Solution for Assignment #4.4

November 21, 2024

Write $A - B$ for $A \setminus B$ when $A \supseteq B$.

Exercise 1. *Suppose that $A \subseteq \omega^\omega$ has the property of Baire. Show that A is nonmeager iff there is a non-empty open set $O \subseteq \omega^\omega$ such that $O \setminus A$ is meager.*

Solution. By the definition of Bair property, there is an open set O such that $O \triangle A$ is meager. If A is nonmeager, then $\varnothing \triangle A = A$ is nonmeager, so O must be non-empty.

On the other hand, if there is a non-empty open set $O \subseteq \omega^\omega$ such that $O \setminus A$ is meager. By Baire Category Theorem, every non-empty open set in $\mathcal{N}$ is nonmeager, so $O = O \triangle A \cup A$ is nonmeager, thus A is nonmeager. $\square$

Exercise 2. *Show that for any $A \subseteq \omega^\omega$, $C_A - A$ contains no nonmeager set, where C_A is defined as follow.*

$$C_A = \{\, O_s \mid O_s - A \text{ is meager} \,\}$$

Solution. Note that $C_A - A = \bigcup_{O_s \in C_A}(O_s - A)$ is a countable union of meager sets (since there are only countably many s's), then is a countable union of nowhere dense sets, is also meager. Also note that any subset of a meager set is meager. So $C_A - A$ contains no nonmeager set. $\square$

Exercise 3. *Show that $2^\omega - \mathrm{ran}(\pi)$ is countable, where π is the embedding defined as follow.*

$$\pi(x) = s_{x(0)} \frown s_{x(1)} \frown s_{x(2)} \frown \cdots$$

where $s_{x(k)}$ is the 01-sequence with $x(k)$ many 1's in the front and one 0 in the back for even k, and $x(k)$ many 0's in the front and one 1 in the back.

Solution. Note that $2^\omega - \mathrm{ran}(\pi)$ only consists of sequence with only finitely many 0's or finitely many 1's, which is clearly countable. $\square$

Exercise 4. *Assume AD. Then $\mathsf{AC}_\omega(\omega^\omega)$, i.e. every countable set consisting of non-empty sets of reals has a choice function. Consequently, ω_1 is regular.*

Solution. Suppose that $X_0, X_1, \cdots$ are non-empty subsets of ω^ω. Define operations on ω^ω as follows.

$$\begin{aligned} a \oplus b &= \langle a(0), b(0), a(1), b(1), \cdots \rangle \\ \pi_0(a) &= \langle a(0), a(2), a(4), \cdots \rangle \\ \pi_1(a) &= \langle a(1), a(3), a(5), \cdots \rangle \end{aligned}$$

Define sets

$$Y_n = \{\, a \oplus b \mid a \in O_{\langle n \rangle} \wedge b \in X_n \,\}$$

and

$$Y = \bigcup_{n<\omega} Y_n = \{\, a \oplus b \mid b \in X_{a(0)} \,\}\,.$$

By AD, game $G(Y)$ has a winning strategy τ for someone. Clearly player II wins the game $G(Y)$, since once player I plays $a(0)$, player II just need to choose some $b \in X_{a(0)}$. Then $f(X_n) = \pi_1(\tau * \langle n, 0, 0, \cdots \rangle)$ is a choice function as required. □