Elementary Set Theory

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Axioms of Set Theory

Need to know:

- 9 axioms, expressed by the formulas of Set Theory.
- Applications of axioms: defining new sets, deriving contradictions, etc.
- *Concepts*: set/class, partition/equivalence relation

Ordinal Numbers

Need to know:

- *Concepts*: partial/linear/well ordering, order type, ordinal, successor/limit ordinal, addition/multiplication/exponentiation of ordinals, etc.
- *Techniques*: transfinite recursion (for definition), transfinite induction, argument with least element.

Cardinal Numbers

Need to know:

- *Concepts*: $|X| \leq |Y|$, $|X| = |Y|$, cardinal, cardinal addition/multiplication/exponentiation, cofinality
- *Techniques*:
  - Cantor's diagonalization argument
  - verify properties of cardinal arithmetic
  - do transfinite counting
- *Theorems*: Cantor-Bernstein, Theorem 3.8 (Cantor), 3.31 (König)

Real Numbers

Need to know:

- *Concepts*: open/closed/perfect subsets of $\mathbb{R}$ (and ω^ω)
- *Techniques*:
  - Tree representation of Baire space, $\mathcal{N}$.
- *Theorems*:
  - Theorem 4.3 (Cantor-Dedekind)[1]
  - Cantor-Bendixson,
  - Baire Category Theorem

[1] $\mathbb{R}$ is the unique complete dense unbounded separable linear order.

The Axiom of Choice

Need to know:

- *Concepts*:
  - Statements of AC, WO, ZL, MP, AC$_\omega$, DC;
- *Theorems*:
  - Implications among AC, WO, ZL, MP, AC$_\omega$, DC;
  - König Theorem ($\Sigma_i \kappa_i < \prod_i \lambda_i$) and its consequences:
    - $2^\kappa > \kappa$,
    - $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, (simple version of König)
    - $\mathrm{cf}(2^\kappa) > \kappa$.

Exercises

- There are arbitrarily large singular cardinals.
- There are arbitrarily large singular cardinals $\aleph_\alpha$ such that $\aleph_\alpha = \alpha$.
- About cofinality
  - $\mathrm{cf}(\alpha + \beta) = \mathrm{cf}(\beta)$.
  - $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$, α is a limit ordinal.
  - $\mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$.
- Cardinal exponentiations under GCH: for any $\kappa, \lambda \geq \omega$,
  $\kappa^\lambda = \kappa$, if $\lambda < \mathrm{cf}(\kappa)$;
  $\kappa^\lambda = \kappa^+$, if $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$; and
  $\kappa^\lambda = \lambda^+$, if $\kappa < \lambda$.

Cardinality

- If a linearly ordered set P has a countable dense subset, then $|P| \leq 2^{\aleph_0}$.
- The cardinality of the set of all null sets.
- The set of all 1-1 function from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

A set or a proper class?

- $\mathrm{Ord} =_{\mathsf{def}} \{\alpha \mid \alpha$ is an ordinal$\}$
- $\mathrm{Card} =_{\mathsf{def}} \{\alpha \mid \alpha$ is a cardinal$\}$
- $\{X \mid X$ is a wellordered set$\}$
- $\{X \subseteq \mathbb{R} \mid X$ is wellordered$\}$
- $\{P \mid P$ is a partially ordered set and $|P| < \aleph_\omega\}$
- the range of $\aleph$-function : $\aleph(i) =$ the i-th cardinal in ordinals.
- Assume $\kappa \in \mathrm{Ord}$ is a strongly inaccessible cardinal. The collection of wellordered sets in V_κ.

Yes, No, or ...?

Assume ZFC, determine the truth of the following statement, if you can.

- Every dense subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$
- $\mathrm{cf}(2^{\aleph_\omega}) = \aleph_\omega$
- Every real in $(0,1)$ is uniquely represented by a $\{0,1\}$-sequence of length ω
- Every real in $(0,1)$ is uniquely represented by a continuous fraction
- Every wellordering has no nontrivial automorphism.

Yes, No, or ...?

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F, F, F, T, T

Yes, No, or ...?

Cont'd

- Every linear order L with the following property is a well order: if $f : L \to L$ is ordering preserving, then $f(x) \geq x$ for every $x \in L$.
- There are more than $\aleph_1$ many reals.
- There is no Suslin line, i.e. a linear ordering that is dense, unbounded, complete and has the countable chain condition but is not nonseparable.
- There is a unique complete ordered field.
- For any set A, $\bigcap A \subset A$.
- For any set A, $\bigcup A \supset A$.

Yes, No, or ...?

Cont'd

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F, I (for Indepedent), I, T, F, T

Counting

Compute the cardinalities of the following sets.

- $\{F \mid F = (A, +, \cdot, 0, 1, <)$ is a complete ordered field$\}$
- The collection of comeager subsets of the Baire space $\mathcal{N}$
- The collection of all Lebesgue measure zero sets of reals.
- The collection of all Borel sets that are of Lebesgue measure zero.
- The collection of all meager sets of reals
  [Hint: The Cantor set is nowhere dense.]

Cofinality

- $\mathrm{cf}(\aleph_\omega)$
- $\mathrm{cf}(\aleph_{\omega+\omega^2+3})$
- Given continuous increasing ordinal function f, for $A \subset$ Ord with no maximal element,

$$\mathrm{cf}(f(\sup A)) = \mathrm{cf}(A).$$

- For instance, the cofinality of $(\omega_1)^{\omega^\omega}$ (as ordinal exponentiation) is ω.

$$(\omega_1)^{\omega^\omega} = \sup_{n<\omega} (\omega_1)^{\omega^n}$$

- But as cardinal exponentiation,

$$\mathrm{cf}((\aleph_1)^{\aleph_0^{\aleph_0}}) > \aleph_0^{\aleph_0}.$$

Miscellaneous

- Every comeager set of reals is dense.
- Every comeager set of reals contains a perfect subset.
- The Cantor set $\mathbb{C}$ is nowhere dense and null.
  - $\mathbb{C}$ is closed. $\mathbb{C}$ contains no intervals, so its interior is empty.
  - Given $\varepsilon > 0$, let $n < \omega$ be such that $(\frac{2}{3})^n < \varepsilon$, then the collection $\bigcup\{O_s \mid s \in {}^{<\omega}\omega \wedge |s| = n\}$ is an open set containing $\mathbb{C}$.