Elementary Set Theory

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Axiom of Choice

The Axiom of Choice asserts that: Every family of nonempty sets has a choice function.

Axiom of Choice (AC)

Let $\mathcal{A}$ be such that $\forall a \in \mathcal{A}\,(a \neq \varnothing)$. Then there exists a function f such that $\mathrm{dom}(f) = \mathcal{A}$ and

$$\text{for every } a \in \mathcal{A},\ f(a) \in a.$$

- AC(X) denotes the version that $\bigcup \mathcal{A} = X$.
- Suppose κ is an (infinite) cardinal. Let AC$_\kappa$ denote the version that $|\mathcal{A}| \leq \kappa$.

So AC $\equiv (\forall\kappa)$AC$_\kappa$.

The Choice Function

ZF-cases that a choice function exists:

- For each $a \in \mathcal{A}$, $|a| = 1$.
- AC_n holds for every $n < \omega$. i.e. $|\mathcal{A}| < \omega$.
- Each $a \in \mathcal{A}$ is a finite set of reals.

The existence of a choice function is not certain even for the case that $\mathcal{A}$ is infinite and for all $a \in \mathcal{A}$, $|a| = 2$.

Remark

The point is that: the choice function needs to be well defined relative to known parameters, such as $\mathcal{A}$ and, if exists, a well ordering of $\bigcup \mathcal{A}$.

The Axiom of Well Orderings

The Axiom of Well Orderings (WO)

Every set can be well ordered.

Theorem

AC $\Leftrightarrow$ WO.

PROOF.

WO $\Rightarrow$ AC is trivial. For the other direction, fixing a set $X \neq \varnothing$, we need a choice function

$$f : \mathscr{P}(X) - \{\varnothing\} \to X.$$

and the enumerating process to well order X.[1] □

[1] We showed $\mathsf{WO}(X) \Rightarrow \mathsf{AC}(X)$ and $\mathsf{AC}_{2^{|X|}}(X) \Rightarrow \mathsf{WO}(X)$.

Other Equivalent Versions in Set Theory

- If A is an infinite set, then $|A| = |A \times A|$.
- Any two sets can be compared by their cardinalities.
- The Cartesian product of any nonempty family of nonempty sets is nonempty.
- Every surjective function has a right inverse, i.e. if $f : A \to B$ is onto, then $|B| \leq |A|$.
- (**König's Theorem**). $\sum_{\alpha<\lambda} \kappa_\alpha < \prod_{\alpha<\lambda} \kappa_\alpha$, where $\lambda > 1$ and each $\kappa_\alpha > 2$.

Next are two equivalent versions in the theory of orderings.

Zorn's Lemma and Maximal Principal

Two more well-known equivalent version of AC.

Zorn's Lemma (ZL)

Let $(P, <)$ be a partial order. If every chain in P has an upper bound, then P has a maximal element.

The Maximum Principle (MP)

Every partial order $(P, <)$ has a maximal chain.

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Theorem

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Proof.

MP $\Rightarrow$ ZL: The upper bound of a maximal chain is a maximal (not necessarily the greatest!) element for the whole partial ordered set.

ZL $\Rightarrow$ MP: Consider the partial order $(P^*, \subset)$:

$$P^* = \{A \subset P \mid (A, <) \text{ is a chain in } (P, <)\}$$

- Every $\subset$-chain A^* in P^* has an upper bound: $(\bigcup A^*, <)$.
- A $\subset$-maximal element of P^* is a maximal $<$-chain in P. □

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<u>PROOF</u>.

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WO $\Rightarrow$ MP: Use an enumeration (a well ordering) of P to construct a maximal chain.

ZL $\Rightarrow$ WO: Given $X \neq \varnothing$, consider the partial order $(P_X, <)$:
$P_X = \{(A, \prec) \mid (A, \prec) \text{ is a well ordered subset of } X\}$, and
$(A_1, \prec_1) < (A_2, \prec_2)$ iff

$$(A_1, \prec_1) \text{ is a proper initial segment of } (A_2, \prec_2)$$

Every maximal element P_X is a well ordering of X. $\square$

Equivalent Versions of AC in Other Area

- Every vector space has a basis.
- Every nontrivial unitary ring contains a maximal ideal.
- (**Tychonoff Theorem**). Any product of compact spaces is compact in the product topology.
- In the product topology, the closure of a product of subsets is equal to the product of the closures.
- Any product of complete uniform spaces is complete.

Weaker Consequences of AC, I

- The union of a countable family of countable sets is countable. (AC_ω)
- For each property $P \in \{$ Perfect Set Property, Lebesgue Measurable, Baire Property $\}$, there is a set without property P.
- The Lebesgue measure of a countable disjoint union of measurable sets is equal to the sum of the measures of the individual sets. (σ-additivity)
- (**Banach-Tarski Paradox**). A solid ball in $\mathbb{R}^3$ can be split into several disjoint pieces, which can be reassembled only by shifting and rotating (without changing their shapes) to yield two identical copies of the original ball.

Weaker Consequences of AC, II

- Every field has a unique algebraic closure.
- Every field extension has a transcendence basis.
- Every subgroup of a free group is free.
- (**Hahn-Banach Extension Theorem**). Every bounded linear functional on a subspace of some vector space can be extended to the whole space.
- The Baire Category Theorem.
- On every infinite-dimensional topological vector space there is a discontinuous linear map.
- Every Tychonoff space has a Stone-Čech compactification.

Weaker Versions of AC

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- The union of countably many countable sets is countable.
- The collection of all countable subsets of $\mathbb{R}$ form a proper ideal.
- $\aleph_1$ is regular.
- Every $\undertilde{\Sigma}^0_\alpha$ is closed under countable union. In particular, the union of countably many F_σ sets ($\undertilde{\Sigma}^0_2$) is F_σ.
- The Lebesgue measure is countably additive.

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However, AC_ω does not imply that $\mathbb{R}$ can be well ordered.

The Principle of Dependent Choice

The following consequence of AC is more preferred in modern Descriptive Set Theory.

Let A be nonempty. Let DC(A) be the following statement:

Suppose $\prec \subseteq A \times A$. If for every $a \in A$, there is a $b \in A$ s.t. $b \prec a$,[2] then there is a $\prec$-descending ω-sequence $\langle a_n : n < \omega \rangle$ contained in A.[3]

The Principle of Dependent Choices (DC)

$\forall A$, DC(A) holds.

[2] Or equivalently, "for any $n < \omega$ and any $\prec$-descending $\langle a_i : i < n \rangle$ contained in A, there is a $b \in A$ s.t. $b \prec a_{n-1}$".

[3] This sequence can start with any $a_0 \in A$.

Corollary 1 (DC)

1. *A linear ordering $(P, <)$ is a well ordering iff there is no infinite $<$-descending sequence in P.*
2. *A relation E on P is well-founded iff there is no infinite E-descending sequence in P.*

PROOF.

1. "$\Rightarrow$": A $<$-descending ω-sequence is a nonempty subset without $<$-least element.

   "$\Leftarrow$": Suppose $(P, <)$ is ill-ordered, and $\varnothing \neq A \subset P$ contains no $<$-minimal element. Then for any $p \in A$, there is a $q \in A$ such that $q < p$.

2. The same argument. $\square$

REMARK. AC $\Rightarrow$ DC is a strict implication.[4]

Recall $\mathsf{AC}_\omega(X)$ is the assertion that:

If $\{X_n \mid n < \omega\}$ is a family of nonempty subsets of X, then there is a choice function $f : \omega \to X$ such that $f(n) \in X_n$.

Theorem 2

If $|X \times \omega| = |X|$, *then* $\mathsf{DC}(X)$ *implies* $\mathsf{AC}_\omega(X)$.

SKETCH OF PROOF.

For disjoint family $\{X_n \mid n < \omega\}$, set

$$y \prec x \iff \exists n \, [x \in X_n \wedge y \in X_{n+1}].$$

Use $|X \times \omega| = |X|$ to convert X_n to $\{n\} \times X_n$. □

[4] $\mathsf{WO}(X) \Rightarrow \mathsf{DC}(X)$.

AC and Regularity Properties

AC produces many unpleasant sets: assuming AC,

- there is a set that is not Lebesgue measurable.
- there is a set that does not have the Baire property.
- there is a set that does not have the Perfect set property.

Bernstein Set

Theorem 3 (Bernstein)

Assume AC. *There is a set* $B \subset \mathbb{R}$ *such that both* B *and its complement* $\bar{B}$ *meet every perfect (hence every uncountable closed) set.*

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Theorem 4 (AC)

An Bernstein set B *is not Lebesgue measurable and lacks the property of Baire and the perfect set property.*

Proof

- B (so is $\bar{B}$) does not have the PSP, by definition.
- In fact, every Lebesgue measurable subset of B ($\bar{B}$ as well) has measure zero. We need the fact that every Lebesgue measurable set A can be written as $A = F \cup P$, where F is F_σ and P is null. The key is that every closed subset of B (or $\bar{B}$) is a null set.
- Similarly, we show that every subset of B (or $\bar{B}$) that has the Baire property is meager. We use the fact that every set A that has the Baire property can be written as $A = G \cup P$, where G is G_δ and P is meager. The key is that every uncountable G_δ set contains a closed set, which is a homeomorphic copy of Cantor set. □

Cardinal Arithmetic, Cont'd

We continue to calculate the sums and products of infinite cardinals. We assume AC for the rest of this chapter.

Plan

- Infinite sums and products.
- Calculating the continuum function, 2^κ.
- Calculating the cardinal exponentiation, κ^λ.

Lemma 5

For $\lambda \leq \kappa$, the set of all size-λ subset of κ, $[\kappa]^\lambda$ has size κ^λ.

PROOF.

- $|[\kappa]^\lambda| \leq \kappa^\lambda$ is trivial.
- Every $f : \lambda \to \kappa$ is a subset of $\lambda \times \kappa$ and $|f| = \lambda$. Thus

$$\kappa^\lambda \leq |[\lambda \times \kappa]^\lambda| \leq |[\kappa]^\lambda|.$$

□

Notation

- $\kappa^{<\lambda} = \sup\{\kappa^\mu \mid \mu \in \text{Card} \wedge \mu < \kappa\}$.
- Let κ be an infinite cardinal and $|A| \geq \kappa$. Let
$$[A]^{<\kappa} = \mathscr{P}_\kappa(A) = \{X \subset A \mid |X| < \kappa\}.$$

By definition,
$$\kappa^{<\lambda} \leq \kappa^\lambda.$$

By the next lemma, we'll see that
$$|[A]^{<\kappa}| = |A|^{<\kappa}.$$

Infinite Sums and Products

AC is needed to ensure that the following definitions are well defined. (See textbook Ex.5.9, 5.10)

Definition 6

Let $\{\kappa_i\}_{i \in I}$ be an infinite set of cardinals, and $\mathscr{X} = \{X_i\}_{i \in I}$ be a family of sets such that each $|X_i| = \kappa_i$. Define

- $\sum_i \kappa_i = |\bigcup_i X_i|$,

  where X_i's, in addition, are pairwise disjoint.

- $\prod_i \kappa_i = |\prod_i X_i|$,

  where $\prod_i X_i = \{f \mid f \text{ is a choice function over } \mathscr{X}\}$.

Infinite Sums

Lemma 7

If $\lambda \geq \omega$ and $\kappa_i > 0$, for each $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$

Infinite Sums

Lemma 7

If $\lambda \geq \omega$ *and* $\kappa_i > 0$*, for each* $i < \lambda$*, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$

PROOF.

For the nontrivial direction,

$$\lambda \leq \sum_{i<\lambda} 1 \leq \sum_{i<\lambda} \kappa_i$$

$$\kappa_j \leq \sum_{i<\lambda} \kappa_i, \quad \text{for each } j < \lambda.$$

□

Infinite Products

Lemma 8

1. $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$.
2. $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$.
3. *If* $I = \bigcup_{j \in J} A_j$, *where* A_j *are pairwise disjoint. then*
$$\prod_{i \in I} \kappa_i = \prod_{j \in J} (\prod_{i \in A_j} \kappa_i).$$
4. *If* $\kappa_i \geq 2$ *for each* i, *then* $\sum_i \kappa_i \leq \prod_i \kappa_i$.
5. *Suppose* $\lambda \geq \omega$ *and* $\langle \kappa_i \mid i < \lambda \rangle$ *is a nondecreasing sequence of cardinals* > 0. *Then*
$$\prod_{i < \lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

Proof

4. Let $\mathscr{X} = \{X_i \mid i \in I\}$ be pairwise disjoint and each $|X_i| = \kappa_i$. Fix a choice function g over $\mathscr{X}$. For $a \in \bigcup_i X_i$, define $F(a) = (i, f_a)$, where

$$f_a(i) = \begin{cases} a & \text{if } a \in X_i \\ g(i) & \text{if } a \notin X_i \end{cases}$$

$F : \bigcup_i X_i \to I \times \prod_i X_i$ is an injection. Note that

$$\prod_i \kappa_i \geq 2^{|I|} > |I|.$$

$\sum_i \kappa_i \leq |I| \cdot \prod_i \kappa_i = \prod_i \kappa_i$.

Remark. Note that König Theorem is equivalent to AC.

Proof, Cont'd

5. Let $\kappa = \sup_{i<\lambda} \kappa_i$. For the nontrivial direction, we use a partition of λ: $\{A_i \mid i < \lambda\}$ with each $|A_i| = \lambda$. Note that for each $j < \lambda$,

$$\prod_{i \in A_j} \kappa_i \geq \sum_{i \in A_j} \kappa_i = \sup_{i \in A_j} \kappa_i = \kappa.$$

Then by the associativity of infinite products, we have

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} (\prod_{i \in A_j} \kappa_i) \geq \prod_{j<\lambda} \kappa = \kappa^\lambda.$$

König's Theorem

Theorem 9 (König)

If $\kappa_i < \lambda_i$ *for each* $i \in I$, *then* $\sum_i \kappa_i < \prod_i \lambda_i$.

Corollary 10

1. $\kappa < 2^\kappa$, *for any cardinal* κ.
2. $\mathrm{cf}(\kappa^\lambda) > \lambda$, *for any cardinals* $\kappa > 1$ *and* $\lambda \geq \omega$. *In particular,* $\mathrm{cf}(2^\lambda) > \lambda$, *for any infinite cardinal* λ.
3. $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, *for any infinite cardinal* κ.

Proof of König's Theorem

PROOF.

We prove the strict part. Let $F : \bigcup_i X_i \to \prod_i \lambda_i$, where X_i's are pairwise disjoint and each $|X_i| = \kappa_i$. We construct an

$$f \in \prod_i \lambda_i - \mathrm{ran}(F).$$

For each $i \in I$, let p_i be the projection function for the i-th coordinate. Define

$$f(i) = \min(\lambda_i - p_i(F[X_i])).$$

Then each $f(i)$ witnesses that $f \notin F[X_i]$. □

Cardinal Exponentiations under GCH

Theorem 11

Assume GCH. *Let* κ, λ *be infinite cardinals. Then*

$$\kappa^\lambda = \begin{cases} \kappa, & \textit{if } \lambda < \mathrm{cf}(\kappa); \\ \kappa^+, & \textit{if } \mathrm{cf}(\kappa) \leq \lambda < \kappa; \\ \lambda^+, & \textit{if } \kappa \leq \lambda. \end{cases}$$

PROOF.

Only the case $\lambda < \mathrm{cf}(\kappa)$ needs proof. In this case, for every $f \in \kappa^\lambda$, $\mathrm{ran}(f)$ is bounded by some $\alpha < \kappa$. So $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$, and then $\kappa^\lambda \leq \sum_{\alpha<\kappa} |\alpha|^\lambda$.[5] For each $\alpha < \kappa$,

$$\alpha^\lambda \leq 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa.$$

So $\kappa^\lambda \leq \kappa \cdot \kappa = \kappa$. □

[5] In fact, it is equal (see Homework). The case $\kappa = \aleph_{\alpha+1}$ is Hausdorff formula: $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

Continuum Function, without GCH

Beth function:

- $\beth_0 = \aleph_0$.
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$.
- $\beth_\lambda = \sup_{\alpha<\lambda} \beth_\alpha$, for limit ordinal λ.

Continuum function: $\mathfrak{C}(\kappa) = 2^\kappa$.

Gimel function: $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$.

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Continuum function: $\mathfrak{C}(\kappa) = 2^\kappa$.

Gimel function: $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$.

Let $\mathfrak{S}(\aleph_\alpha) = \aleph_{\alpha+1}$. Then

$$\mathrm{GCH} \;\Rightarrow\; \mathfrak{S} = \mathfrak{C} = \gimel, \text{ and } \aleph = \beth.$$

Next we work without GCH.

Proposition 12

1. $\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$.
2. $\mathrm{cf}(2^\kappa) > \kappa$.
3. *If* κ *is a limit cardinal, then* $(2^{<\kappa})^{\mathrm{cf}(\kappa)} = 2^\kappa$.
   In particular,
4. *If* κ *is singular and there exists* μ_0 *s.t.* $2^{\mu_0} = 2^\mu$ *for all* $\mu_0 \leq \mu < \kappa$, *then* $2^\kappa = 2^{\mu_0}$.

PROOF OF 3..

First, $(2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa$. Let $\kappa = \sup_{i<\mathrm{cf}(\kappa)} \kappa_i$. Then

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i (\sup_j 2^{\kappa_j}) = (2^{<\kappa})^{\mathrm{cf}(\kappa)}.$$

□

Corollary 13

1. *If κ is a successor cardinal, then $2^\kappa = \gimel(\kappa)$.*
2. *If κ is a limit cardinal, there are two cases:*
   - 2.1 *if there exists $\mu_0 < \kappa$ s.t. $2^\mu = 2^{\mu_0}$ for all $\mu_0 \leq \mu < \kappa$, then $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$;*
   - 2.2 *otherwise, $2^\kappa = \gimel(2^{<\kappa})$.*

PROOF.

1. Trivial, since $\kappa = \mathrm{cf}(\kappa)$ and $2^\kappa = \kappa^\kappa$.

For 2.2, the key is that $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}(\kappa)$.

For 2.1, clearly $2^\kappa \geq 2^{<\kappa} \cdot \gimel(\kappa)$.

If κ is singular, $2^\kappa \leq 2^{<\kappa}$;

if κ is regular, then $2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cf}(\kappa)}$. □

Cardinal Exponentiation

Theorem 14

Let κ, λ be two infinite cardinals. Then

$$\kappa^\lambda = \begin{cases} 2^\lambda, & \text{(a). } \kappa \leq \lambda; \\ \mu^\lambda, & \text{(b). } \mu^\lambda \geq \kappa, \text{ for some } \mu < \kappa; \\ \kappa, & \text{(c). neither (a) nor (b), and } \operatorname{cf}(\kappa) > \lambda; \\ \kappa^{\operatorname{cf}(\kappa)}, & \text{(d). neither (a) nor (b), and } \operatorname{cf}(\kappa) \leq \lambda; \end{cases}$$

Cardinal Exponentiation

Theorem 14

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Corollary 15

κ^λ is either 2^λ, or κ, or $\gimel(\mu)$ for some μ s.t. $\mathrm{cf}(\mu) \le \lambda < \mu$.

Proof

(b). $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$.

(c). If $\mathrm{cf}(\kappa) > \lambda$, every $f : \lambda \to \kappa$ is bounded in κ, so

$$\kappa^\lambda = \kappa \cdot \sup_{\alpha<\kappa} \alpha^\lambda = \kappa.$$

(d). If $\mathrm{cf}(\kappa) \leq \lambda$, then

$$\begin{aligned} \kappa^\lambda &= (\textstyle\sum_{i<\mathrm{cf}(\kappa)} \kappa_i)^\lambda \leq (\prod_{i<\mathrm{cf}(\kappa)} \kappa_i)^\lambda \\ &= (\textstyle\prod_{i<\mathrm{cf}(\kappa)} \kappa_i^\lambda) \leq (\sup_{i<\mathrm{cf}(\kappa)} \kappa_i^\lambda)^{\mathrm{cf}(\kappa)} \\ &\leq \kappa^{\mathrm{cf}(\kappa)}. \end{aligned}$$

The last inequality is because for all $\mu < \kappa$, $\mu^\lambda < \kappa$.

Singular Cardinal Hypothesis

- Easton (1970) showed that for regular cardinals κ, the value of 2^κ could be any $\aleph_\alpha$, as long as $\mathrm{cf}(\aleph_\alpha) > \kappa$.
  — GCH can fail at all regular cardinals.
- The **Singular Cardinals Hypothesis (SCH)** arose from the question of whether the least cardinal number for which the generalized continuum hypothesis (GCH) might fail could be a singular cardinal.

Singular Cardinal Hypothesis (two versions)

- If κ is any singular strong limit cardinal, then $2^\kappa = \kappa^+$.
- (Stronger) If κ is singular and $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$.

SCH is a consequence of GCH. It reduces values of κ^λ to values of the continuum function at regular cardinals.

Theorem 16

Assume SCH.

1. *If κ is a singular cardinal, then*
    1.1 $2^\kappa = 2^{<\kappa}$, *if the continuum function is eventually constant below κ.*
    1.2 $2^\kappa = (2^{<\kappa})^+$, *otherwise.*
2. *If κ, λ are infinite cardinals, then*
    2.1 *If $\kappa \leq 2^\lambda$, then $\kappa^\lambda = 2^\lambda$.*
    2.2 *If $2^\lambda < \kappa$ and $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.*
    2.3 *If $2^\lambda < \kappa$ and $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^+$.*

Homework

The rest problems are from Textbook Exercise for Chapter 5.

1. 5.4, 5.11-5.13

Assume AC.

2. 5.8 (only the case $\kappa = \omega$), 5.9
3. 5.17
   (Hint: Discuss the $\geq$-direction in two cases: λ is finite and λ is infinite.)
4. 5.18
   (Hint: $\aleph_\omega^{\aleph_1} \subset {}^{\aleph_0}\aleph_1 \cdot \prod_n {}^{\aleph_1}\aleph_{n+1}$ and Hausdroff formula.)