Elementary Set Theory

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Models of Set theory

– A super-brief introduction –

Universe of sets

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$$\begin{aligned} V_0 &= \varnothing \\ V_{\alpha+1} &= \mathscr{P}(V_\alpha) \\ V_\lambda &= \bigcup\{V_\alpha \mid \alpha < \lambda\} \qquad \text{(for limit } \lambda > 0) \end{aligned}$$

Models of set theory

- For all α, $V_\alpha \models$ Extensionality, Regularity, and Union.
- Let α be a limit ordinal. Then
$$V_\alpha \models \text{Pairing, Power set and Choice.}$$
- If $\alpha > \omega$, then $V_\alpha \models$ Infinity.
- If κ is a regular strong limit cardinal, then[1]
$$V_\kappa \models \text{Replacement.}$$

[1] ω is a regular and strong limit cardinal.

Models of set theory

Theorem (ZFC)

1. $V_\omega \models \mathsf{ZFC} - \mathrm{Inf}$.
2. *Assume κ is a strongly inaccessible cardinal.*[2] *Then*

$$V_\kappa \models \mathsf{ZFC}.$$

[2] Strongly inaccessible cardinals are uncountable regular strongly inaccessible cardinals.

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Theorem (ZFC)

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So by Gödel's 2-nd Incompleteness Theorem,
$\mathsf{ZFC} \not\Rightarrow \mathrm{Con}(\mathsf{ZFC})$,

Theorem

ZFC $\nvdash$ *"there exists a strongly inaccessible cardinal".*

[2] Strongly inaccessible cardinals are uncountable regular strongly inaccessible cardinals.

- A proof of ZFC $\nvdash \exists\kappa(\kappa$ is strongly inaccessible$)$ without using Gödel's Incompleteness.

<u>PROOF</u>.

If not, let κ_0 be the least strongly inaccessible cardinal. Then $V_{\kappa_0} \models$ ZFC implies that $V_{\kappa_0} \models \exists\lambda(\lambda$ is strongly inaccessible$)$. Let λ_0 be such a λ. Then $\lambda_0 < \kappa_0$. The key fact is that λ_0 is also strongly inaccessible in V. This contradicts to the hypothesis that κ_0 is the least strongly inaccessible cardinal in V. □

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- In fact, neither does ZFC imply the existence of weakly inaccessible cardinals.
- A cardinal whose existence cannot be proved in ZFC but whose existence has not been shown to be inconsistent with ZFC is called a **large cardinal.**
- Thus weakly inaccessible and strongly inaccessible cardinals are large cardinals.

Constructibility

Definition

- A set y is **definable over a structure** M iff there is a formula $\varphi(\bar{x}) \in \mathcal{L}^M$ and parameters $\bar{p} \in M$ such that:
$$y = \{z \in M \mid M \models \varphi[z, \bar{p}]\}.$$
- For any set M, let
$$\text{Def}(M) = \{y \subseteq M \mid y \text{ is definable over } (M, \in)\}.$$

$$\begin{aligned} L_0 &= \varnothing \\ L_{\alpha+1} &= \text{Def}(L_\alpha) \\ L_\lambda &= \bigcup\{L_\alpha \mid \alpha < \lambda\} \qquad (\text{for limit } \lambda > 0) \end{aligned}$$

- L was discovered by Gödel (1937) as a way to prove that
$$\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZFC}).$$
- L is a (definable) class, the class of **constructible sets**, and the assertion $V = L$, i.e. $\forall x\,(x \in L)$, is the **Axiom of Constructibility**.
- By transfinite recursion within L one can well-order L level-by-level, well ordering $L_{\alpha+1} - L_\alpha$ according to definitions and the previous well-ordering of the parameters from L_α.
- Gödel also provided a construction of L by transfinite recursion in terms of 8 elementary set operators.

- Each L_α is transitive.
- Every ordinal is an element of L.
- $V_\alpha = L_\alpha$, for $\alpha \leq \omega$.
- $L \models$ Pairing, Union, Power set, Replacement etc.

Pairing. $a, b \in X \implies \{a, b\}$ is definable from X.

Union. $a \in X \wedge X$ is transitive $\implies \bigcup a$ is definable from X.

Power set. If $a \in L$,

$$\mathscr{P}(a)^L = \mathscr{P}(a) \cap L = \{x \in L \mid x \subseteq a\} \subseteq \mathscr{P}(a).$$

By Replacement, $\mathscr{P}(a) \cap L \subseteq L_\beta$ for some β.

In fact, for each $x \in \mathscr{P}(a)^L$, let α_x be least α such that $x \in \mathscr{P}(a) \cap L_{\alpha_x}$, take $\beta \geq \sup\{\alpha_x \mid x \in \mathscr{P}(a)^L\}$. So $\mathscr{P}(a)^L = \{x \in L_\beta \mid x \subseteq a\} \in L_{\beta+1}$. ⊠

- Since $\mathrm{Ord} \subseteq L \subseteq V$, Π_1^{ZF}-properties of ordinals are preserved when going down from V to L.
- Hence regular cardinals remain regular in L; similarly, limit cardinals, inaccessible cardinals, Mahlo cardinals, etc.
- More generally, any large cardinal property weaker than $0^\sharp$ will be retained in L.
- (Gödel, 1937) $L \models \mathsf{ZFC} + \mathrm{GCH}$.
- If κ is weakly inaccessible (in V), $L_\kappa \models \mathsf{ZFC} + \mathrm{GCH}$.

  Key point. GCH *implies that weakly inaccessible is the same as strongly inaccessible.*

- Jensen (1972) developed a "**fine structure**" theory for L of intrinsic interest, and the study of constructibility and its generalizations has become one of the mainstreams of modern set theory.
- $0^\sharp$ is a large cardinal principle isolated by Jensen. The existence of $0^\sharp$ implies $V \neq L$, i.e. there is a non-constructible set.
- Since $L \models$ AC, GCH, $V = L$, these statements can not be refuted within L, more generally, by taking inner models. These can also be done by Cohen's method of **forcing** (1963).

Inner model

Definition

For a proper class M, M is an **inner model** iff M is a transitive $\in$-model of ZF with $\mathrm{Ord} \subset M$.

Gödel also showed that

Theorem

1. *L is the smallest inner model of* ZF*,*
   i.e. any M if $M \models$ ZF *and $M \supset \mathrm{Ord}$, then $M \supseteq L$.*
2. *In fact, $L^M = L$,*
   where L^M is the constructible universe defined within M.

More L-like models

Definition 1

$$L(A) \quad \left| \begin{aligned} L_0(A) &= \mathrm{TC}(\{A\}) \\ L_{\alpha+1}(A) &= \mathsf{Def}(L_\alpha(A)) \\ L_\lambda(A) &= \bigcup\{L_\alpha(A) \mid \alpha < \lambda\} \quad \text{(for limit } \lambda > 0\text{)} \end{aligned} \right.$$

$$L[A] \quad \left| \begin{aligned} L_0[A] &= \varnothing \\ L_{\alpha+1}[A] &= \mathsf{Def}^A(L_\alpha[A]) \\ L_\lambda[A] &= \bigcup\{L_\alpha[A] \mid \alpha < \lambda\} \quad \text{(for limit } \lambda > 0\text{)} \end{aligned} \right.$$

where $\mathrm{TC}(x)$ is the smallest transitive set containing x,

$$\mathsf{Def}^A(x) = \{y \subseteq x \mid y \text{ is definable over } (x, \in, A \cap x)\}$$

- For a given set A, $L(A)$ is the constructible closure of A, i.e. the smallest inner model M such that $A \in M$.
  - For a given set A, $L[A]$ is the constructible universe relative to A, i.e. the smallest inner model M such that $A \cap M \in M$.
- $L(A)$ does not satisfy AC, unless $\mathrm{TC}(\{A\})$ can be well ordered.
  - $L[A]$ satisfies AC.
- For $\alpha \geq \omega$, $|L_\alpha(A)| = |\mathrm{TC}(\{A\})| \cdot |\alpha|$.
  - For $\alpha \geq \omega$, $|L_\alpha[A]| = |\alpha|$.
- More L-like models: $L(\mathbb{R})$, $L(\mathrm{Ord}^\omega)$, $L[0^\sharp]$, $L[\widetilde{E}]$, etc.

OD and HOD

- A set X is **ordinal definable** if there are an ordinal $\alpha \in \mathrm{Ord}$ and a formula φ such that
$$X = \{x \mid V_\alpha \models \varphi(x, \bar{\alpha})\}$$
for some ordinal parameters $\bar{\alpha} \in \mathrm{Ord}^{<\omega}$.
- $\mathrm{OD} = \{X \mid X \text{ is ordinal definable}\}$.

OD has a natural well-ordering, however it is not transitive.

- Let HOD denote the class of **hereditarily ordinal-definable** sets, i.e.
$$\mathrm{HOD} = \{x \mid \mathrm{TC}(\{x\}) \subseteq \mathrm{OD}\}$$
- The class HOD is a transitive model of ZFC.[3]

[3] This gives another proof of $\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZFC})$.

- Both notions of OD and HOD are expressible in the language of set theory.
- Both OD and HOD can be relativized to a given set A: $\mathrm{OD}[A], \mathrm{OD}(A), \mathrm{HOD}[A]$ and $\mathrm{HOD}(A)$.
- By definition, $L \subseteq \mathrm{HOD}$.
- In contrast to the Constructible universe L, which may consist of only a very small fraction of the universe V,[4] HOD is very close to V.
- HOD-like models play important roles in contemporary foundational study of set theory.

[4] assuming suitable large cardinals.