Solutions for Assignment # 4.2

November 21, 2024

1. Prove proposition 5. Let T be a tree.

   (a) If $s, t, u \in T$, then $R_{stu} = \{\delta_{st}, \delta_{tu}, \delta_{su}\}$ has ≤ 2 elements, and $p, q \in R_{stu} \to p \subset q \vee q \subset p$.

   (b) $\prec$ is a linear ordering of T which extends $\sqsubseteq$.

   (c) For every $t \in T$, $T^t = \{s \in T \mid t \sqsubseteq s\}$ is an interval in $(T, \prec)$.

   Solution:

   (a) It is to easy to prove that δ_{st} is an initial segment of $(\cdot, s)_T$. Fix $x \in \delta_{st}$, for any $x' < x$, $x' < s$ and $x' < t$ ($<$ is a partial order), i.e $x' \in \delta_{st}$.
   Without losing generality, suppose $n_{st} \leq n_{tu} \leq n_{su}$. Since δ_{st}, δ_{tu} are both initial segments of well-ordered set $(\cdot, t)_T$. So $\delta_{st} \subset \delta_{tu}$. Similar, we have $\delta_{st} \subset \delta_{tu} \subset \delta_{su}$. Then $\delta_{st} = \delta_{st} \cap \delta_{su} = (\cdot, s)_T \cap (\cdot, t)_T \cap (\cdot, u)_T = \delta_{tu} \cap \delta_{su} = \delta_{tu}$. Hence $R_{stu} = \{\delta_{tu}, \delta_{su}\}$ and $\delta_{tu} \subset \delta_{su}$.

   (b) i. (irreflective) Suppose $s \prec t$. if $s \sqsubseteq t$, obviously $t \not\prec s$. Otherwise, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge s(n_{st}) <_X t(n_{st})$, $t \not\prec s$ since X is linear ordered.

   ii. (transitive) Suppose $s \prec t \wedge t \prec u$. It is easier when $s \sqsubseteq t$ or $t \sqsubseteq u$. Now prove the other case. If $n_{st} < n_{tu}$, then $n_{su} = n_{st}$(see the proof of (a)), $s(n_{su}) = s(n_{st}) <_X t(n_{st}) = u(n_{st}) = u(n_{su})$. If $n_{tu} < n_{st}$, then $n_{su} = n_{tu}$, similar, $s(n_{su}) <_X u(n_{su})$. If $n_{tu} = n_{st}$, then $n_{tu} = n_{st} = n_{su}$, since X is linear ordered, $s(n_{su}) <_X u(n_{su})$

   iii. (trichotomous) Suppose $s \neq t$. There are exactly four cases $s \sqsubseteq t$, $t \sqsubseteq s$, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge s(n_{st}) <_X t(n_{st})$, $s \not\sqsubseteq t \wedge t \not\sqsubseteq s \wedge t(n_{st}) <_X s(n_{st})$

   (c) It suffices to prove that $s_1 \prec s \prec s_2$, $t \sqsubseteq s_1 \wedge t \sqsubseteq s_2$ implies $t \sqsubseteq s$. By definition, $t \sqsubseteq s \Rightarrow t \in (\cdot, s)_T$. So we have $t \in \delta_{s_1 s_2}$. Suppose $t \notin (\cdot, s)_T$. Then $\delta_{s_1 s} \neq \delta_{s_1 s_2} \wedge \delta_{s_2 s} \neq \delta_{s_1 s_2}$. By (a), $\delta_{s_1 s} = \delta_{s_2 s} \subsetneq \delta_{s_1 s_2}$. So $s_2(n_{s_2 s}) = s_1(n_{s_1 s}) < s(n_{s_1 s}) = s(n_{s_2 s})$ (contradiction!).

2. Prove Proposition 6. Let T, B_T be as above.

   (a) $\prec$ is a linear ordering of $T \cup B_T$.

   (b) For every $t \in T$, $B_t = \{f \in T \cup B_T \mid t \in f\}$ is an interval in $(T \cup B_T, \prec)$.

   Solution: Consider $(T \cup B_T, \sqsubseteq^*)$, where $\sqsubseteq^* = \sqsubseteq \cup \{(t, f) \mid t \in T \wedge f \in B_T \wedge t \in f\}$. If $t \in T$, $(\cdot, t)_{T \cup B_T} = (\cdot, t)_T$ is well-ordered. If $t \in B_T$, $(\cdot, t)_{T \cup B_T} = \{s \in T \mid s \in t\} = t$ is also well-ordered. From this we can say $(T \cup B_T, \sqsubseteq^*)$ is a tree.

   (a) By 2(b), we can define $\prec^*$ as a linear ordering of the tree $T \cup B_T$ which extends $\sqsubseteq^*$. Notice that for $f, g \in T \cup B_T, f \sqsubseteq g \Rightarrow f(n_{fg}) = \varnothing \leq_X g n_{fg}$. So $\prec = \prec^*$ is a linear ordering of $T \cup B_T$.

   (b) By 2(c), $(T \cup B_T)^t = \{s \in T \cup B_T \mid t \sqsubseteq^* s\}$ is an interval in $(T \cup B_T, \prec)$. So $B_t = (T \cup B_T)^t$ is an interval in $(T \cup B_T, \prec)$.

3. If X is a Suslin line, then X^2 is not c.c.c.

   Solution: We make the interval by recursion on α.

   First, pick $a_0 < b_0 < c_0$ and make the first interval $(a_0, b_0) \times (b_0, c_0)$.

For $\alpha < \omega_1$, having constructed $(a_\beta, b_\beta) \times (b_\beta, c_\beta)$ for any $\beta < \alpha$, since $\{b_\beta \mid \beta < \alpha\} \in X$ is countable, it can't be dense in X. So there is $a_\alpha < c_\alpha$ s.t $\{b_\beta \mid \beta < \alpha\} \cap (a_\alpha < c_\alpha) = \varnothing$. We can choose $b_\alpha \in (a_\alpha, c_\alpha)$ since X is dense. It is obvious that $(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \cap (a_\beta, b_\beta) \times (b_\beta, c_\beta) = \varnothing$ for any $\beta < \alpha$.

Thus, we get $M = \{(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \mid \alpha < \omega_1\}$ is a pairwise-disjoint collection of open intervals of X^2, but $|M| = \omega_1 > \omega$.

4. *Ch4: 8.* If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \varnothing$, then $|P \cap (a, b)| = c$.

SOLUTION: Obviously, $|P \cap (a, b)| \leq c$ since it is a subset of $\mathbb{R}$.

Choose $x_0 \in P \cap (a, b)$, if $[x_0, b) \subset P$ or $(a, x_0] \subset P$, we can easily get $|P \cap (a, b)| \geq c$. Now, suppose that $[x_0, b) \nsubseteq P$ and $(a, x_0] \nsubseteq P$. Pick $a_1 \in (a, x_0] - P$ and $b_1 \in [x_0, b) - P$, Then $P \cap [a_1, b_1]$ is a perfect. Since $P \cap [a_1, b_1]$ is a nonempty closed set, it suffices to prove the set has no isolated point. But $P \cap [a_1, b_1] = P \cap (a_1, b_1)$. For any $x \in P \cap (a_1, b_1)$ and any open neighborhood I of x, $(P \cap (a_1, b_1)) \cap I - \{x\} = P \cap ((a_1, b_1) \cap I) - \{x\} \neq \varnothing$, since x is a limit point of P. This means that $P \cap [a_1, b_1]$ is a perfect set. Thus $|P \cap (a, b)| \geq |P \cap [a_1, b_1]| = c$.

5. *Ch4: 9.* If $P_2 \nsubseteq P_1$ are perfect sets, then $|P_2 - P_1| = c$.

SOLUTION: Pick $x \in P_2 - P_1$. Since P_1 is closed, there exists $\delta > 0$ such that $B(x, \delta) \subset P_1^c$. By exercise 4, $|P_2 \cap B(x, \delta)| = c$. Then $P_2 \cap B(x, \delta) \subset P_2 - P_1 \subset \mathbb{R}$, so $|P_2 - P_1| = c$.

6. *Ch4: 10.* If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

If P is perfect then $P = P^*$.

SOLUTION: For any $x \in P$ and $\delta > 0$, $B(x, \delta) \cap P \neq \varnothing$. Using the conclusion of exercise 4, $|B(x, \delta) \cap P| = \mathfrak{c} > \aleph_0$, i.e $x \in P^*$.

On the other hand, $P' = \{x \in \mathbb{R} \mid \forall \delta > 0 (B(x, \delta) \cap P - \{x\} \neq \varnothing)\}$. Clearly $P^* \subset P'$. But because P is closed, we have $P' \subset P$. Thus $P^* \subset P$.

So $P = P^*$ holds.

7. *Ch4: 11.* If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.

SOLUTION: We have prove that $P = P^*$ in the above exercise, so it is sufficient to show that $P^* \subset F^*$. For any $x \in P^*$, by definition, $\forall \delta > 0$, $|B(x, \delta) \cap P| > \aleph_0$. Then $|B(x, \delta) \cap F| \geq |B(x, \delta) \cap P| > \aleph_0$, which implies that $x \in F^*$.

8. *Ch4: 12.* If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$, thus $F^* = P$.

SOLUTION: For any $x \in F^*$ and $\delta > 0$, $|B(x, \delta) \cap F| > \aleph_0$. By theorem 4.6, $F = P \cup S$, where P is a perfect set and S is at most countable.

$$B(x, \delta) \cap F = (B(x, \delta) \cap P) \cup (B(x, \delta) \cap S)$$

$|B(x, \delta) \cap P|$ must be larger than $\aleph_0$. Otherwise, both of $B(x, \delta) \cap P$ and $B(x, \delta) \cap S$ are at most countable, which leads to that $B(x, \delta) \cap F$ is at most countable (contradiction). So we get $x \in P^* = P$. Thus $F^* \subset P$.

On the other hand, we have proved $P \subset F^*$ in exercise 7. So $F^* = P$.

9. *Ch4: 13.* If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.

SOLUTION: In the above exercise, we have proved that $F = F^* \cup (F - F^*)$ is a partition required. Here we prove the uniqueness. Suppose $F = P_1 \cup S_1 = P_2 \cup S_2$ are two partitions. If $P_1 \neq P_2$, without lost of generality, assume $P_2 \nsubseteq P_1$. By exercise 5, $|P_2 - P_1| = c$. But $|P_2 - P_1| = |S_1 - S_2| \leq \aleph_0$, which is a contradiction. So the partition is unique.

10. *Ch4: 15.* If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

SOLUTION: For each $\alpha < \omega_1$,

$$\begin{aligned}\Sigma_1^0 &= \text{the collection of all open sets}\\ \Pi_1^0 &= \text{the collection of all closed sets}\\ \Sigma_\alpha^0 &= \{\bigcup B_n \mid \text{ each } B_n \in \Pi_\beta^0 \text{ some } \beta < \alpha\}\\ \Pi_\alpha^0 &= \{A^c \mid A \in \Sigma_\alpha^0\}\end{aligned}$$

Then

$$\bigcup_{\alpha<\omega_1} \Sigma_\alpha^0 = \bigcup_{\alpha<\omega_1} \Pi_\alpha^0 = \mathscr{B}$$

Now we prove by induction that each Σ_α^0 is closed under inverse image by continuous function.

Σ_1^0 holds the property by the definition of continuous function.

For any $A \in \Sigma_\alpha^0$,

$$A = \bigcup B_n = \bigcup A_n^c$$

where each $B_n \in \Pi_{\beta_n}^0$, $A_n \in \Sigma_{\beta_n}^0$ for some $\beta_n < \alpha$. $f_{-1}(A) = f_{-1}(\bigcup A_n^c) = \bigcup f_{-1}(A_n^c) = \bigcup f_{-1}(A_n)^c$. By induction, each $f_{-1}(A_n) \in \Sigma_{\beta_n}^0$. Thus $f_{-1}(A) \in \Sigma_\alpha^0$.

So $\mathscr{B}$ is closed under inverse image by continuous function, i.e if B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

11. *Ch4: 18.* The tree T_F has no maximal note, i.e, $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential tree without maximal nodes.

SOLUTION:

$$T_F = \{s \in \text{Seq} : s \subset f \text{ for some } f \in F\}$$

For any $s \in T$, there exists an f in F such that $s \subset f$, i.e $s = f \restriction n$ for some $n < \omega$. Let $t = f \restriction (n+1)$, thus we have $s \in t$ and $s \subset t$.

It suffices to prove that the map $F \mapsto T_F$ is one-to-one. It is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f \restriction n \in T_F$ for all $n \in \mathbb{N}$, then for each n, there is some $g \in F$ such that $g \restriction n = f \restriction n$; and since F is closed, it follows that $f \in F$. Let F_1, F_2 be closed sets in $\mathcal{N}$. If $F_1 \neq F_2$, say $g \in F_1 - F_2$, then for all n, $g \restriction n \in T_{F_1}$ (by definition of T_{F_1}), but $g \restriction m \notin T_{F_2}$ for some m (since F_2 is closed), $T_{F_1} \neq T_{F_2}$.