Solutions for Assignment # 4.1

November 21, 2024

1. Define $+_Q$, $\cdot_Q$ and $<_Q$ and verify that your definitions doesn't depend on the choice of representatives.

   SOLUTION: Define $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}_+ / _\approx$, where $(p, q) \approx (r, s)$ iff $p \cdot_\mathbb{Z} s = q \cdot_\mathbb{Z} r$. For rest of the solution, we write (a, b) for equivalent class $[(a, b)]_\approx$, and $+, \cdot, <$ for $+_\mathbb{Z}, \cdot_\mathbb{Z}, <_\mathbb{Z}$.

   - Define $(a, b) +_\mathbb{Q} (c, d) = (a \cdot d + b \cdot c, b \cdot d)$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

   $$\begin{aligned}(a_1 \cdot d_1 + b_1 \cdot c_1, b_1 \cdot d_1) &\approx ((a_1 \cdot d_1 + b_1 \cdot c_1) \cdot b_2 \cdot c_2, (b_1 \cdot d_1) \cdot b_2 \cdot c_2) \\ &= ((a_2 \cdot d_2 + b_2 \cdot c_2) \cdot b_1 \cdot c_1, (b_2 \cdot d_2) \cdot b_1 \cdot c_1) \\ &\approx (a_2 \cdot d_2 + b_2 \cdot c_2, b_2 \cdot d_2).\end{aligned}$$

   - Define $(a, b) \cdot_\mathbb{Q} (c, d) = (a \cdot c, b \cdot d)$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

   $$\begin{aligned}(a_1 \cdot c_1, b_2 \cdot d_2) &\approx (a_1 \cdot c_1 \cdot b_2 \cdot d_2, c_1 \cdot d_1 \cdot b_2 \cdot d_2) \\ &= (a_2 \cdot c_2 \cdot b_1 \cdot d_1, b_2 \cdot d_2 \cdot b_1 \cdot d_1) \\ &\approx (a_2 \cdot c_2, b_2 \cdot d_2).\end{aligned}$$

   - Define $(a, b) <_\mathbb{Q} (c, d)$ iff $a \cdot d < b \cdot c$. Suppose that $(a_1, b_1) \approx (a_2, b_2)$ and $(c_1, d_1) \approx (c_2, d_2)$, then

   $$\begin{aligned}a_1 \cdot d_1 < b_1 \cdot c_1 &\Leftrightarrow a_1 \cdot b_2 \cdot d_1 \cdot d_2 < c_1 \cdot d_2 \cdot b_1 \cdot b_2 \\ &\Leftrightarrow a_2 \cdot b_1 \cdot d_1 \cdot d_2 < c_2 \cdot d_1 \cdot b_1 \cdot b_2 \\ &\Leftrightarrow a_2 \cdot d_2 < b_2 \cdot c_2.\end{aligned}$$

2. *Ch4: 1.* The set of all continue functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality c (while the set of all functions has cardinality 2^c).

   SOLUTION: Denote the set of all continue functions by $c(\mathbb{R})$.

   For any $a \in \mathbb{R}$, let $g_a(x) = a(\forall x \in \mathbb{R})$. Obviously, $g_a \in c(\mathbb{R})$. Define

   $$\begin{aligned}G : \quad \mathbb{R} &\longrightarrow c(\mathbb{R}) \\ a &\longmapsto g_a\end{aligned}$$

   G is an injection, so $|c(\mathbb{R})| \geq |\mathbb{R}| = c$.

   $\mathbb{Q}$ is countable, denoted by $\{r_i \mid i < \omega\}$. Let

   $$\begin{aligned}F : \quad c(\mathbb{R}) &\longrightarrow {}^\omega\mathbb{R} \\ f &\longmapsto \langle f(r_0), f(r_1), \dots \rangle\end{aligned}$$

   F is injection. For any $f \neq g \in c(\mathbb{R})$, there exists $x \in \mathbb{R}$, s.t $f(x) \neq g(x)$. Since f and g are continue, there exists an interval I, s.t $f(y) \neq g(y)(\forall y \in I)$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, so $f(r_i) \neq g(r_i)$ for some $i < \omega$, i.e $F(f) \neq F(g)$. From this, we can get

   $$|c(\mathbb{R})| \leq |{}^\omega\mathbb{R}| = |\mathbb{R}|^\omega = (2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega = c$$

   Hence, $|c(\mathbb{R})| = c$.

   The set of all functions on $\mathbb{R}$ is ${}^\mathbb{R}\mathbb{R}$. $|{}^\mathbb{R}\mathbb{R}| = (2^\omega)^{2^\omega} = 2^{\omega \cdot 2^\omega} = 2^{2^\omega} = 2^c$.

3. *Ch4: 2.* There are at least c countable order-types of linearly ordered sets.

<u>SOLUTION</u>: For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers, let

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \ldots$$

where ξ is the order-type of the integers.

If $a \neq b$, then $\tau_a \neq \tau_b$. Suppose not, let $\varphi : \tau_a \to \tau_b$ be the isomorphism and i be the least s.t $a_i \neq b_i$. Since φ is order preserving, it must be that $\varphi(a_0 + \xi + a_1 + \xi + \cdots + a_{i-1} + \xi) = b_0 + \xi + b_1 + \xi + \cdots + b_{i-1} + \xi$ (by induction). Without lose of generality, suppose $a_i < b_i$. Let $\eta_i = a_0 + \xi + a_1 + \xi + \cdots + a_{i-1} + \xi$, and we identify η_i with $\varphi(\eta_i)$. Then $\varphi[\eta_i + a_i]$ is an initial part of $\eta_i + b_i$. Suppose c is the k-th element of b_i. Let c^* denote the element such that

$$\text{ordertype}(\{d \in \tau_b \mid d <_b c^*\}) = \eta_i + k.$$

Note that $\varphi^{-1}(a_i^*) > c$, for any $c \in \eta_i + a_i$. Thus $\varphi^{-1}(a_i^*) \in (\eta_i + a_i, +\infty)$. Then for any $c \in (\eta_i + a_i, \varphi^{-1}(a_i^*)]$, there is a $d \in (\eta_i + a_i, c)$. But this is not true on the τ_b side – there is a least element in $(\eta_i + a_i, a_i^*]$. This contradicts to that φ is isomorphism.

The map $a \to \tau_a$ is a injection from ${}^{\omega}\mathbb{N}$ into the set of order-types of linearly ordered sets. Hence

$$|\{\text{order-types of linearly ordered sets}\}| \geq |{}^{\omega}\mathbb{N}| = \omega^{\omega} = 2^{\omega} = \mathfrak{c}.$$

4. *Ch4: 3.* The set of all algebraic reals is countable.

<u>SOLUTION</u>: Since every algebraic real is one element of the finite roots of some polynomials in $\mathbb{Z}[x]$, and $|\mathbb{Z}[x]| = |\mathbb{Z}^{<\omega}|$ is countable, there are only countably many algebraic reals.

5. *Ch4: 4.* If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$.

<u>SOLUTION</u>: Let S^* be a countable subset of $\mathbb{R}^2$. Let $PS^* = \{x \in \mathbb{R} \mid \exists y((x, y) \in S^*)\}$. Since $PS^* \subset \mathbb{R}$ is countable, there exists an $x_0 \notin PS^*$. Then $(\{x_0\} \times \mathbb{R}) \subset ((\mathbb{R} - PS^*) \times \mathbb{R}) \subset (\mathbb{R}^2 - S^*)$. So $|\mathbb{R}^2 - S^*| \geq \mathfrak{c}$. But $|\mathbb{R}^2 - S^*| \leq \mathfrak{c}$ since it is a subset of $\mathbb{R}^2$. We can get that $|\mathbb{R}^2 - S^*| = \mathfrak{c}$.

Now, since $\phi : \mathbb{R} \approx \mathbb{R}^2$, there exists a countable sunset $S^* = \phi(S)$ of $\mathbb{R}^2$. So $(\mathbb{R} - S) \approx (\mathbb{R}^2 - S^*)$. Hence $|\mathbb{R} - S| = \mathfrak{c}$.

6. *Ch4: 5.*

   (a) The set of all irrational numbers has cardinality $\mathfrak{c}$.

   (b) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

<u>SOLUTION</u>: This is the direct conclusion of previous two exercises.