Solution for Assignment #4.3

November 21, 2024

Write $A - B$ for $A \setminus B$ when $A \supseteq B$.

Exercise 1. *$F \subseteq \mathcal{N}$ is closed iff $F = [T]$ for some tree $T \subseteq \omega^{<\omega}$.*

Solution. $\Rightarrow$. Suppose that $F \subseteq \mathcal{N}$ is closed. Let tree $T \subseteq \omega^{<\omega}$ be the set $\{ f \restriction n \mid f \in F \wedge n \in \omega \}$. Clearly every $f \in F$ is a path through T, and hence $F \subseteq [T]$. On the other hand, since F is closed, if $g \notin F$, then there is an $s \in \omega^{<\omega}$ such that $s \sqsubset g$, but $s \not\sqsubset f$ for all $f \in F$. So such an s is not in T and g is not a path through T. Thus $F = [T]$.

$\Leftarrow$. Note that

$$\begin{aligned}[T] &= \{ f \mid \forall n\, (f \restriction n \in T) \} \\ &= \mathcal{N} - \{ f \mid \exists n\, (f \restriction n \notin T) \} \\ &= \mathcal{N} - \{ f \mid \exists s \notin T\, (s \sqsubset f) \} \\ &= \mathcal{N} - \left(\textstyle\bigcup_{s \notin T} O_s \right)\end{aligned}$$

is clearly closed. □

Exercise 2. *If f is an isolated point of a closed set $F \subseteq \mathcal{N}$, then there is $n \in \omega$ such that $\forall g \in F\, (f \neq g \to f \restriction n \neq g \restriction n)$.*

Solution. Suppose that f is an isolated point of a closed set $F \subseteq \mathcal{N}$, then there is $s \in \omega^{<\omega}$ such that $O_s \cap F = \{ f \}$. Then $n = |s|$ is as required. □

A tree $T \subseteq \omega^{<\omega}$ is perfect iff for every $t \in T$, there exists $s_1, s_2 \in T$ such that $t \sqsubset s_1$ and $t \sqsubset s_2$, but s_1, s_2 are incomparable.

Exercise 3. *A closed set $F \subseteq \mathcal{N}$ is perfect iff*

$$T_F = \{ f \restriction n \mid f \in F \wedge n < \omega \}$$

is a perfect tree.

Solution. $\Rightarrow$. Suppose that T_F is not a perfect tree, i.e., there is an $t = f \restriction n \in T_F$, such that if $s_1, s_2 \in T_F$ are extensions of t, then s_1, s_2 are comparable. Then such s_i can only be the initial segment of f, and $O_t \cap T_F = \{ f \}$. Hence f is an isolated point of F.

$\Leftarrow$. Suppose that F is not perfect, i.e., there is an isolated point $f \in F$. By previous exercise, there is $n \in \omega$ such that $\forall g \in F\, (f \neq g \to f \restriction n \neq g \restriction n)$. Then $t = f \restriction n \in T_F$ witnesses that T_F is not a perfect tree. □

Exercise 4. *Show that $\bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$ is the collection of all Borel sets.*

SOLUTION. We first show that every Σ^0_α set is Borel by induction. Cleary every set in Σ^0_1 is open and Borel. Suppose that every Σ^0_β set is Borel for every $\beta < \alpha$. Then by complement operation, every Π^0_β set is Borel. And then by countable union, every Σ^0_α set is Borel.

Now it suffices to show that $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$ is a σ-algebra. For every Σ^0_α set X, $X^c \in \Pi^0_\alpha \subseteq \Sigma^0_{\alpha+1}$. Suppose that $X_0, X_1, \cdots$ are sets in $\Sigma^0_{\alpha_0}, \Sigma^0_{\alpha_1}, \cdots$. Let $\alpha = \sup\{\, \alpha_i \mid i < \omega \,\}$. By AC, ω_1 is regular and so $\alpha < \omega$. Thus $\bigcup_{i<\omega} X_i \in \Pi^0_\alpha \subseteq \Sigma^0_{\alpha+1}$. Hence $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$ is closed under complement and countable intersection. □

Write F_σ set for Σ^0_2 set (countable union of closed sets), and G_δ set for Π^0_2 set (countable intersection of open sets).

EXERCISE 5. *Show that the collection of Lebesgue measurable sets (of reals) forms a σ-algebra.*

SOLUTION. Complement. Suppose that A is Lebesgue measurable. Then there are F_σ set F and G_δ set G such that $F \subseteq A \subseteq G$ and $\mu^*(G - F) = 0$. Note that $G^c \subseteq A^c \subseteq F^c$, $G^c \in F_\sigma$, $F^c \in G_\delta$, and $\mu^*(F^c - G^c) = \mu^*(F - G) = 0$. So A^c is also Lebesgue measurable.

Countable union. Suppose that for $i = 0, 1, \cdots$, $F_i \in F_\sigma$, $G_i \in G_\delta$, $F_i \subseteq A_i \subseteq G_i$, $\mu^*(F_i - G_i) = 0$. Fix i. Assume that $G_i = \bigcap_{j<\omega} X_{ij}$ where all X_{ij}'s are open. Without loss of generity, we may assume that $X_{i0} \supseteq X_{i1} \supseteq \cdots \supseteq G_i$. And by $\mu^*(G_i - F_i) = 0$, we may further assume that $\mu^*(X_{ij} - F_i) \leqslant 2^{-ij}$. Let $H_j = \bigcup_{i<\omega} X_{ij}$ be a Σ^0_1 set, and $F = \bigcup_{i<\omega} F_i$ be a F_σ set. Then $H_j \supseteq \bigcup_{i<\omega} G_i$ and

$$\mu^*(H_j - F) = \mu^*\left(\bigcup_{i<\omega}(X_{ij} - F_i)\right) \leqslant 2^{-j}.$$

Let $H = \bigcap_{j<\omega} H_j$ be a G_δ set. Then $F \subseteq \bigcup_{i<\omega} A_i \subseteq H$, $\mu^*(H - F) = \lim_{j<\omega} \mu^*(H_j - F) = 0$. □

EXERCISE 6. *Show that the collection of sets (of reals) having the property of Baire forms a σ-algebra.*

SOLUTION. Complement. Note that if G is open, then $\bar{G} - G$ is nowhere dense. So if $A \triangle G$ is meager, then $A^c \triangle \bar{G}^c \subseteq (A \triangle G) \cup (\bar{G} - G)$ is meager. Thus A has property of Baire implies A^c has property of Baire.

Countable union. Suppose that $A_i \triangle G_i = \bigcup_{j<\omega} B_{ij}$ is meager, G_i is open, and B_{ij} is nowhere dense. Then

$$\left(\bigcup_{i<\omega} A_i\right) \triangle \left(\bigcup_{i<\omega} G_i\right) \subseteq \bigcup_{i<\omega}(A_i \triangle G_i) = \bigcup_{i<\omega} \bigcup_{j<\omega} B_{ij}.$$

Note that subset of nowhere dense set is also nowhere dense, and subset of meager set is also meager. Then $(\bigcup_{i<\omega} A_i) \triangle (\bigcup_{i<\omega} G_i)$ is meager and $\bigcup_{i<\omega} A_i$ has the property of Baire. □