Solutions for Assignment # 4

October 29, 2024

1. Write an explicit formula for this bijection.

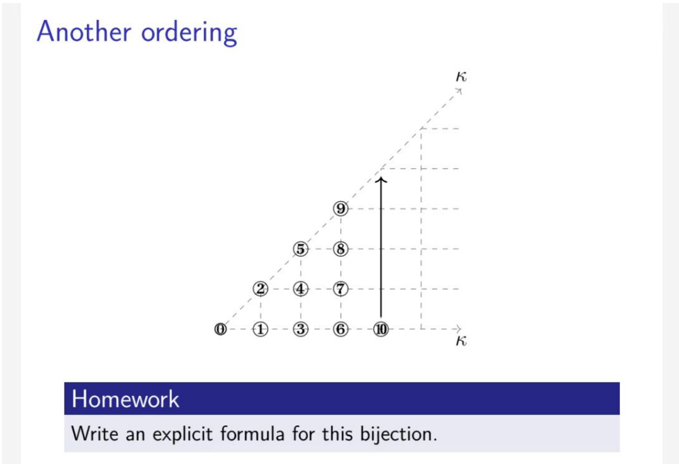

Figure 1: Another ordering

SOLUTION:

The area in the picture is $\kappa \times \kappa$ affine transformation $A = \{(\alpha, \beta) \in \kappa \times \kappa \mid \beta \leq \alpha\}$. And there is a nature bijection

$$\begin{aligned} f : \kappa \times \kappa &\to A \\ (\alpha, \beta) &\mapsto (\alpha + \beta, \beta) \end{aligned}$$

The picture give us a well-ordering of A:

$$\begin{aligned} g : A &\to \kappa \\ (\alpha, \beta) &\mapsto \sum_{\gamma < \alpha} \gamma + \alpha + \beta \end{aligned}$$

Where $\sum_{\gamma<\alpha} \gamma$ is the sum of all ordinals $< \alpha$, which can be defined recursively:
For all ordinal number α,

(a) $\sum_{\gamma<0} \gamma = 0$.
(b) $\sum_{\gamma<\alpha+1} \gamma = \sum_{\gamma<\alpha} \gamma + \alpha$.
(c) $\sum_{\gamma<\alpha} \gamma = \lim_{\xi\to\alpha} \sum_{\gamma<\xi} \gamma$, for limit $\alpha > 0$.

And one can easily verify that g is a bijection. So the bijection from $\kappa \times \kappa \to \kappa$ should be

$$\begin{aligned} g \circ f : \kappa \times \kappa &\to \kappa \\ (\alpha, \beta) &\mapsto \sum_{\gamma < \alpha+\beta} \gamma + \alpha + \beta + \beta \end{aligned}$$