Solutions for Assignment # 2

October 15, 2024

1. Show that the function f given in the proof of Theorem 11 is an isomorphism.

   <u>Solution</u>:

$$f = \{(x, y) \mid x \in U \wedge y \in V \wedge (U_x, (<_U)_x) \cong (V_y, (<_V)_y)\}$$

   It suffices to show that $x_1 <_U x_2 \Leftrightarrow f(x_1) <_V f(x_2)$, since this implies that f is an injective function as well as order-preserving. We only need to consider the case $x_1 <_U x_2$ and $x_1 = x_2$.

   Suppose $x_1 = x_2$. Then $U_{x_1} \cong V_{f(x_1)} \cong V_{f(x_2)}$, since there are no two distinct initial segments of $(V, <_V)$ are isomorphic, it must be that $f(x_1) = f(x_2)$. This implies that f is a (well-defined) function between its domain and range.

   Suppose $x_1 <_U x_2$. Since U_{x_1} is an initial segment of U_{x_2}, $V_{f(x_1)}$ is isomorphic to an initial segment of $V_{f(x_2)}$, say V_y, $y < f(x_2)$. Since no well-ordering is isomorphic to its proper initial segments, it must be that $f(x_1) = y < f(x_2)$.

2. The relation "$(P, <) \cong (Q, <)$" is an equivalence relation (on the class of all partially ordered sets).

   <u>Solution</u>:

   (a) (reflexive) For any partially ordered sets P, id is the natural automorphism.

   (b) (symmetric) Suppose $f : P \to Q$ is an isomorphism, then so does $f^{-1} : Q \to P$, since

$$y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$$

   (c) (transitive) For any partially ordered sets P, Q, R, Suppose $f : P \to Q$ and $g : Q \to R$ are isomorphisms, then $g \circ f$ is an isomorphism between P and R.

3. Let $\mathcal{A}$ denote the class of all well orderings. For any $a, b \in \mathcal{A}$, $a \prec b$ iff a is isomorphic to an initial segment of b. Show that $\prec$ is a well ordering on $\mathcal{A}/_{\cong}$, where $\cong$ is the equivalence relation given in Ex.2.

   <u>Solution</u>:

   It is obvious that $a \prec b \Leftrightarrow [a] \prec [b]$, in which $[a]$ denotes the equivalence class containing a.

   (a) (irreflexive) $a \not\prec a$ since any a is isomorphic to itself, therefore, can't be isomorphic to its own initial segment.

   (b) (transitive) For any $a, b, c \in \mathcal{A}$, suppose $a \prec b$ and $b \prec c$. It follows that a is isomorphic to an initial segment of an initial segment of c, which is still an initial segment of c. Thus $a \prec c$.

   (c) (trichotomous) For any $a, b \in \mathcal{A}$, by theorem 2. 3, we have $a \prec b$, $a = b$ (actually, $a \cong b$) or $a \succ b$

   (d) (well-ordering) Suppose NOT, there is a nonempty subclass $P \subseteq \mathcal{A}$, such that P has no least element, i.e. there exists a infinite sequence $\{U^i\} \subseteq P$

$$U^0 \succ U^1 \succ U^2 \succ \ldots$$

   By definition, for every $i \in \mathbb{N}^*$, there exist a $x_i \in U^0$ such that $U^i \cong U^0_{x_i}$. It is obvious that $\{x_i\}$ is an infinite decreasing sequence of U^0 (Contradict to the fact that U^0 is well-ordering).

4. Prove Proposition 6.

(a) If $(W, <)$ is a well ordering and $U \subseteq W$, then $(U, < \cap (U \times U))$ is a well ordering.

(b) If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, $W_1 \cap W_2 = \emptyset$, then $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$ is a well ordering, where

$$\prec = <_1 \cup <_2 \cup \{(a, b) \mid a \in W_1 \wedge b \in W_2\}$$

(c) If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, then $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$ is a well ordering, where

$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \vee (b_1 = b_2 \wedge a_1 <_1 a_2)$$

Solution:

(a) Let $<_1$ denote $< \cap (U \times U)$. Notice the fact that for any $p, q \in U$, $p <_1 q \Leftrightarrow p < q$.
- (irreflexive) For any $p \in U$, $p \not<_1 p$ because $p \not< p$.
- (transitive) For any $p, q, r \in U$, $p <_1 q \wedge q <_1 r \Rightarrow p < q \wedge q < r \Rightarrow p < r \Rightarrow p <_1 r$
- (trichotomous) For any $p, q \in U$, $p < q \vee p = q \vee q < p \Rightarrow p <_1 q \vee p = q \vee q <_1 p$
- (well-ordered) For any nonempty subset $P \subseteq U$, it is also a nonempty subset of W. Since W is a well-ordering, P has a least element p. For all element $x \in P$, $p \leq x$, which implies that $p \leq_1 x$. So p is the least element in $(P, <_1)$.

(b)
- (irreflexive) For any $a \in W_1 \cup W_2$, $a \in W_1$ or $a \in W_2$. In either case, $a \not\prec a$.
- (transitive) Suppose $a, b, c \in W_1 \cup W_2$ are such that $a \prec b \prec c$. We show that $a \prec c$. Two cases:

CASE 1: $a, c \in W_i$, $i = 1$ or 2. Then $a \prec c$ follows from $a <_i b <_i c$ and the transitivity of $<_i$.

CASE 2: $a \in W_1$, $c \in W_2$. Then $a \prec c$ follows from the definition of $\prec$.
- (trichotomous) For any $a, b \in W_1 \cup W_2$, if $a \in W_1$, $b \in W_2$ or $a \in W_2$, $b \in W_1$, then a, b are comparable according to the definition of $\prec$; otherwise if $a, b \in W_i$ ($i = 1$ or 2), then a, b are comparable by the trichotomy of $<_i$.
- (well-ordered) Let P be a nonempty subset of $W_1 \cup W_2$. If $P \cap W_1 \neq \varnothing$, then $<_1$-min of $P \cap W_1$ gives the $\prec$-min element of P. Otherwise $P \subset W_2$, hence the $\prec$-min element of P is in fact its $<_2$-min.

(c) It is trivial to prove that $W_1 \otimes W_2$ is a linear order. So it suffices to show that for any nonempty subset $P \subseteq W_1 \times W_2$, P has a least element. Let

$$U = \{a \in W_1 \mid (a, b) \in P\}$$

U is a nonempty subset of W_1, therefore has a least element a_0. Let

$$V = \{b \in W_1 \mid (a_0, b) \in P\}$$

V is a nonempty subset of W_2, therefore has a least element b_0. Then (a_0, b_0) is a least element of P.