北京师范大学 2023 ～ 2024 学年第一学期期末考试试卷（A 卷）

课程名称：集合论基础　　　　任课老师姓名：施翔晖

卷面总分：100 分　　　　考试时长：120 分钟　　　　考试类别：闭卷 ☒　开卷 ☐　其他 ☐

院（系）：__________　　专业：__________　　年级：__________

姓名：__________　　学号：__________

题号	一	二	三	四	五	六	总分
得分							

阅卷老师（签字）：__________

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1. [8 + 8pts] Let $\mathcal{M}$ denote the set of all meager sets of reals. For $x, y \subset \mathcal{N}$, define $x \subseteq^* y$ if $x \setminus y \in \mathcal{M}$.
   (a) Write $x \equiv^* y$ when $x \subseteq^* y$ and $y \subseteq^* x$. Show that $\equiv^*$ is an equivalence relation.

   解答. Omitted. ☐

   (b) Let X be the quotient set $\{[x]_{\equiv^*} \mid x \subset \mathcal{N}\}$ and $[x]_{\equiv^*} \prec [y]_{\equiv^*}$ be the induced relation on the equivalence classes. Show that $\prec$ is well defined and is a partial order on X.

   解答. omitted. ☐

2. [10pts] Show that if $(A, \prec)$ is a well-order, then $\mathscr{P}(A)$ can be linearly ordered.

   解答. For $X \neq Y$ in $\mathscr{P}(A)$, define $X \prec Y$ iff $\min(X \mathbin{\Delta} Y) \in X$. Clearly $\prec$ is irreflective and trichotomous. Suppose that $X \prec Y \prec Z$. Say $x_0 = \min(X \Delta Y) \in X \setminus Y$, and $y_0 = \min(Y \Delta Z) \in Y \setminus Z$. If $x_0 \in X \cap Z$, then $(Y \mathbin{\Delta} Z) \ni x_0 \prec y_0 = \min(Y \mathbin{\Delta} Z)$, contradiction! Thus $x_0 \in X \setminus Z$ and $x_0 = \min((X \mathbin{\Delta} Y) \cup (Y \mathbin{\Delta} Z)) \leqslant \min(X \mathbin{\Delta} Z) \leqslant x_0$. And so $x_0 = \min(X \mathbin{\Delta} Z) \in X$ and $X \prec Z$. ☐

3. [5 × 7pts] Compute the cardinality of the following sets. No justication is needed. Answers are a finite cardinal number, or in the forms of $\aleph_*$, $2^{\aleph_*}$ or even $2^{2^{\aleph_*}}$.
   (a) The set $\{m/2^n \mid m \in \mathbb{Z} \wedge n \in \mathbb{N}\}$.

   解答. $\aleph_0$. ☐

   (b) The set of all closed sets of reals that contain no perfect subset.

   解答. $2^{\aleph_0}$. ☐

   (c) The set of all countable subset of $\aleph_\omega$, assuming GCH.

解答. $\aleph_{\omega+1}$. □

(d) The ordinal exponential $\varepsilon_0^{\varepsilon_0}$, where ε_0 is the least $\alpha \in \mathsf{Ord}$ such that $\omega^\alpha = \alpha$.

解答. $\aleph_0$. □

(e) The set of all sets of reals that are Lebesgue measurable.

解答. $2^{2^{\aleph_0}}$. □

(f) $V_{\omega+7}$, assuming GCH.

解答. $\aleph_7$. □

(g) $[\mathcal{P}(V_{\omega+5} \setminus V_{\omega+2})]^{V_{\omega_2}}$, assuming GCH.

解答. $\aleph_{\omega_2+1}$. □

4. [5 × 3pts] Compute the cofinalities of the following ordinals. The additions, multiplications and exponentiations below are ordinal operations. No justification is needed.

(a) $\aleph_{\omega^2+\omega+1}$.

解答. $\aleph_{\omega^2+\omega+1}$. □

(b) $\aleph_{\omega^\omega+1} + (\aleph_1)^{\omega\cdot\omega}$.

解答. ω. □

(c) η, the least $\eta \in \mathsf{Ord}$ such that $\omega^\eta = \eta$.

解答. ω. □

5. [10pts] Show that $\prod_{n<\omega} \aleph_{\omega\cdot n^2+1} = (\aleph_{\omega^2})^{\aleph_0}$.

解答. $\aleph_\omega^{\aleph_0} = \prod_{n<\omega} \aleph_n \leq \prod_{n<\omega} \aleph_{\omega\cdot n^2+1} \leq (\sup_n \aleph_{\omega\cdot n^2+1})^{\aleph_0} = (\aleph_{\omega^2})^{\aleph_0}$. □

6. [6 × 4pts] Let $\mathcal{N} = {}^\omega\omega$ denote the Baire space.

(a) Please write down

(i) the characterization of a closed subset of $\mathcal{N}$ in terms of subtree of ${}^{<\omega}\omega$.

(ii) the definition of perfect subtree of ${}^{<\omega}\omega$.

(b) Write down the characterization of a perfect subset of $\mathcal{N}$ in terms of perfect subtree of ${}^{<\omega}\omega$ and prove it.

(c) Show that if $P \subset \mathcal{N}$ is perfect and $P \cap O_s \neq \varnothing$ for some $s \in {}^{<\omega}\omega$, then $P \cap O_s$ is perfect. Here $O_s = \{f \in \mathcal{N} \mid s \sqsubset f\}$.

(d) Suppose that $P_1, P_2 \subset \mathcal{N}$ are perfect. Show that if $P_1 \setminus P_2 \neq \varnothing$, then $P_1 \setminus P_2$ contains a perfect subset.

解答. (a),(b) are omitted. We prove (c) and (d).

(c) Let T_P be the tree associated to P. Suppose $f \in P \cap O_s$. Then $s \sqsubset f$, so $s \in T_P$. Let $T_P[s] = \{t \in T_P \mid s \sqsubset t\}$. $T_P[s]$ is perfect and $[T_P[s]] = P \cap O_s$. So $P \cap O_s$ is perfect.

(d) Suppose $f \in P_1 \setminus P_2$. As $f \in P_2$, let $n \in \omega$ be least such that $s = f{\upharpoonright}n \notin T_{P_2}$, then $P_2 \cap O_s = \varnothing$. But $f \in P_1 \cap O_s$, so $P_1 \cap O_s \neq \varnothing$. Use (c), we have $P_1 \cap O_s$ is perfect and is contained in $P_1 \setminus P_2$. □