Elementary Set Theory

Xianghui Shi

School of Mathematical Sciences
Beijing Normal University

Fall 2024

Axiom of Regularity

Axiom of Regularity

Every nonempty set has an $\in$-minimal element:

$$\forall S \, (S \neq \varnothing \to (\exists x \in S) \; [S \cap x = \varnothing].$$

This axiom asserts that the universe of sets is $\in$-wellfounded.

Some Lemmas

Lemma

- *There is no infinite $\in$-descending sequence:*

$$\cdots \in x_n \in \cdots \in x_2 \in x_1 \in x_0$$

- *Every set S has a* **transitive closure**,[1]

$$\begin{aligned}\mathrm{TC}(S) &= \bigcap\{T \mid T \supset S \wedge T \text{ is transitive}\} \\ &= \textstyle\bigcup_{n<\omega} \bigcup^n S.\end{aligned}$$

- *Every nonempty class C has an $\in$-minimal element.*

[1] Closed under "$\in$": $x \in b \wedge b \in S \rightarrow x \in S$, or equivalently, $\bigcup S \subset S$.

The Cumulative Hierarchy of Sets

$$V_0 = \varnothing, \qquad V_{\alpha+1} = \mathscr{P}(V_\alpha)$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha,\ \lambda \text{ is a limit ordinal.}$$

The sets V_α have the following properties:

1. If $\alpha < \beta$ then $V_\alpha \subset V_\beta$.
2. Each V_α is transitive.
3. $\alpha \subset V_\alpha$.

<u>PROOF</u>.

1. $V_\alpha \subset V_{\alpha+1}$ implies $\mathscr{P}(V_\alpha) \subset \mathscr{P}(V_{\alpha+1})$, ($\therefore V_\alpha \subset V_\beta$) and
2. $\bigcup V_{\alpha+1} = \bigcup \mathscr{P}(V_\alpha) = V_\alpha \subset V_{\alpha+1}$. At limit λ, $\bigcup V_\lambda = \bigcup_{\alpha<\lambda} V_{\alpha+1} = V_\lambda$. Therefore every V_α is transitive.
3. $\alpha \subset V_\alpha$ implies that $\alpha + 1 \subset V_\alpha \cup \{\alpha\} \subset V_{\alpha+1}$. At limit λ, $\lambda = \bigcup_{\alpha<\lambda} \alpha \subset \bigcup_{\alpha<\lambda} V_\alpha = V_\lambda$.

□

Von Neumann universe

The **Axiom of Regularity** implies that

Proposition 1

$V = \bigcup_\alpha V_\alpha$, *i.e. for every* x *there is* $\alpha \in \mathrm{Ord}$ *s.t.* $x \in V_\alpha$.

PROOF.

- Let $C = V \setminus \bigcup_\alpha V_\alpha$. Assume $C \neq \varnothing$. Take a $c \in \min C$.
- $c \neq \varnothing$, as $\varnothing \in V_1$. Take any $x \in c$. $x \notin C$ as c is minimal. So there is an $\alpha_x \in \mathrm{Ord}$ such that $x \in V_{\alpha_x+1} \setminus V_{\alpha_x}$.
- Let $A_c = \{\alpha_x + 1 \in \mathrm{Ord} \mid x \in V_{\alpha_x+1} \setminus V_{\alpha_x} \wedge x \in c\}$.
- C is a set, by Axiom of Replacement, A_c is a set of ordinals.
- Let $\beta = \sup A_c$. Then $c \in \mathscr{P}(V_\beta) = V_{\beta+1}$. Contradiction! □

The Rank Function

For every $x \in V$, we may define the **rank** of x:

$$\mathrm{rank}(x) = \text{ the least } \alpha \text{ such that } x \in V_{\alpha+1}.$$

Thus

- For every $x \in V_\alpha$, $\mathrm{rank}(x) < \alpha$.
- If $x \in y$ then $\mathrm{rank}(x) < \mathrm{rank}(y)$.
- $\mathrm{rank}(\alpha) = \alpha$.

Equivalence classes over a proper class

- Given a class C, let
$$C^* = \{x \in C \mid x \text{ has minimal rank}\}.$$
Then $C \neq \varnothing \implies C^* \neq \varnothing$ and $C^* \in V_{\alpha+2}$ for some α.
- Given an equivalence relation $\equiv$ over a proper class C, the previous definition of quotient class: $C/_{\equiv} = \{[x] \mid x \in C\}$ is a class of classes.
- With the rank function, one can refer to each equivalence class $[x]$ via $[x]^*$ (as sets). In particular, one can define isomorphism types for a given isomorphism.

$\in$-induction

The method of transfinite induction, which is along Ord, can be extended to arbitrary transitive class.

$\in$-induction

The method of transfinite induction, which is along Ord, can be extended to arbitrary transitive class.

Theorem 2 ($\in$-induction)

Let T be a transitive class, let φ be a property. Assume that

1. $\varphi(\varnothing)$ *holds;*
2. *if $x \in T$ and $\varphi(z)$ holds for every $z \in x$, then $\varphi(x)$ holds.*

Then $\varphi(x)$ holds for every $x \in T$.

$\in$-recursion

Theorem 3 ($\in$-Recursion)

Let T be a transitive class and G be a function over V. Then there is a function F on T s.t. $F(x) = G(F \upharpoonright x)$ for every $x \in T$.

PROOF.

$$\begin{aligned} F(x) = y \leftrightarrow & \exists f \text{ s.t. } \operatorname{dom}(f) \subset T \text{ is transitive and} \\ & (i)\ \forall u \in \operatorname{dom}(f)\, (f(u) = G(f \upharpoonright u)) \\ & (ii)\ f(x) = y \end{aligned}$$

The uniqueness of F follows from $\in$-induction. □

Nontrivial $\in$-isomorphism

Theorem 4

Let T_1, T_2 be two classes, $\pi : T_1 \to T_2$ be an $\in$-isomorphism. If T_1, T_2 are transitive, then $T_1 = T_2$ and $\pi(u) = u$ for every $u \in T_1 = \mathrm{TC}(T_1)$.

PROOF.

Assume that $\pi(z) = z$ for every $z \in x$. We show that $\pi(x) = x$.

- By inductive hypothesis, $x \subseteq \pi(x)$.
- Suppose $t \in \pi(x)$. As π is an isomorphism, $\pi(u) = t$ for some $u \in T_1$. $\pi(u) \in \pi(x)$ implies $u \in x$. So $t = \pi(u) = u \in x$. □

Well-Founded Relations

We extend these to a broader class: well-founded relations.
Let E be a binary relation on a class P, for each $x \in P$, define $\mathrm{Ext}_E(x) = \{z \in P \mid z \mathrel{E} x\}$ to be the E-**extension** of x.

Well-Founded Relations

We extend these to a broader class: well-founded relations.
Let E be a binary relation on a class P, for each $x \in P$, define $\mathrm{Ext}_E(x) = \{z \in P \mid z \, E \, x\}$ to be the E-**extension** of x.

Definition 5

A relation E on P is **well-founded** if

1. every nonempty set $x \subset P$ has an E-minimal element.
2. $\mathrm{Ext}_E(x)$ is a set, for every $x \in P$.

The **Axiom of Regularity** implies that $\in$ is well-founded.

E-minimal element

Lemma 6

If E is a well-founded on P, then every nonempty class $C \subset P$ has an E-minimal element.

<u>Proof</u>.

For an $S \in P$, define $\mathrm{TC}_E(S) = \bigcup S_n$, where

$$S_0 = \mathrm{Ext}_E(S) \quad \text{and} \quad S_{n+1} = \bigcup\{\mathrm{Ext}_E(x) \mid x \in S_n\}.$$

Then pick an $S \in C$ and an E-minimal of $\mathrm{TC}_E(S) \cap C$ is an E-minimal of C. □

Well-Founded Induction

Theorem 7 (E-induction)

Let T be a transitive class on P, φ be a property. Assume that

1. *Every E-minimal element x has property φ.*
2. *if $x \in P$ and if $\varphi(z)$ holds for every z s.t. $z \, E \, x$, then $\varphi(x)$ holds.*

Then every $x \in P$ has property φ.

Well-Founded Recursion

Theorem 8 (Well-Founded Recursion)

Let E be a well-founded relation on P. Let G be a function on $V \times V$. Then there is a unique function F on P s.t.

$$F(x) = G(x, F \restriction \mathrm{Ext}_E(x)), \quad \text{for every } x \in P.$$

Rank Function

Example 9 (The Rank Function)

Let E be a well-founded relation on P. Define, by induction, for all $x \in P$.

$$\rho^E(x) = \sup\{\rho^E(z) + 1 \mid z \, E \, x\}.$$

For all $x, y \in P$, $x \, E \, y \leftrightarrow \rho^E(x) < \rho^E(y)$. The range of ρ^E is either an ordinal or the class Ord.

Mostowski Collapse

Definition 10

A well-founded relation E on a class P is **extensional** if $\mathrm{Ext}_E(x) \neq \mathrm{Ext}_E(y)$ whenever $x, y \in P$ and $x \neq y$.

Mostowski Collapse

Definition 10

A well-founded relation E on a class P is **extensional** if $\mathrm{Ext}_E(x) \neq \mathrm{Ext}_E(y)$ whenever $x, y \in P$ and $x \neq y$.

Theorem 11

If E is a well-founded and extensional relation on a class P, then there is a unique transitive class M and a unique isomorphism $\pi : (P, E) \cong (M, \in)$.