# Elementary Set Theory

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# Course Information

## Webpage

- http://math0.bnu.edu.cn/∼shi/teaching/fall2024/ST/

## Textbook

- Thomas Jech. *Set Theory: The Third Millennium Edition, Revised and Expanded, Part I*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

## Suggested Readings


[1] Judith Roitman. *Introduction to Modern Set Theory*. John Wiley & Sons, 1990.

[2] Karel Hrbacek, Thomas Jech. *Introduction to set theory*. M. Dekker, New York, 1978.

[3] Wikipedia (en). http://en.wikipedia.org

## Advanced Readings


[1] Kenneth Kunen, *Set theory, An introduction to independence proofs*. Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Company, 1980.

[2] Kenneth Kunen. Set theory, volume 34 of Studies in Logic (London). College Publications, London, 2011.

[3] Akihiro Kanamori, The higher infinite. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[4] Ralf Schindler. Set theory. Universitext. Springer, Cham, 2014. Exploring independence and truth.

# Coming up next

# Introduction

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- Axiom system for set theory: formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set-theoretic principles.
- From such axioms, all mathematics may be derived.
- Unfortunately, usual axioms can not settle all problems.
  (Continuum Hypothesis)

# History

## Before axiomatization

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- Modern set theory: infinity.
  Antient Greeks: the idea of infinity.
  Antient China: 不积跬步，无以至千里，不积细流，无以成江海。[1]

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- Cantor's work between 1874 and 1884 mark the birth of modern set theory.
- 1895, 1899 and 1902: three paradoxes,[2] especially **Russell paradox**, were discovered and shaked the foundations of mathematics.
  – the 3rd crisis in foundations of mathematics

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- Zermelo (1908) was the first to attempt an axiomatization of set theory. Then followed by Fraenkel, von Neumann, Bernays and Gödel etc. (ZF, ZFC)
- (late 1930s) limitations of naïve axiomatic approaches. (Gödel's 1st & 2nd Incompleteness)

- Gödel (1940) showed that the Axiom of Choice (AC) and the Continuum of Hypothesis (CH) cannot be disproved from ZF. (Constructible universe)

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- W. Hugh Woodin (late 1990s) proposed a stronger form of logic, $\Omega$-logic, for set theory transcending forcing, and recently, an ultimate universe for large cardinal and inner model program, meanwhile, settling CH.

# Areas of study[3]

- Combinatorial set theory
- Descriptive set theory
- Fuzzy set theory
- Inner model theory
- Large cardinals
- Determinacy
- Forcing
- Cardinal invariants
- Set-theoretic topology

[3] See https://en.wikipedia.org/wiki/Set_theory#Areas_of_study

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- In **axiomatic set theory**, the concepts of sets and set membership are defined indirectly by first postulating certain axioms which specify their properties. In this conception, sets and set membership are primitive concepts like point and line in Euclidean geometry, and are not themselves directly defined.

# Paradoxes in Naïve Set Theory

## Need for axiomatic approach

In Naive Set Theory, one's first intuition might be that we can form any sets we want, but this view leads to inconsistencies. Define

$$Z = \{x \mid x \notin x\}.$$

Then

$$Z \in Z \;\; \Leftrightarrow \;\; Z \notin Z.$$

This is the famous **Russell's paradox**.

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- **Zermelo**$^{1908}$-**Fraenkel**$^{1922}$ set theory is the most commonly used system of set-theoretic axioms, based on Zermelo set theory and further developed by Abraham Fraenkel and Thoralf Skolem.

- **Von Neumann**$^{1920s}$-**Bernays**$^{1937}$-**Gödel**$^{1940}$ set theory is an axiom system for set theory designed to yield the same results as Zermelo-Fraenkel set theory, together with the axiom of choice (AC), but with only a finite number of axioms, that is without axiom schemata.

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- **New Foundations** (W. Quine, 1937) and **positive** set theory are among the alternative set theories which have been proposed.

# People in Set Theory, I

Georg Cantor

1845 - 1918

Ernst Zermelo

1871 - 1953

Kurt Gödel

1906 - 1978

# People in Set Theory, II

Adolf Fraenkel

1891 - 1965

Thoralf Skolem

1887 - 1963

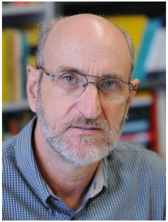
Bertrand Russell

1872 - 1970

# People in Set Theory, III


Saharon Shelah

1945 -



Alexander S. Kechris

1946 -

W. Hugh Woodin

1955 -

Stevo Todorcěvić

1955 -

# Assignment

- Read https://en.wikipedia.org/wiki/Set_theory.
- Follow links of your interest in that article.