Solutions for Assignment # 1

October 15, 2024

1. Using only $\hat{\in}$ and $\hat{=}$ to express the following formulas:

   (a) $z \hat{=} ((x,y),(u,v))$

   (b) $\forall x[\neg(x \hat{=} \varnothing) \to (\exists y \hat{\in} x)(x \cap y \hat{=} \varnothing)]$

   (c) $\forall u[\forall x \exists y (x,y) \hat{\in} u \to \exists f \forall x (x, f(x)) \hat{\in} u]$

   <u>Solution</u>:

   (a) Note that $(x,y) = \{\{x\},\{x,y\}\}$. The formula $z \hat{=} (x,y)$ can be expressed as:

   $$\varphi(z,x,y) \equiv \forall u(u \hat{\in} z \leftrightarrow \forall v(v \hat{\in} u \leftrightarrow v \hat{=} x) \vee \forall v(v \hat{\in} u \leftrightarrow v \hat{=} x \vee v \hat{=} y))$$

   So $z \hat{=} ((x,y),(u,v))$ can be expressed as:

   $$\exists z_1 \exists z_2 (\varphi(z_1,x,y) \wedge \varphi(z_2,u,v) \wedge \varphi(z,z_1,z_2))$$

   (b) Note that $x \cap y \hat{=} \varnothing$ iff $\neg\exists z(z \hat{\in} y \wedge z \hat{\in} x)$, $\exists y \hat{\in} x$ means $\exists y(y \hat{\in} x)$. This formula can be expressed as:

   $$\forall x\,(\exists v\,(v \in x) \to \exists y\,(y \hat{\in} x \wedge \neg\exists u\,(u \in x \wedge u \in y))$$

   (c) Note that function f is a binary relation and $(\forall(x,y),(x,z) \hat{\in} f)(y \hat{=} z)$. "$f$ is a binary relation" can be expressed as:

   $$\varphi_1(f) \equiv \forall z(z \hat{\in} f \leftrightarrow \exists x \exists y \varphi(z,x,y))$$

   $\varphi(z,x,y)$ means $z \hat{=} (x,y)$ defined in (a).
   "f is a function" can be expressed as:

   $$\varphi_2(f) \equiv \varphi_1(f) \wedge \forall x \forall y \forall z \exists u \exists v(\varphi(x,y,u) \wedge \varphi(x,z,v) \wedge (u \hat{\in} f \wedge v \hat{\in} f \to y \hat{=} z))$$

   So the formula $\forall u[\forall x \exists y (x,y) \hat{\in} u \to \exists f \forall x (x,f(x)) \hat{\in} u]$ can be expressed as:

   $$\forall u(\forall x \exists y \exists z(\varphi(z,x,y) \wedge z \hat{\in} u) \to \exists f(\varphi_2(f) \wedge \forall x \exists y \exists z(\varphi(z,x,y) \wedge z \hat{\in} f \wedge z \hat{\in} u)))$$

2. Suppose that R, S are two relations. Show that R_{-1} and $S \circ R$ exist.

   <u>Solution</u>: Since $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are two sets, so are $\mathrm{ran}(R) \times \mathrm{dom}(R)$. By Comprehension Schema,

   $$R_{-1} = \{(u,v) \in \mathrm{ran}(R) \times \mathrm{dom}(R) \mid (v,u) \in R\}$$

   exists. $R_{-1} \subset \mathscr{P}(\mathscr{P}(\bigcup\bigcup R))$ with Comprehension also shows R_{-1} is a set.

   Since $\mathrm{dom}(R)$ and $\mathrm{ran}(S)$ are two sets, so does $\mathrm{dom}(R) \times \mathrm{ran}(S)$. By Comprehension Schema,

   $$S \circ R = \{(u,v) \in \mathrm{dom}(R) \times \mathrm{ran}(S) \mid \exists w((u,w) \in R \wedge (w,v) \in S)\}$$

   exists. $S \circ R \subset \mathscr{P}(\mathscr{P}(\bigcup\bigcup(R \cup S)))$ with Comprehension also shows $S \circ R$ is a set.

3. There is no set X such that $\mathscr{P}(X) \subseteq X$.

   <u>Solution</u>: Suppose NOT. There exists a set X s.t. $\mathscr{P}(X) \subseteq X$.

Method I We have $X \in X \in X \cdots$, since X is a subset of itself $X \in \mathscr{P}(X) \subseteq X$. But it contradicts Regularity/Well-foundedness axioms.

Method II Let $W = \{x \in X \mid x \notin x\}$. $W \subset X$, thus $W \in \mathscr{P}(X) \subseteq X$. But $W \in W \leftrightarrow W \notin W$. Contradiction!

Let $N = \bigcap\{X \mid X \text{ is inductive}\}$. N is the smallest inductive set. Let us use the following notation:

$$0 = \varnothing, \quad 1 = \{0\}, \quad 2 = \{0,1\}, \quad 3 = \{0,1,2\}$$

If $n \in N$, let $n+1 = n \cup \{n\}$. And for $n, m \in N$,

$$n < m \leftrightarrow n \in m$$

A set T is *transitive* if $x \in T$ implies $x \subseteq T$.

4. If X is inductive, then the set

$$\{x \in X \mid x \subseteq X\}$$

is inductive. Hence N is transitive, and for each n, $n = \{m \in N \mid m < n\}$.

<u>Solution</u>: Let $E = \{x \in X \mid x \subseteq X\}$.

(a) It is clear that $\varnothing \in X$(X is inductive) and $\varnothing \subseteq X$ (trivial). So $\varnothing$ belongs to E.

(b) For all $x \in E$, $x \cup \{x\} \in X$ because x is an element of X and X is inductive. Since both x and $\{x\}$ are subsets of X, we have $x \cup \{x\} \subseteq X$. Hence $x \cup \{x\} \in E$.

According to (a) and (b), $\{x \in X \mid x \subseteq X\}$ is inductive.

Let $E_0 = \{x \in N \mid x \subseteq N\} \subseteq N$. But E_0 is inductive, so N is a subset of E_0. That means $\{x \in N \mid x \subseteq N\} = N$, thus N is transitive.

It is obvious that $\{m \in N \mid m < n\} = \{m \in N \mid m \in n\} \subseteq n$. On the other hand, since $n \in N$ and N is transitive, we have $n \subseteq N$. Then $m \in n \to m \in N$ which equals $n \subseteq \{m \in N \mid m < n\}$. Hence $n = \{m \in N \mid m < n\}$.

5. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive}\}$$

is inductive. Hence every $n \in N$ is transitive.

<u>Solution</u>: Let $E = \{x \in X \mid x \text{ is transitive}\}$.

(a) $\varnothing \in E$ since $\varnothing$ is transitive.

(b) For all $x \in X$, x is transitive. Our goal is to show that $x \cup \{x\}$ is transitive, too. For all $y \in x \cup \{x\}$, no matter wether $y \in x$ or $y \in \{x\}$, we have $y \subseteq x \cup \{x\}$. Thus $x \cup \{x\}$ is transitive.

According to (a) and (b), $\{x \in X \mid x \text{ is transitive}\}$ is inductive.

Since N is the smallest inductive set, $N \subseteq \{x \in N \mid x \text{ is transitive}\}$. Thus every element n of N is transitive.

6. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive and } x \notin x\}$$

is inductive. Hence $n \notin n$ and $n \neq n+1$ for each $n \in N$.

<u>Solution</u>: According to the conclusion above, it is sufficient to prove that $x \cup \{x\} \notin x \cup \{x\}$ if x is transitive and $x \notin x$. Suppose NOT, we have $x \cup \{x\} \in x$ or $x \cup \{x\} \in \{x\}$. Both of them lead to $x \cup \{x\} \subseteq x$ (x is transitive). But

$$x \cup \{x\} \subseteq x \to \{x\} \subseteq x \to x \in x$$

It is a contradiction.

Since N is the smallest inductive set, $N \subseteq \{x \in X \mid x \text{ is transitive and } x \notin x\}$. Thus $n \notin n$. $n+1 = n \cup \{n\}$ by definition. Since there is n s.t. $n \notin n$ but $n \in n+1$, we have $n \neq n+1$.

7. If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\!-\text{minimal element}\}$ is inductive. (t is $\in$-*minimal* in z if there is no $s \in z$ such that $s \in t$.)

   SOLUTION: It is sufficient to show that every nonempty $z \subseteq x \cup \{x\}$ has an $\in$-minimal element if x belongs to the above set. For any nonempty $z \subseteq x \cup \{x\}$, Suppose $z = \{x\}$, x is the $\in$-minimal element in z (Otherwise $x \in x$, but then $\{x\}$, as a nonempty subset of x, has no $\in$-minimal element). Otherwise, $z \subseteq x$, there would exist a $y \in z \backslash \{x\} \subseteq x$ is a $\in$-minimal element in $z \backslash \{x\}$. Meanwhile, $x \notin y$ (Otherwise $x \in y \in x \to x \in x$, since x is transitive). That implies y is an $\in$-minimal element in z.

8. Every nonempty $X \subseteq N$ has an $\in$-minimal element.

   SOLUTION: Since N is the smallest inductive set, $N \subseteq \{x \in N \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in \text{-minimal element}\}$. For all $X \subseteq N$, pick $n \in X$. If $X \cap n = \varnothing$, $(\forall m < n)(m \notin X)$. So n is an $\in$-minimal element. If $X \cap n \neq \varnothing$, $X \cap n \subseteq n$ has an $\in$-minimal element. It is an $\in$-minimal element in X.

9. If X is inductive then so is $\{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq \varnothing$ is $m + 1$ for some m.

   SOLUTION: Let $E = \{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$. Suppose an nonempty set $x \in E$. Then $x \cup \{x\} = y \cup \{y\}$ for $y = x$. Thus $x \cup \{x\} \in E$. So the above set is inductive.

   Since $N \subseteq \{x \in N \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}$, each $n \neq 0$ is $m + 1$ for some m.

10. (Induction) Let A be a subset of N such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = N$.

    SOLUTION: By definition, A is inductive. So N is a subset of A. But $A \subseteq N$ naturally. Hence $A = N$.