]$ via f

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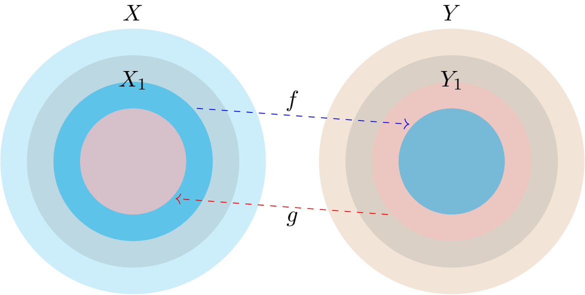
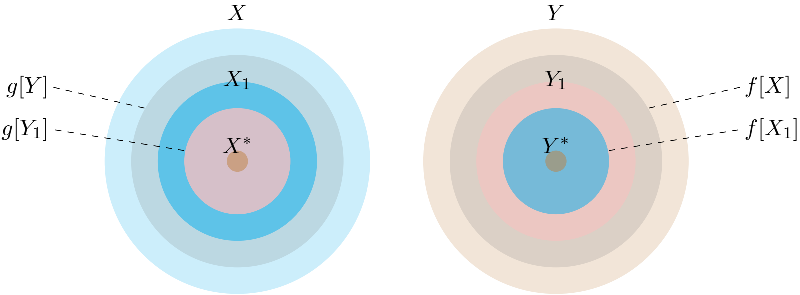


Thus $X - X_1 \approx Y - Y_1$,
also we have $f : X_1 \to Y_1$, $g : Y_1 \to X_1$.

Proof of Cantor-Bernstein



$X \supset X_1 \supset X_2 \cdots \supset X_n \supset \cdots$ $\qquad$ $Y \supset Y_1 \supset Y_2 \cdots \supset Y_n \supset \cdots$

Let $X^* = \bigcap_i X_i$ and $Y^* = \bigcap_i Y_i$. By induction, $X - X^* \approx Y - Y^*$

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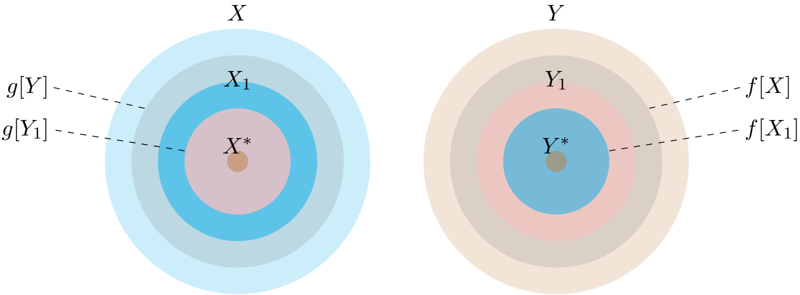

$X \supset X_1 \supset X_2 \cdots \supset X_n \supset \cdots \qquad Y \supset Y_1 \supset Y_2 \cdots \supset Y_n \supset \cdots$

Let $X^* = \bigcap_i X_i$ and $Y^* = \bigcap_i Y_i$. By induction, $X - X^* \approx Y - Y^*$

But $f[X] \supset Y_1 \supset f[X_1] \supset Y_2 \supset \cdots$. Thus $Y^* = \bigcap_i f[X_i] = f[X^*]$. $\square$

Coming up next

Cardinal Numbers

Cardinal

Cardinal arithmetic, I

Cofinality

Cardinal

Thus we can assign to each set X its **cardinal number** $|X|$ so that

$$X \approx Y \quad \text{iff} \quad |X| = |Y|$$

Cardinal numbers can be defined

- either via equivalence classes (need **Regularity**),
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- either via equivalence classes (need **Regularity**),
- (von Neumann) or using ordinals (need AC).
  — We shall use this definition.

Cardinality

One determines the size of a finite set by counting it. More generally,

Definition 4

If X can be well-ordered, then $X \approx \alpha$ for some $\alpha \in \mathrm{Ord}$, and the least such α is called the **cardinality** of X, $|X|$.

Some simple facts.

- If $X \preccurlyeq \alpha$, then X can be well-ordered.
- $|\alpha| \leq \alpha$, for all $\alpha \in \mathrm{Ord}$.
- Under AC, every set can be well-ordered, so $|X|$ is defined for every X.

For the rest of this Chapter, we assume AC.

Cardinal

Definition 5

An ordinal α is a **cardinal** if $|\alpha| = \alpha$.

We use κ, λ, δ etc to denote cardinals.

Some simple facts.

- α is a cardinal iff $\forall \beta < \alpha \, (\beta \not\approx \alpha)$.
- If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.
- Every infinite cardinal is a limit ordinal.
- For every $n \in \omega$, $n \not\approx n + 1$.
- If $n \in \omega$, then for all α, $\alpha \approx n \to \alpha = n$.

Finite-Countable-Uncountable

Corollary 6

ω is a cardinal and each $n \in \omega$ is a cardinal.

Finite-Countable-Uncountable

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Definition 7

- X is **finite** iff $|X| < \omega$. **Infinite** means not finite.
- X is **countable** iff $|X| \leq \omega$. **Uncountable** means not countable.

Finite-Countable-Uncountable

Example

- Every $n \in \omega$ is finite.
- $\omega, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is countable. (To be discussed later)
- (Cantor). $\mathbb{R}$ is uncountable. (To be proved in Chapter 4)

Finite-Countable-Uncountable

Example

- Every $n \in \omega$ is finite.
- $\omega, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is countable. (To be discussed later)
- (Cantor). $\mathbb{R}$ is uncountable. (To be proved in Chapter 4)

Remark. One cannot prove on the basis of ZFC $-$ **Power Set** that uncountable sets exist. In fact, it is consistent with ZFC $-$ **Power Set** that the only infinite cardinal is ω.

Uncountable Cardinal

Before Cantor's proof of "$\mathbb{R}$ is uncountable", it was not known that there are more than one infinite cardinal.

Theorem 8

For any set X, $X \prec \mathscr{P}(X)$.

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PROOF.

- Identify every set X with its characteristic function $C_X : X \to \{0,1\}$. Hence $\mathscr{P}(X) \approx {}^X 2$.
- Suppose $F : X \to \mathscr{P}(X)$ is an arbitrary injection. Construct an $Z \in \mathscr{P}(X) - \operatorname{ran}(F)$ by diagonalization:

$$C_Z(x) = 1 \quad \text{iff} \quad C_{f(x)}(x) = 0,$$

i.e. $Z = \{x \in X \mid x \notin f(x)\}$. F is not surjective!

$\square$

In fact, Card is "unbounded" along Ord.

Corollary 9

For any set $S \subset \mathrm{Card}$, there is a cardinal κ s.t.

$$\forall \lambda \in S \, (\lambda < \kappa).$$

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Without assume AC, the following is not easy to prove.

Theorem (Halbeisen and Shelah, 1994)

For all infinite set A,

$$\mathrm{fin}(A) \prec \mathscr{P}(A),$$

where $\mathrm{fin}(A) := \{x \subseteq A \mid x$ is finite$\}$.

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Operations on Cardinals

The arithmetic operations on cardinals are defined as follows

Definition 10

1. $\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$
2. $\kappa \cdot \lambda = |\kappa \times \lambda|$.
3. $\kappa^\lambda = |{}^\lambda\kappa|$.

κ, λ on the right are referred as sets.

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Exercise

Verify that these definitions are well defined.

We've shown that $|\mathscr{P}(X)| = 2^{|X|}$ and $\forall \kappa \, (\kappa < 2^{\kappa})$.

Simple Facts About Cardinal Arithmetics

- Unlike the ordinal operations, $+$ and $\cdot$ are associative, commutative and distributive.
- $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
- $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
- $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.
- If $\kappa \leq \lambda$, then $\kappa + \mu \leq \lambda + \mu$, $\kappa \cdot \mu \leq \lambda \cdot \mu$ and $\kappa^\mu \leq \lambda^\mu$.
- If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.
- $\kappa^0 = 1$, $1^\kappa = 1$, $0^\kappa = 0$ if $\kappa > 0$.
- When $\kappa, \lambda < \omega$, $\kappa + \lambda$, $\kappa \cdot \lambda$ and κ^λ are the same as the corresponding operations on natural numbers.

Alephs

Since $\mathrm{Card} \subset \mathrm{Ord}$, Card is well-ordered and the elements of Card can be enumerated with Ord as indices. Consider infinite cardinals only.

Definition 11

For any cardinal κ, κ^+ denotes the least cardinal $> \kappa$. The Aleph function $\aleph$ is define by the transfinite recursion:

$$\begin{aligned} \aleph_0 &= \omega, \\ \aleph_{\alpha+1} &= \aleph_\alpha^+, \\ \aleph_\sigma &= \lim_{\alpha \to \sigma} \aleph_\alpha, \quad \lambda \text{ is a limit ordinal.} \end{aligned}$$

An infinite cardinal is called a **successor** cardinal if it is of the form $\aleph_{\alpha+1}$ for some α, otherwise is called a **limit** cardinal.

Alephs

$\aleph_\alpha$ are often written as ω_α.

This definition is legitimate due to the following facts

- For every κ, there is a λ s.t. $\kappa < \lambda$.
  Hence, κ^+ exists for every cardinal κ.
- For every set $S \subset \mathrm{Card}$, $\sup(S)$ is a cardinal.
  In particular, $\lim_{\alpha<\sigma} \aleph_\alpha$ is a cardinal.

These ensure that $\mathrm{dom}(\aleph) = \mathrm{Ord}$. Since for each $\alpha \in \mathrm{Ord}$,

$$\aleph_\alpha = \min\{\kappa \in \mathrm{Card} \mid \forall \beta < \alpha\, (\aleph_\beta < \kappa)\},$$

$\mathrm{ran}(\aleph) = \mathrm{Card} \setminus \omega$.

Alephs

REMARK. The existence of κ^+ (κ infinite) can be shown without referring to 2^κ and AC:

$$\kappa^+ = \sup\{\mathrm{ordertype}(\prec) \mid (\kappa, \prec) \text{ is a well-ordering.}\}$$

Alephs

<u>REMARK</u>. The existence of κ^+ (κ infinite) can be shown without referring to 2^κ and AC:

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Lemma 12

Card *is a proper class.*

In general, $A \subset \mathrm{Ord}$ is unbounded iff A is proper.

Cardinality of Sets,

Corollary 13

The following sets are countable:

- $\mathbb{Z}, \mathbb{Q}$ *are countable.*
- *The set of all algebraic numbers,* $\mathbb{A}$*, is countable.*

Assume that $|\mathbb{R}| = 2^{\aleph_0}$*. Then the following sets are of size* $2^{\aleph_0}$*.*

- *The set of all points in the* n*-dimensional space,* $\mathbb{R}^n$*.*
- *The set of all complex numbers,* $\mathbb{C}$*.*
- *The set of all* ω*-sequences of natural numbers,* ω^ω*.*
- *The set of all* ω*-sequences of real numbers,* $\mathbb{R}^\omega$

Lemma 14 (AC)

If $|A| < |B|$ *then* $|B - A| = |B|$.

In fact, one can prove the following without using AC.

Lemma 15

If $A \subseteq B$, $|A| = \aleph_0$ *and* $|B| = 2^{\aleph_0}$, *then* $|B - A| = 2^{\aleph_0}$.

Hint: View $A \subseteq \mathbb{R} \times \mathbb{R} \approx B$. $\exists r \in \mathbb{R}$ s.t. $A \cap (\{r\} \times \mathbb{R}) = \varnothing$.

Corollary 16

The set of irrationals, $\mathbb{R} - \mathbb{Q}$*, and the set of transcendental numbers,* $\mathbb{R} - \mathbb{A}$*, are of cardinality* $2^{\aleph_0}$.

Addition and Multiplication are trivial

Theorem 17 (AC)

Let κ, λ be infinite cardinals. Then

1. $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.
2. $|^{<\omega}\kappa| = \kappa$.

They follow from the lemma on next page.

Lemma 18 (AC)

For every $\alpha \in \mathrm{Ord}$, $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

<u>Proof of Theorem</u>.

(1) follows immediately from Lemma 18. Below is for (2).

- For each $n \in \omega$, pick an injection $f_n : {}^n\kappa \to \kappa$.
- Combining them gives us an injection
$$f : \bigcup_n {}^n\kappa \to \omega \times \kappa, \quad f(\sigma) = (|\sigma|, f_{|\alpha|}(\sigma))$$
whence $|{}^{<\omega}\kappa| \leq \omega \cdot \kappa = \kappa$. □

Next, we prove the lemma via pictures.

A Well-Ordering of $\kappa \times \kappa$, $\kappa = \aleph_{\delta+1}$

Proof of Lemma

$$\begin{aligned}(a_1, b_1) \prec (a_2, b_2) \leftrightarrow\ & \max(a_1, b_1) < \max(a_2, b_2) \\ & \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 < b_2) \\ & \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 = b_2 \wedge a_1 < a_2)\end{aligned}$$

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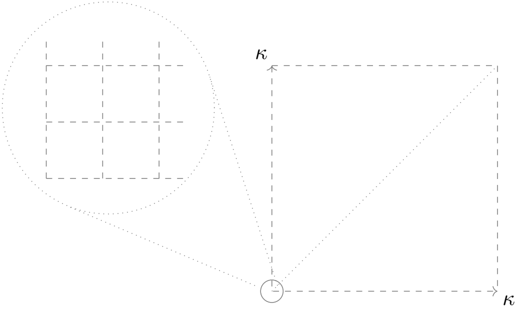

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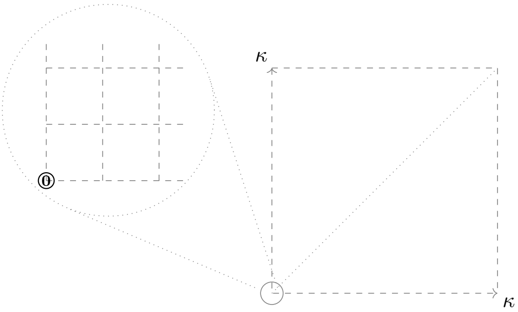

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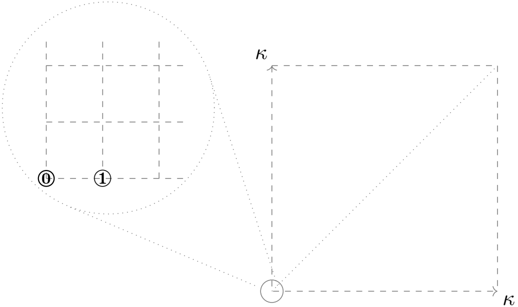

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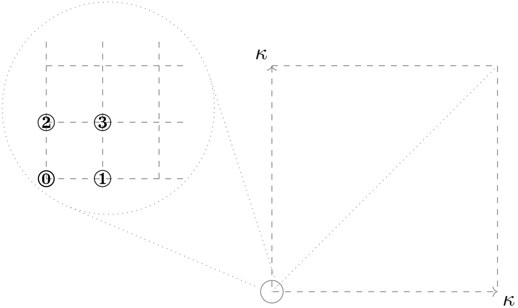

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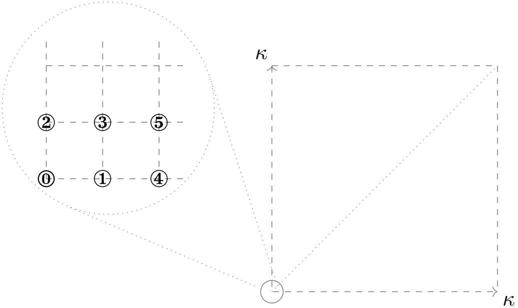

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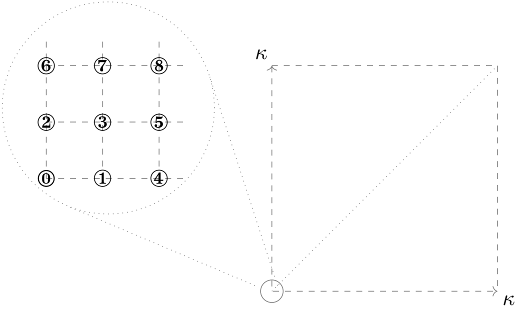

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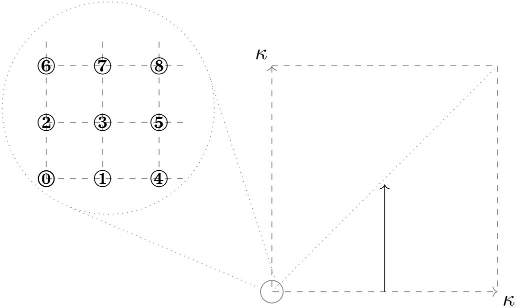

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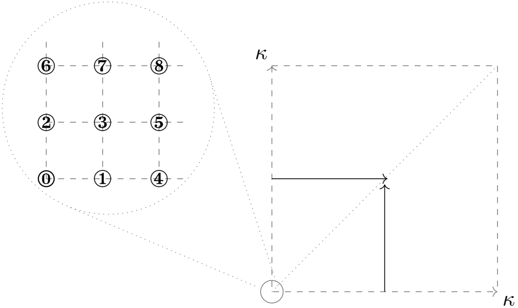

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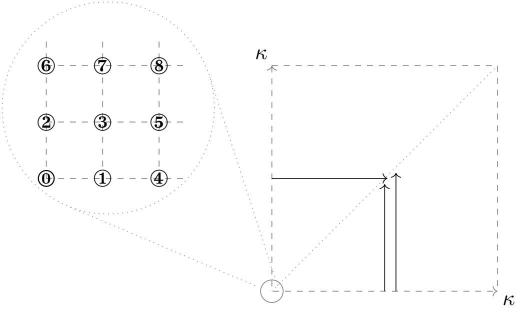

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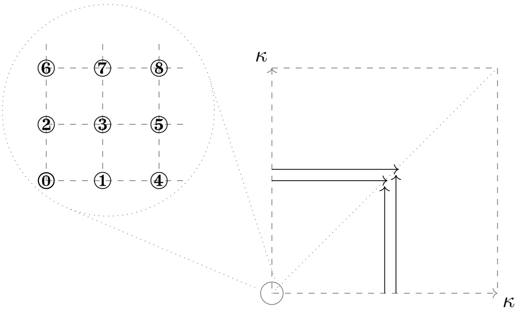

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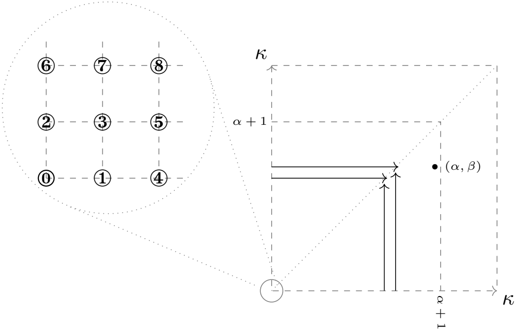
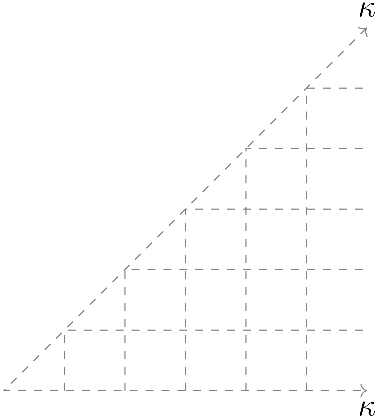
& \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 < b_2) \\
& \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 = b_2 \wedge a_1 < a_2)
\end{aligned}
$$

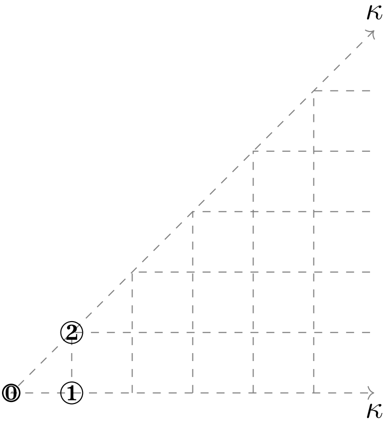
At any $(a, b) \in \aleph_{\delta+1} \times \aleph_{\delta+1}$, $|$the initial segment of $\prec$ up to $(a, b)| \leq \aleph_\delta$



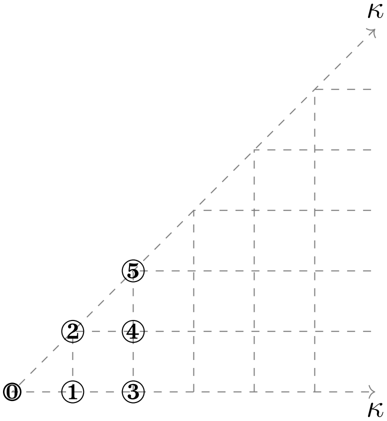
Another ordering



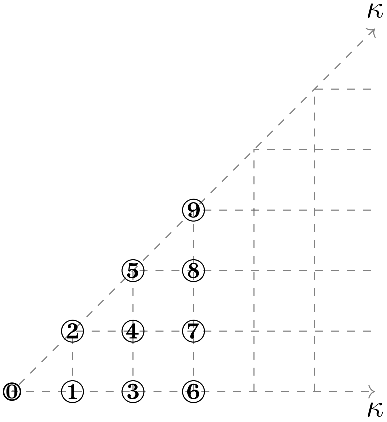
Another ordering



Another ordering



Another ordering



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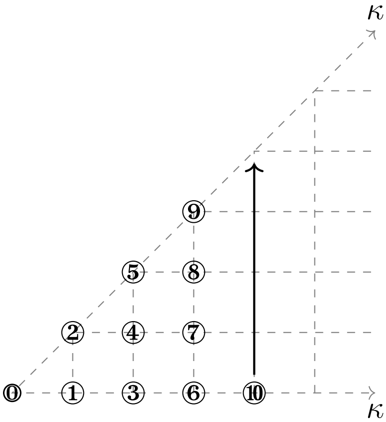


Homework I

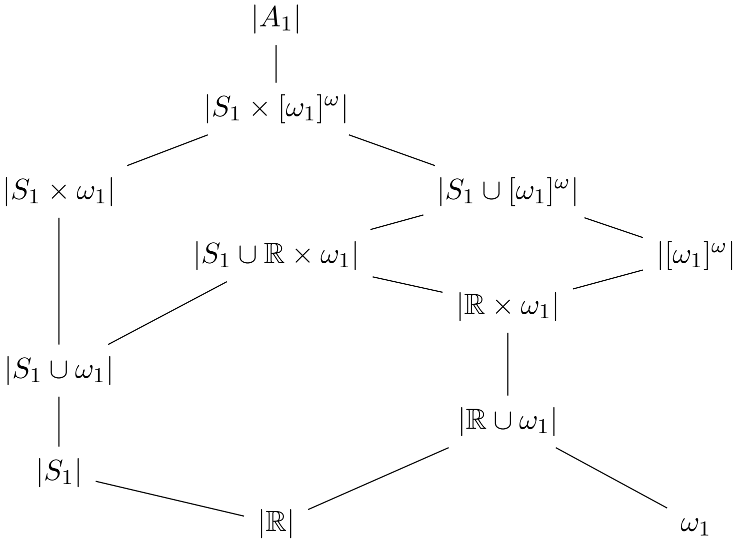
1. Write an explicit formula for this bijection.
2. **Instructions:** State the value of the following expressions.
   (i) $\aleph_0 + \aleph_0$
   (ii) $\aleph_0 \cdot \aleph_0$
   (iii) $\mathfrak{c} + \mathfrak{c}$
   (iv) $\mathfrak{c} \cdot \mathfrak{c}$
   (v) $2^{\aleph_0}$
   (vi) $\mathfrak{c}^{\aleph_0}$

   Where $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$.

Homework II

3. Assuming the **Generalized Continuum Hypothesis (GCH)**, which states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α, evaluate the following expressions.
   (i) $2^{\mathfrak{c}}$
   (ii) $\aleph_1^{\aleph_0}$
   (iii) $\mathfrak{c}^{\aleph_1}$
4. Determine the relationship between the two cardinal numbers ($\leq$, $\geq$, or $=$). You may use the fact that for infinite cardinals κ and λ, if $\kappa \leq \lambda$, then $\kappa + \lambda = \lambda$ and $\kappa \cdot \lambda = \lambda$.
   (i) $\aleph_0 + \mathfrak{c}$ _______ $\mathfrak{c}$
   (ii) $\aleph_0 \cdot \mathfrak{c}$ _______ $\mathfrak{c}$
   (iii) $2^{\mathfrak{c}}$ _______ $2^{\aleph_0} + \mathfrak{c}$
   (iv) $(\aleph_0 + \mathfrak{c})^{\aleph_0}$ _______ $\mathfrak{c}$

Small cardinals, when no full AC (Woodin, 2006)



Impact of AC

AC is equivalent to the assertion that

$$\text{“Every set can be well-ordered”.} \qquad \text{(WO)}$$

Many of the basic properties of cardinals need AC.

Write $X \preccurlyeq^* Y$ if $X = \varnothing$ or there is a surjection $f : Y \xrightarrow{\text{onto}} X$.

Lemma 19 (AC)

1. *If* $X \preccurlyeq^* Y$, *then* $X \preccurlyeq Y$.
2. *If* $\kappa \geq \omega$ *and* $X_\alpha \preccurlyeq \kappa$ *for all* $\alpha < \kappa$, *then* $\bigcup_{\alpha<\kappa} X_\alpha \preccurlyeq \kappa$.

PROOF.

1. Let $\prec$ well-orders Y. Suppose $f : Y \to X$ is surjective. Define $g : X \to Y$ as
$$g(x) = \prec\text{-least element of } f^{-1}(\{x\}).$$

2.
   - For each α, pick an injection $f_\alpha : X_\alpha \to \kappa$.
     f_α are selected via a well-ordering of $\mathscr{P}(\bigcup X_\alpha \times \kappa)$.
   - For $t \in \bigcup X_\alpha$, let $F(t) = (f_\alpha(t), \alpha)$, where
     $$\alpha_t = \text{least } \alpha \text{ such that } t \in X_\alpha.$$
   Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection. $\pi \circ F$ works. $\square$

PROOF.

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     Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection. $\pi \circ F$ works. $\square$

An important application of Lemma 19-2 is the **Downward Löwenheim-Skolem-Tarski Theorem** in model theory.

An Application, Definitions

Definition 20

1. An n**-ary operation** on X is a function $f : X^n \to X$ if $n > 0$, or an element of X if $n = 0$.
2. If $Y \subset X$, Y is **closed under** f iff $f[Y^n] \subset Y$ (or $f \in B$ when $n = 0$).
3. A **finitary operation** is an n-ary operation for some $n < \omega$.
4. If $\mathcal{E}$ is a set of finitary operations on X and $Y \subset X$, the closure of Y under $\mathcal{E}$, denoted as $\mathrm{cl}_{\mathcal{E}}(Y)$, is the least $Y^* \subset X$ such that $Y \subset Y^*$, and Y^* is closed under all the operations in $\mathcal{E}$.

An Application, Theorem

Theorem 21 (AC)

Let κ be an infinite cardinal. Suppose $Y \subset X$, $Y \leq \kappa$, and $\mathcal{E}$ is a set of $\leq \kappa$ finitary operations on X. Then $|\mathrm{cl}_{\mathcal{E}}(Y)| \leq \kappa$.

An Application, Theorem

Theorem 21 (AC)

Let κ be an infinite cardinal. Suppose $Y \subset X$, $Y \leq \kappa$, and $\mathcal{E}$ is a set of $\leq \kappa$ finitary operations on X. Then $|\mathrm{cl}_{\mathcal{E}}(Y)| \leq \kappa$.

Example. Every infinite group has a countably infinite subgroup.

An Application, Theorem

<u>PROOF</u>.

- Let $E_0 \subset \mathcal{E}$ be the set of all 0-ary operations in $\mathcal{E}$.
- Let $C_0 = Y \cup E_0$. We may assume that $\mathcal{E}$ has no 0-ary operations.
- By induction on $n < \omega$, define
$$C_{n+1} = C_n \cup (\bigcup\{f[^k C_n] \mid f \in \mathcal{E},\ f \text{ is } k\text{-ary.}\})$$
- Take $C_\omega = \bigcup_n C_n$. Check that $C_\omega = \mathrm{cl}_{\mathcal{E}}(Y)$.

$\square$

Homework

1. Prove the following statements.
   1.1 If $x \cap y = \varnothing$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.
   1.2 If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.
2. Ex.3.1-3.3 in textbook.
3. Prove that $\kappa^\kappa \leq 2^{\kappa \cdot \kappa}$.
4. If $A \preccurlyeq B$, then $A \preccurlyeq^* B$.
5. If $A \preccurlyeq^* B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.[2]
6. Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

[2] Don't forget the case $A = \varnothing$.

REMARK.

- Assuming AC, the converse of (4) is true (see Lemma 19).
- (6) is related to so called "Dedekind-infinite". (see textbook Ex.3.14-3.16)

Exercises*

1. α is called an **epsilon number** iff $\alpha = \omega^\alpha$ (ordinal exponentiation). Show that
   - the first epsilon number ε_0 is countable.
   - for each $\alpha \in \mathrm{Ord} - \{0\}$, $\aleph_\alpha$ is an epsilon number.
   - for each $\alpha \in \mathrm{Ord} - \{0\}$, the set of epsilon numbers is unbounded below $\aleph_\alpha$. Hence, there are $\aleph_\alpha$ epsilon numbers below $\aleph_\alpha$.
2. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

Continuum Hypothesis

Since Cantor could show (under AC) that $\aleph_1 \leq 2^{\aleph_0}$, and had no way producing cardinals between $\aleph_1$ and $2^{\aleph_0}$, he conjectured that

Continuum Hypothesis (CH)

$$\aleph_1 = 2^{\aleph_0}?$$

Continuum Hypothesis

More generally,

Generalized Continuum Hypothesis (GCH)

For every $\alpha \in \mathrm{Ord}$,

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}?$$

Continuum Hypothesis

More generally,

Generalized Continuum Hypothesis (GCH)

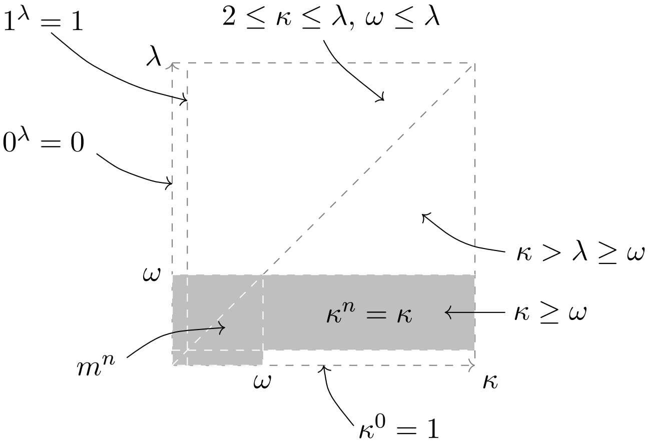
For every $\alpha \in \mathrm{Ord}$,

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}?$$

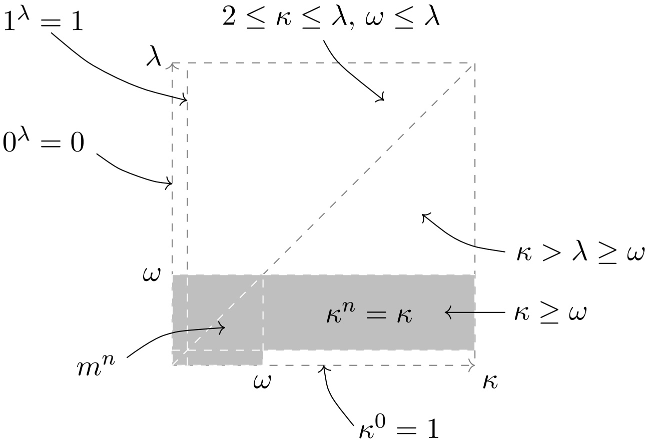
<u>Remark</u>. Without AC, it is possible that $\aleph_1 \not\leq 2^{\aleph_0}$; however, one can still show that $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$, for every $\alpha \in \mathrm{Ord}$. (see textbook Ex.3.7-3.11)

Coming up next

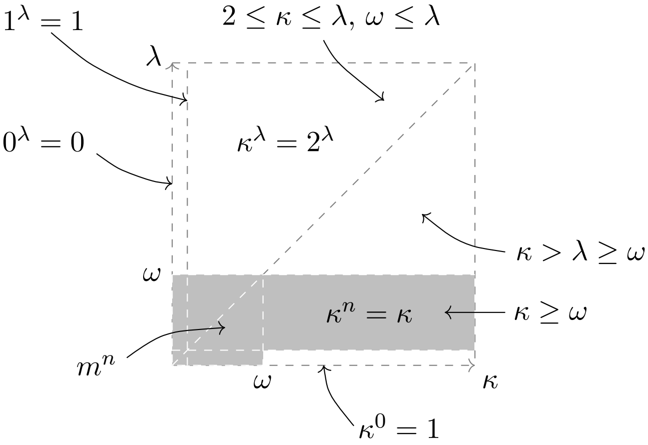
Exponentiation of Cardinals



Exponentiation of Cardinals



Exponentiation of Cardinals



Exponentiation of Cardinals

Lemma 22

If $\lambda \geq \omega$ *and* $2 \leq \kappa \leq \lambda$, *then* $\kappa^\lambda = 2^\lambda$.

Exponentiation of Cardinals

Lemma 22

If $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.

Under GCH, κ^λ can be easily computed, but the notion of **cofinality** is needed.

Cofinality

Definition 23

- If $f : \alpha \to \beta$, f maps α **cofinally (into β)** iff $\mathrm{ran}(f)$ is unbounded in β, i.e. $\forall b \in \beta, \exists a \in \alpha, f(a) \geq b$.
- The cofinality of β, $\mathrm{cf}(\beta)$, is the least α s.t. there is a map from α cofinally into β.

Cofinality

Definition 23

- If $f : \alpha \to \beta$, f maps α **cofinally (into β)** iff $\mathrm{ran}(f)$ is <u>unbounded</u> in β, i.e. $\forall b \in \beta, \exists a \in \alpha, f(a) \geq b$.
- The cofinality of β, $\mathrm{cf}(\beta)$, is the <u>least</u> α s.t. there is a map from α cofinally into β.

Revise f to get a strictly increasing function $f|A : A \subset \alpha \to \beta$, and then $g_{f|A} : \mathrm{otp}(A) \to \beta$. Clearly $\mathrm{otp}(A) \leq \alpha$. Thus we have

Lemma 24

There is a cofinal map $f : \mathrm{cf}(\alpha) \to \alpha$ which is strictly increasing, i.e. $\xi < \eta \to f(\xi) < f(\eta)$.

In general, it is not true for $\gamma > \mathrm{cf}(\alpha)$.

Properties of $\mathrm{cf}(\cdot)$

Lemma 25

If α is a limit ordinal and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

PROOF.

"$\geq$": Let $\gamma = \mathrm{cf}(\alpha)$ and $g : \gamma \to \alpha$ be cofinal, then $f \circ g : \gamma \to \beta$ is cofinal. Thus $\gamma \geq \mathrm{cf}(\beta)$, as $\mathrm{cf}(\beta)$ is minimal.

"$\leq$": Let $\gamma = \mathrm{cf}(\beta)$ and $g : \gamma \to \beta$. A map $h : \gamma \to \alpha$ is defined as follows: for $a \in \gamma$,

$$h(a) = \min\{b \in \alpha \mid g(b) > f(a)\}.$$

h is well defined by the strictly-increasing-ness of f.
Verify that h is strictly increasing and cofinal.

Properties of $\mathrm{cf}(\cdot)$

Corollary 26

1. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
2. *If α is a limit ordinal, then $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.*

Properties of $\mathrm{cf}(\cdot)$

Corollary 26

1. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
2. *If α is a limit ordinal, then $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.*

Clearly,

- $\mathrm{cf}(\alpha) \leq \alpha$,
- if α is a successor, $\mathrm{cf}(\alpha) = 1$.
- if α is a limit ordinal, $\mathrm{cf}(\alpha)$ is a limit ordinal $\geq \omega$.

<u>Example</u>. $\mathrm{cf}(\omega^n) = \mathrm{cf}(\aleph_\omega) = \omega$.

Regular Cardinal

Definition 27

α is **regular** iff α is a limit ordinal and $\mathrm{cf}(\alpha) = \alpha$. Otherwise, α is **singular**.

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Lemma 28

1. *For every limit ordinal α, $\mathrm{cf}(\alpha)$ is regular.*
   In particular, ω is regular.
2. *If α is regular, then α is a cardinal.*

Regular Cardinal

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Lemma 28

1. *For every limit ordinal α, $\mathrm{cf}(\alpha)$ is regular.*
   In particular, ω is regular.
2. *If α is regular, then α is a cardinal.*

Proof of (2): Suppose $\gamma < \alpha$ and $\pi : \gamma \to \alpha$ were bijective. π would be unbounded, thus $\gamma \geq \mathrm{cf}(\alpha) = \alpha$. Contradiction!

Singular Cardinal

Lemma 29

Suppose $\kappa = \aleph_\alpha$ *for some* $\alpha \in \mathrm{Ord}$. κ *is singular iff there exists a cardinal* $\lambda < \kappa$ *and a family* $\{S_\xi \mid \xi < \lambda\}$ *of subsets of* κ *with each* $|S_\xi| < \kappa$, $\xi < \kappa$, *such that* $\kappa = \bigcup_{\xi<\lambda} S_\xi$. *The least cardinal* λ *that satisfies the condition is* $\mathrm{cf}(\kappa)$.

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PROOF.

"⇒": Suppose $\lambda < \kappa$ and $f : \lambda \to \kappa$ is cofinal. For each $\xi < \lambda$, let $S_\xi = f(\xi)$ (as subset of κ). Then $\kappa = \sup_{\xi<\lambda} S_\xi = \bigcup_{\xi<\lambda} S_\xi$. Moreover, least such $\lambda \leq \mathrm{cf}(\kappa)$.

"⇐": Let λ be least such. For $\delta < \lambda$, let $f(\delta) = \mathrm{otp}\big(\bigcup_{\xi<\delta} S_\xi\big)$. Each $f(\delta) \leq \kappa$. f is nondecreasing. By the minimality of λ, $f(\delta) < \kappa$, for $\delta < \lambda$. Clearly, $\kappa_f := \sup_{\delta<\lambda} f(\delta) \leq \kappa$.

$$\kappa = \Big|\bigcup_{\xi<\lambda} S_\xi\Big| \leq \lambda \times \kappa_f = \max\{\lambda, \kappa_f\}.$$

Since $\lambda < \kappa$, $\kappa = \kappa_f$. f is cofinal in κ, so $\lambda \geq \mathrm{cf}(\kappa)$.

Singular Cardinal

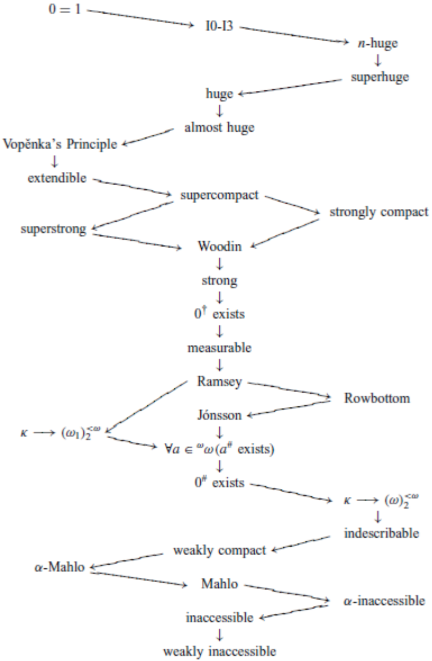
Corollary 30 (AC)

For each α, $\aleph_{\alpha+1}$ is regular.

<u>Remark</u>. Without AC, it is consistent that $\mathrm{cf}(\omega_1) = \omega$, i.e. ω_1 is a countable union of countable sets. In contrast, in ZF one can show that ω_2 cannot be a countable union of countable sets.

Large cardinals

- There are arbitrarily large singular cardinals. For each α, $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$.
- It is unknown whether one can prove in ZF that there exists a cardinal κ with $\mathrm{cf}(\kappa) > \omega$.
- (Hausdroff, 1908) κ is **weakly inaccessible** if κ is a regular limit cardinal ($\forall \lambda < \kappa$, $\lambda^+ < \kappa$). Every weak inaccessible is a fix point of the $\aleph$-sequence ($\aleph_\alpha = \alpha$). The first weakly inaccessible cardinal is rather large. And its existence is independent of ZFC.
- (Sierpiński-Tarski, Zermelo, 1930). κ is **strongly inaccessible** iff $\kappa > \omega$, κ is regular and $\forall \lambda < \kappa \, (2^\lambda < \kappa)$. Strong inaccessibles are weak inaccessibles. Under GCH, these two notions coincide.



From
The Higher Infinite,
by A. Kanamori

König's Theorem

Theorem 31 (König)

If κ *is an infinite cardinal then* $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.

König's Theorem

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If κ is an infinite cardinal then $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.

Proof. Key: "No injection is surjective".

- Let $\{f_\alpha \mid \alpha < \kappa\}$ be an arbitrary subset of ${}^{\mathrm{cf}(\kappa)}\kappa$ of size κ.
- Construct an $f : \mathrm{cf}(\kappa) \to \kappa$ different from all f_α, $\alpha < \kappa$.
- Suppose $\kappa = \lim_{\xi < \mathrm{cf}(\kappa)} \alpha_\xi$. For each $\xi < \mathrm{cf}(\kappa)$, $f(\xi)$ is selected to ensure that at ξ, $f \neq f_\alpha$ for all $\alpha < \alpha_\xi$. □

König's Theorem

Theorem 31 (König)

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PROOF. Key: "No injection is surjective".

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Corollary 32 (AC)

If $\lambda \geq \omega$, then $\mathrm{cf}(2^\lambda) > \lambda$.

König's Theorem

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PROOF. Key: "No injection is surjective".

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Corollary 32 (AC)

If $\lambda \geq \omega$, then $\mathrm{cf}(2^\lambda) > \lambda$.

Hint: Otherwise, $2^\lambda < (2^\lambda)^{\mathrm{cf}(2^\lambda)} \leq (2^\lambda)^\lambda = 2^\lambda$.

Further results in cardinal arithmetics will appear in Chapter 5.

Homework I

1. Calculate the result of the cardinal operation $\aleph_0 + \aleph_2$.
   (a) $\aleph_0$
   (b) $\aleph_2$
   (c) $\aleph_3$
   (d) $2\aleph_2$
2. Calculate the cofinality of the ordinal $\alpha = \omega^4 + 7$.
   (a) ω
   (b) 1
   (c) 7
   (d) 4

Homework II

3. Which of the following cardinal numbers is **singular**?
   (a) $\aleph_1$
   (b) $\aleph_0$
   (c) $\aleph_\omega$
   (d) $\aleph_5$

4. Calculate the cofinality of the ordinal product $\alpha = \omega_1 \cdot \omega^2$.
   (a) ω^2
   (b) ω_1
   (c) 1
   (d) ω

Homework III

5. What is the value of the cardinal exponentiation $2^{\aleph_1}$ under the **Generalized Continuum Hypothesis (GCH)**?
   (a) $\aleph_1$
   (b) $\aleph_0$
   (c) $\aleph_2$
   (d) $\mathfrak{c}$

6. If κ is a **regular** infinite cardinal, what is the cofinality of the sum $\kappa + \kappa$ (cardinal sum)?
   (a) 2
   (b) $\aleph_0$
   (c) κ
   (d) κ^+

Homework IV

7. Calculate the cofinality of the ordinal $\beta = \omega^\omega + 1$.
   (a) ω
   (b) 1
   (c) ω^ω
   (d) 2

8. Let κ be a regular cardinal. What is the value of κ^λ when $\lambda < \kappa$?
   (a) κ
   (b) 2^λ
   (c) λ
   (d) $\kappa^{\mathrm{cf}(\kappa)}$

Homework V

9. What is the cofinality of the limit cardinal $\aleph_{\omega^2}$?
   (a) $\aleph_{\omega^2}$
   (b) $\aleph_0$
   (c) $\aleph_2$
   (d) ω^2

10. Which statement about the cofinality $\mathrm{cf}(\kappa)$ of an infinite cardinal κ is **always** true?
    (a) $\mathrm{cf}(\kappa) = \aleph_0$
    (b) $\mathrm{cf}(\kappa) = \kappa$
    (c) $\mathrm{cf}(\kappa)$ is a regular cardinal.
    (d) $\mathrm{cf}(\kappa) < \kappa$