Elementary Set Theory

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Coming up next

Cardinal Numbers

Cardinal

Cardinal arithmetic, I

Cofinality

Cardinality

We use injective functions to compare the size of sets.

Definition 1

1. $X \approx Y$ iff there is a bijection from X to Y.
2. $X \preccurlyeq Y$ iff there is an injection from X to Y.[1]
3. $X \prec Y$ iff $X \preccurlyeq Y$ and $\neg(Y \preccurlyeq X)$.

[1] Note that empty function is injective.

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Easy to check:

Proposition 2

1. $\approx$ *is an equivalence relation.*
2. $\preccurlyeq$ *is transitive.*

[1] Note that empty function is injective.

Cantor-Bernstein

Next is a much deeper result

Theorem 3 (Cantor-Bernstein-Schröder)

Let X, Y be any two sets. Then

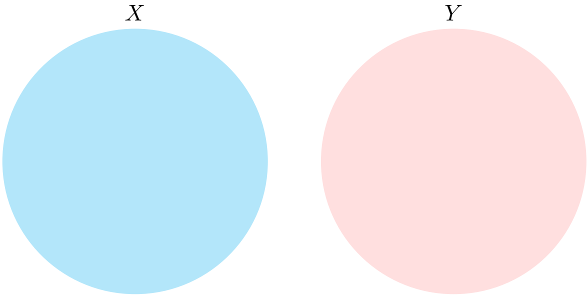
$$X \preccurlyeq Y \wedge Y \preccurlyeq X \implies X \approx Y.$$

A bit history

As it is often the case in mathematics, the name of this theorem does not truly reflect its history.

- The traditional name "Schröder-Bernstein" is based on two proofs published independently in 1898.
- Cantor is often added because he investigated it around 1870s, and first stated it as a theorem in 1895,
- while Schröder's name is often omitted because his proof turned out to be flawed
- and while the name of the mathematician who first proved it (Dedekind, 1887, 1897) is not connected with the theorem.

Proof of Cantor-Bernstein



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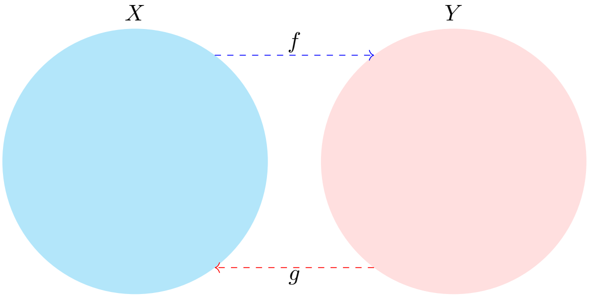

If f (or g) is onto, then we are done!
f (or g^{-1}) is a bijection.

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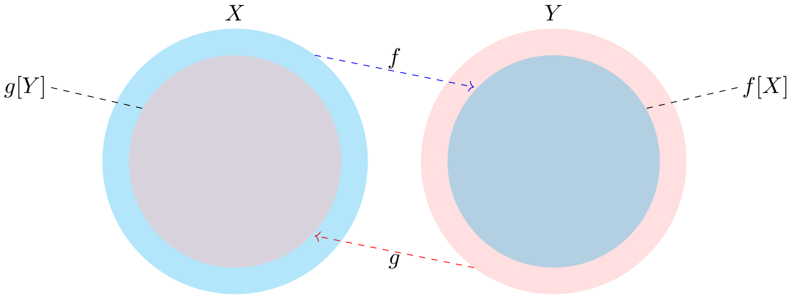

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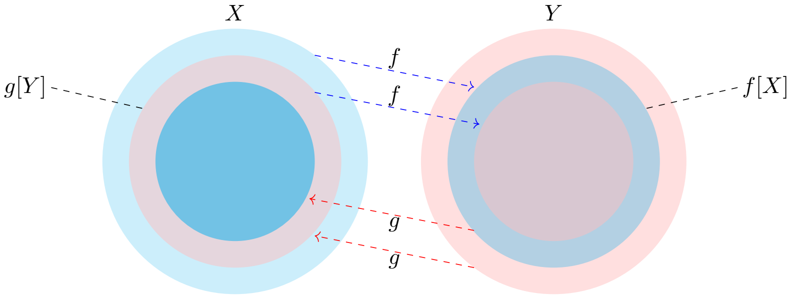

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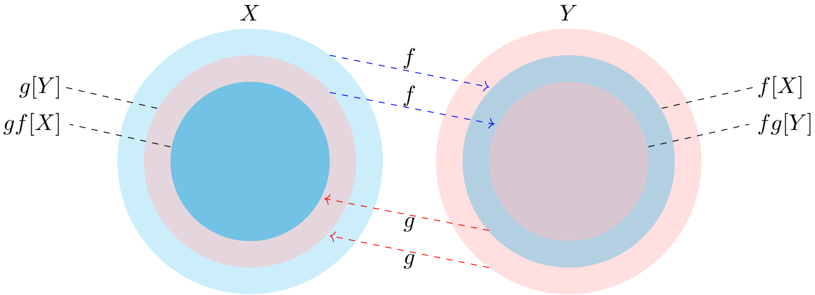

$g[Y] - gf[X] \approx Y - f[X]$ via g^{-1}

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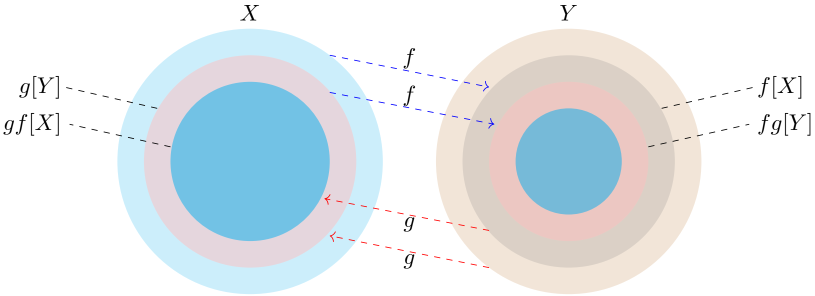
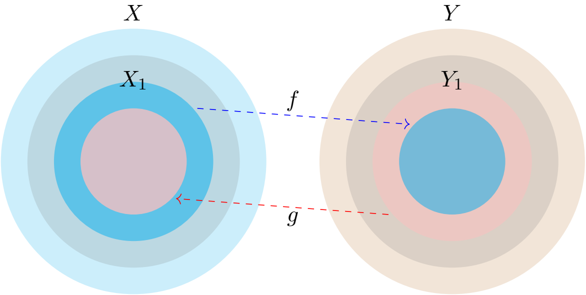


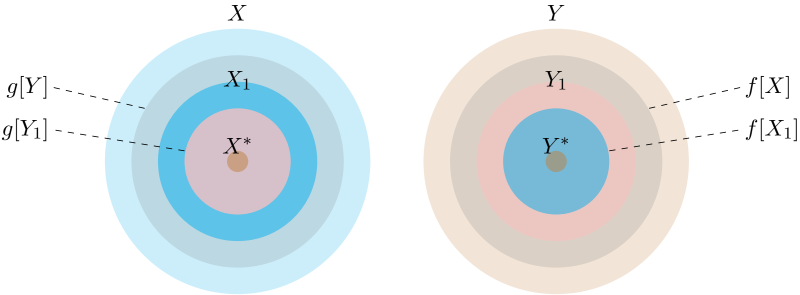
$g[Y] - gf[X] \approx Y - f[X]$ via g^{-1}
$X - g[Y] \approx f[X] - fg[Y]$ via f

Proof of Cantor-Bernstein



Thus $X - X_1 \approx Y - Y_1$,
also we have $f : X_1 \to Y_1$, $g : Y_1 \to X_1$.

Proof of Cantor-Bernstein



$$X \supset X_1 \supset X_2 \cdots \supset X_n \supset \cdots \qquad Y \supset Y_1 \supset Y_2 \cdots \supset Y_n \supset \cdots$$

Let $X^* = \bigcap_i X_i$ and $Y^* = \bigcap_i Y_i$. By induction, $X - X^* \approx Y - Y^*$

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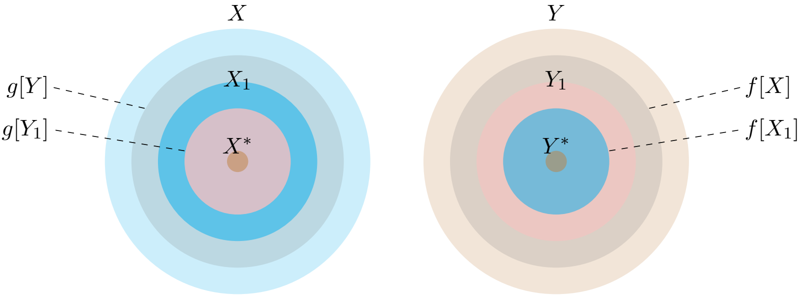

$X \supset X_1 \supset X_2 \cdots \supset X_n \supset \cdots$ $\qquad$ $Y \supset Y_1 \supset Y_2 \cdots \supset Y_n \supset \cdots$

Let $X^* = \bigcap_i X_i$ and $Y^* = \bigcap_i Y_i$. By induction, $X - X^* \approx Y - Y^*$

But $f[X] \supset Y_1 \supset f[X_1] \supset Y_2 \supset \cdots$. Thus $Y^* = \bigcap_i f[X_i] = f[X^*]$. $\square$

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Thus we can assign to each set X its **cardinal number** $|X|$ so that

$$X \approx Y \quad \text{iff} \quad |X| = |Y|$$

Cardinal numbers can be defined

- either via equivalence classes (need **Regularity**),
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- (von Neumann) or using ordinals (need AC).
  — We shall use this definition.

Cardinality

One determines the size of a finite set by counting it. More generally,

Definition 4

If X can be well-ordered, then $X \approx \alpha$ for some $\alpha \in \mathrm{Ord}$, and the least such α is called the **cardinality** of X, $|X|$.

Some simple facts.

- If $X \preccurlyeq \alpha$, then X can be well-ordered.
- $|\alpha| \leq \alpha$, for all $\alpha \in \mathrm{Ord}$.
- Under AC, every set can be well-ordered, so $|X|$ is defined for every X.

For the rest of this Chapter, we assume AC.

Cardinal

Definition 5

An ordinal α is a **cardinal** if $|\alpha| = \alpha$.

We use κ, λ, δ etc to denote cardinals.

Some simple facts.

- α is a cardinal iff $\forall \beta < \alpha \, (\beta \not\approx \alpha)$.
- If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.
- Every infinite cardinal is a limit ordinal.
- For every $n \in \omega$, $n \not\approx n + 1$.
- If $n \in \omega$, then for all α, $\alpha \approx n \to \alpha = n$.

Finite-Countable-Uncountable

Corollary 6

ω is a cardinal and each $n \in \omega$ is a cardinal.

Finite-Countable-Uncountable

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Definition 7

- X is **finite** iff $|X| < \omega$. **Infinite** means not finite.
- X is **countable** iff $|X| \leq \omega$. **Uncountable** means not countable.

Finite-Countable-Uncountable

Example

- Every $n \in \omega$ is finite.
- $\omega, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is countable. (To be discussed later)
- (Cantor). $\mathbb{R}$ is uncountable. (To be proved in Chapter 4)

Finite-Countable-Uncountable

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- Every $n \in \omega$ is finite.
- $\omega, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is countable. (To be discussed later)
- (Cantor). $\mathbb{R}$ is uncountable. (To be proved in Chapter 4)

Remark. One cannot prove on the basis of ZFC $-$ **Power Set** that uncountable sets exist. In fact, it is consistent with ZFC $-$ **Power Set** that the only infinite cardinal is ω.

Uncountable Cardinal

Before Cantor's proof of "$\mathbb{R}$ is uncountable", it was not known that there are more than one infinite cardinal.

Theorem 8

For any set X, $X \prec \mathscr{P}(X)$.

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<u>Proof</u>.

- Identify every set X with its characteristic function $C_X : X \to \{0,1\}$. Hence $\mathscr{P}(X) \approx {}^X 2$.
- Suppose $F : X \to {}^X 2$ is an arbitrary injection. Construct an $Z \in {}^X 2 - \mathrm{ran}(F)$ by diagonalization:

$$C_Z(x) = 1 \quad \text{iff} \quad C_{f(x)}(x) = 0,$$

i.e. $Z = \{x \in X \mid x \notin f(x)\}$. F is not surjective!

$\square$

In fact, Card is “unbounded” along Ord.

Corollary 9

For any set $S \subset \mathrm{Card}$, there is a cardinal κ s.t.

$$\forall \lambda \in S \, (\lambda < \kappa).$$

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For any set $S \subset \mathrm{Card}$, *there is a cardinal* κ *s.t.*

$$\forall \lambda \in S\,(\lambda < \kappa).$$

Without assume AC, the following is not easy to prove.

Theorem (Halbeisen and Shelah, 1994)

For all infinite set A,

$$\mathrm{fin}(A) \prec \mathscr{P}(A),$$

where $\mathrm{fin}(A) := \{x \subseteq A \mid x \text{ is finite}\}$.

Coming up next

Operations on Cardinals

The arithmetic operations on cardinals are defined as follows

Definition 10

1. $\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$
2. $\kappa \cdot \lambda = |\kappa \times \lambda|$.
3. $\kappa^{\lambda} = |{}^{\lambda}\kappa|$.

κ, λ on the right are referred as sets.

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Verify that these definitions are well defined.

We've shown that $|\mathscr{P}(X)| = 2^{|X|}$ and $\forall\kappa\,(\kappa < 2^\kappa)$.

Simple Facts About Cardinal Arithmetics

- Unlike the ordinal operations, $+$ and $\cdot$ are associative, commutative and distributive.
- $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
- $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
- $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.
- If $\kappa \leq \lambda$, then $\kappa + \mu \leq \lambda + \mu$, $\kappa \cdot \mu \leq \lambda \cdot \mu$ and $\kappa^\mu \leq \lambda^\mu$.
- If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.
- $\kappa^0 = 1$, $1^\kappa = 1$, $0^\kappa = 0$ if $\kappa > 0$.
- When $\kappa, \lambda < \omega$, $\kappa + \lambda$, $\kappa \cdot \lambda$ and κ^λ are the same as the corresponding operations on natural numbers.

Alephs

Since $\mathrm{Card} \subset \mathrm{Ord}$, Card is well-ordered and the elements of Card can be enumerated with Ord as indices. Consider infinite cardinals only.

Definition 11

For any cardinal κ, κ^+ denotes the least cardinal $> \kappa$. The Aleph function $\aleph$ is define by the transfinite recursion:

$$\begin{aligned} \aleph_0 &= \omega, \\ \aleph_{\alpha+1} &= \aleph_\alpha^+, \\ \aleph_\sigma &= \lim_{\alpha \to \sigma} \aleph_\alpha, \quad \lambda \text{ is a limit ordinal.} \end{aligned}$$

An infinite cardinal is called a **successor** cardinal if it is of the form $\aleph_{\alpha+1}$ for some α, otherwise is called a **limit** cardinal.

Alephs

$\aleph_\alpha$ are often written as ω_α.

This definition is legitimate due to the following facts

- For every κ, there is a λ s.t. $\kappa < \lambda$.
  Hence, κ^+ exists for every cardinal κ.
- For every set $S \subset \mathrm{Card}$, $\sup(S)$ is a cardinal.
  In particular, $\lim_{\alpha<\sigma} \aleph_\alpha$ is a cardinal.

These ensure that $\mathrm{dom}(\aleph) = \mathrm{Ord}$. Since for each $\alpha \in \mathrm{Ord}$,

$$\aleph_\alpha = \min\{\kappa \in \mathrm{Card} \mid \forall \beta < \alpha \, (\aleph_\beta < \kappa)\},$$

$\mathrm{ran}(\aleph) = \mathrm{Card} \setminus \omega$.

Alephs

<u>Remark</u>. The existence of κ^+ (κ infinite) can be shown without referring to 2^κ and AC:

$$\kappa^+ = \sup\{\text{ordertype}(\prec) \mid (\kappa, \prec) \text{ is a well-ordering.}\}$$

Alephs

<u>REMARK</u>. The existence of κ^+ (κ infinite) can be shown without referring to 2^κ and AC:

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Lemma 12

Card *is a proper class.*

In general, $A \subset \mathrm{Ord}$ is unbounded iff A is proper.

Cardinality of Sets,

Corollary 13

The following sets are countable:

- $\mathbb{Z}, \mathbb{Q}$ *are countable.*
- *The set of all algebraic numbers,* $\mathbb{A}$, *is countable.*

Assume that $|\mathbb{R}| = 2^{\aleph_0}$. *Then the following sets are of size* $2^{\aleph_0}$.

- *The set of all points in the* n*-dimensional space,* $\mathbb{R}^n$.
- *The set of all complex numbers,* $\mathbb{C}$.
- *The set of all* ω*-sequences of natural numbers,* ω^ω.
- *The set of all* ω*-sequences of real numbers,* $\mathbb{R}^\omega$

Lemma 14 (AC)

If $|A| < |B|$ *then* $|B - A| = |B|$.

In fact, one can prove the following without using AC.

Lemma 15

If $A \subseteq B$, $|A| = \aleph_0$ *and* $|B| = 2^{\aleph_0}$, *then* $|B - A| = 2^{\aleph_0}$.

HINT: View $A \subseteq \mathbb{R} \times \mathbb{R} \approx B$. $\exists r \in \mathbb{R}$ s.t. $A \cap (\{r\} \times \mathbb{R}) = \varnothing$.

Corollary 16

The set of irrationals, $\mathbb{R} - \mathbb{Q}$, *and the set of transcendental numbers,* $\mathbb{R} - \mathbb{A}$, *are of cardinality* $2^{\aleph_0}$.

Addition and Multiplication are trivial

Theorem 17 (AC)

Let κ, λ be infinite cardinals. Then

1. $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.
2. $|{}^{<\omega}\kappa| = \kappa$.

They follow from the lemma on next page.

Lemma 18 (AC)

For every $\alpha \in \mathrm{Ord}$, $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

PROOF OF THEOREM.

(1) follows immediately from Lemma 18. Below is for (2).

- For each $n \in \omega$, pick an injection $f_n : {}^n\kappa \to \kappa$.
- Combining them gives us an injection

$$f : \bigcup_n {}^n\kappa \to \omega \times \kappa, \quad f(\sigma) = (|\sigma|, f_{|\alpha|}(\sigma))$$

  whence $|{}^{<\omega}\kappa| \leq \omega \cdot \kappa = \kappa$. □

Next, we prove the lemma via pictures.

A Well-Ordering of $\kappa \times \kappa$, $\kappa = \aleph_{\delta+1}$

Proof of Lemma

$$\begin{aligned}(a_1, b_1) \prec (a_2, b_2) \leftrightarrow\, & \max(a_1, b_1) < \max(a_2, b_2) \\ & \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 < b_2) \\ & \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 = b_2 \wedge a_1 < a_2)\end{aligned}$$

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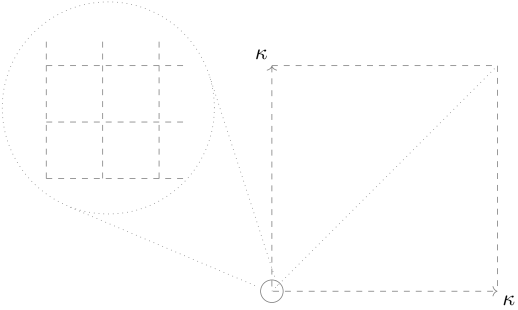

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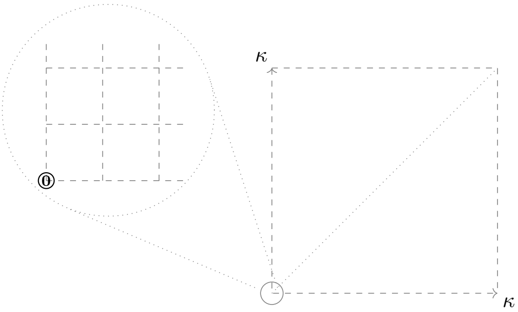

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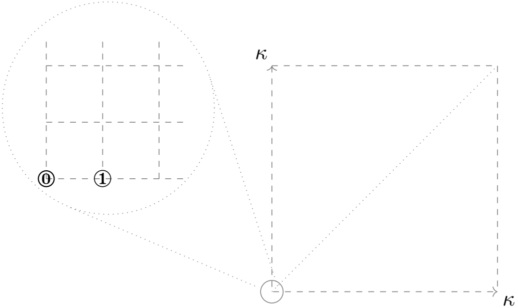

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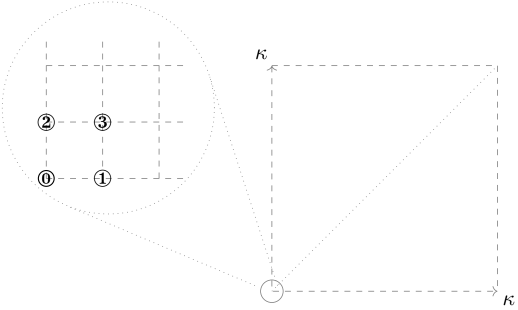

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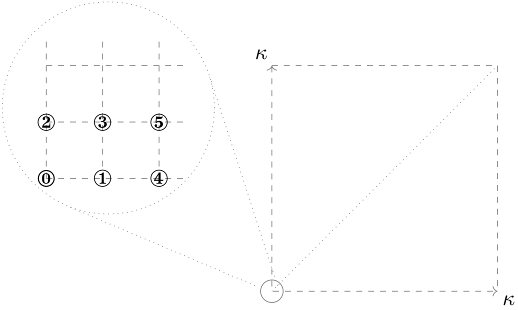

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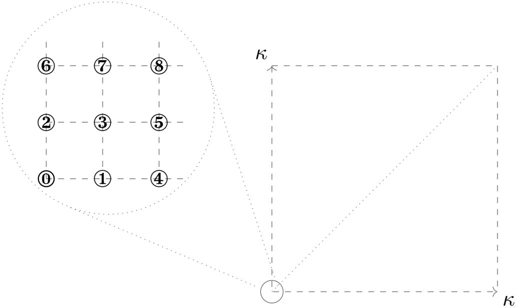

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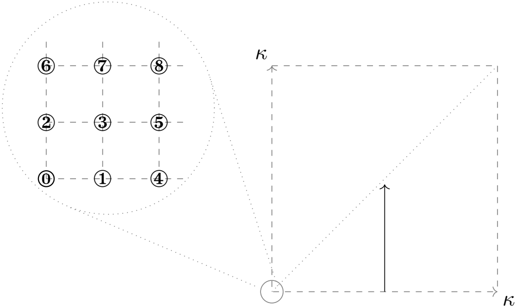

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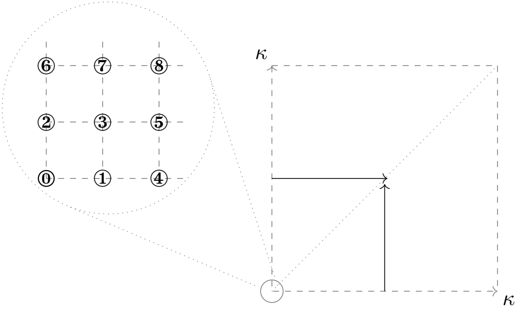

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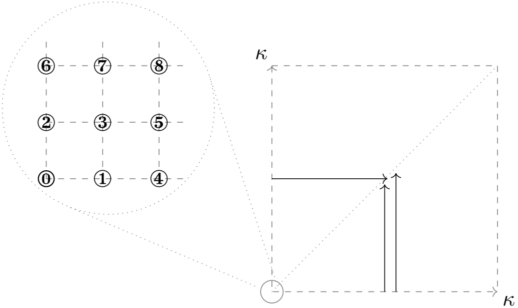

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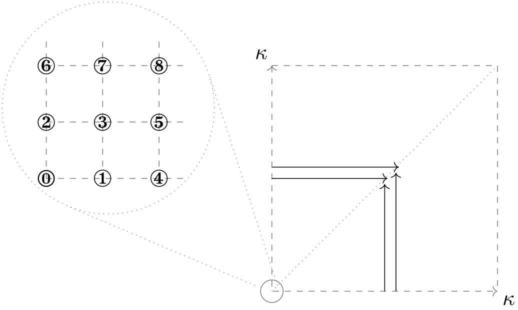

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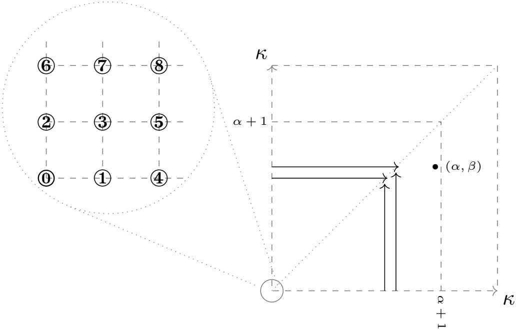
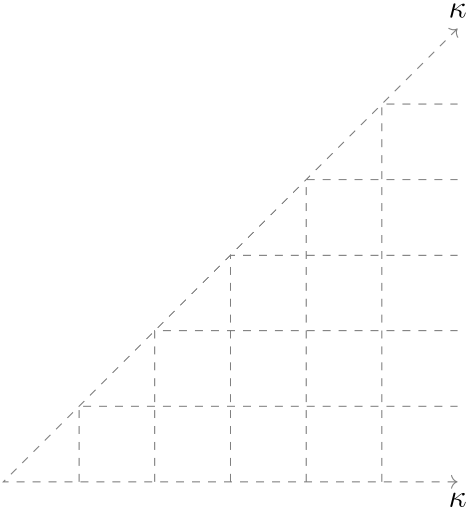
& \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 < b_2) \\
& \vee (\max(a_1, b_1) = \max(a_2, b_2) \wedge b_1 = b_2 \wedge a_1 < a_2)
\end{aligned}
$$

At any $(a, b) \in \aleph_{\delta+1} \times \aleph_{\delta+1}$, $|$the initial segment of $\prec$ up to $(a, b)| \leq \aleph_\delta$



Another ordering



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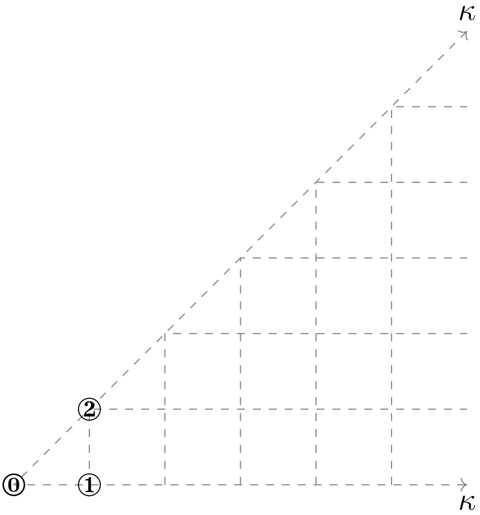

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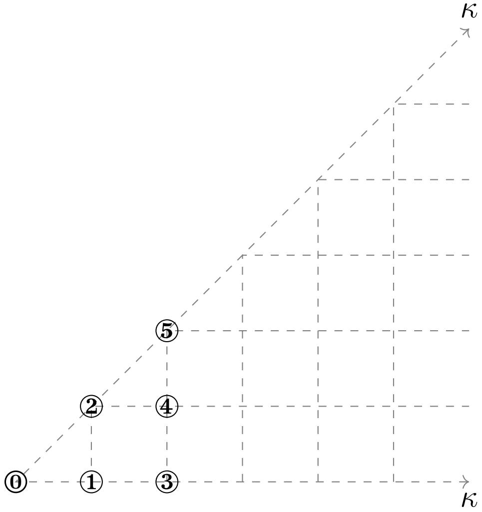

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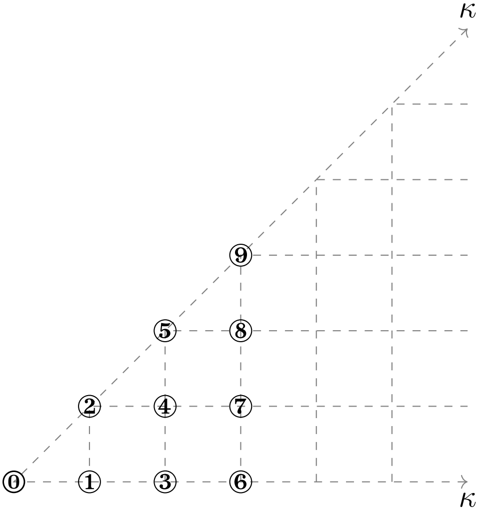

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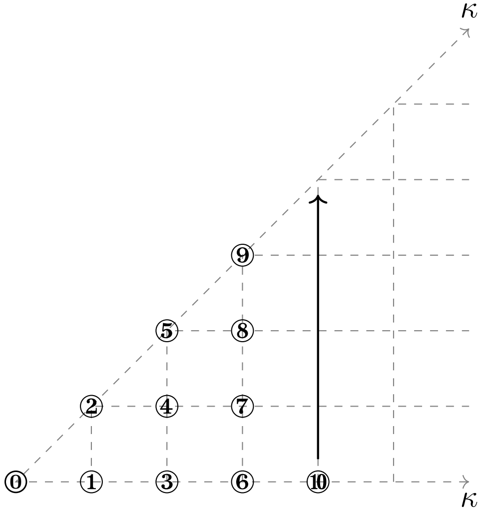

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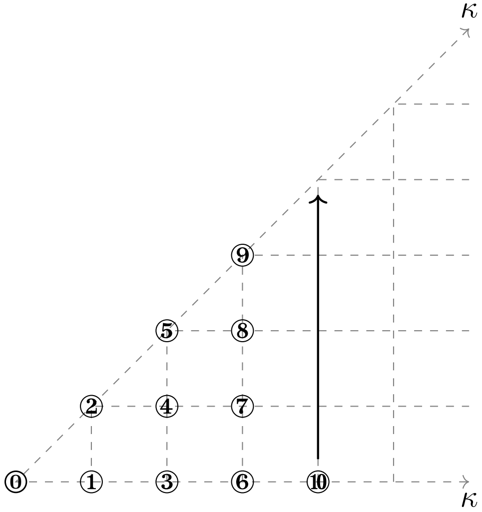
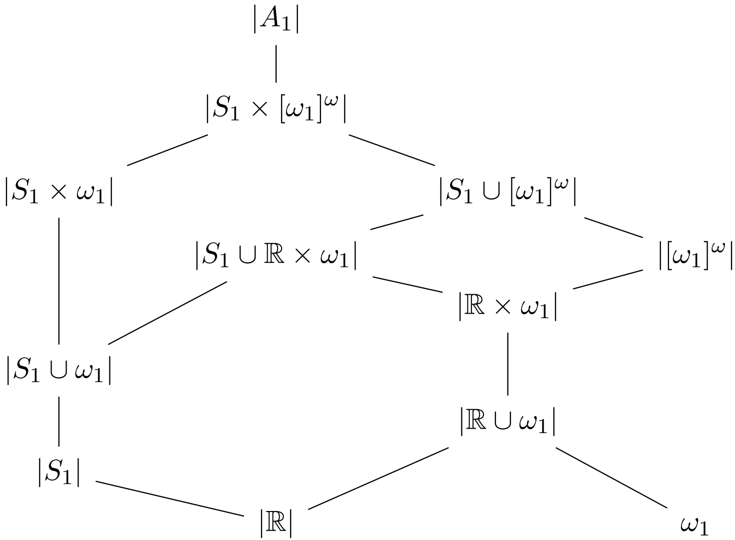


Homework

Write an explicit formula for this bijection.

Small cardinals, when no full AC (Woodin, 2006)



Impact of AC

AC is equivalent to the assertion that

"Every set can be well-ordered". (WO)

Many of the basic properties of cardinals need AC.

Write $X \preccurlyeq^* Y$ if $X = \varnothing$ or there is a surjection $f : Y \xrightarrow{\text{onto}} X$.

Lemma 19 (AC)

1. *If* $X \preccurlyeq^* Y$*, then* $X \preccurlyeq Y$*.*
2. *If* $\kappa \geq \omega$ *and* $X_\alpha \preccurlyeq \kappa$ *for all* $\alpha < \kappa$*, then* $\bigcup_{\alpha<\kappa} X_\alpha \preccurlyeq \kappa$*.*

PROOF.

1. Let $\prec$ well-orders Y. Suppose $f : Y \to X$ is surjective. Define $g : X \to Y$ as
$$g(x) = \prec\text{-least element of } f^{-1}(\{x\}).$$

2.
   - For each α, pick an injection $f_\alpha : X_\alpha \to \kappa$.

     f_α are selected via a well-ordering of $\mathscr{P}(\bigcup X_\alpha \times \kappa)$.
   - For $t \in \bigcup X_\alpha$, let $F(t) = (f_\alpha(t), \alpha)$, where
$$\alpha_t = \text{least } \alpha \text{ such that } t \in X_\alpha.$$

   Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection. $\pi \circ F$ works. □

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An important application of Lemma 19-2 is the **Downward Löwenheim-Skolem-Tarski Theorem** in model theory.

An Application, Definitions

Definition 20

1. An n-**ary operation** on X is a function $f : X^n \to X$ if $n > 0$, or an element of X if $n = 0$.
2. If $Y \subset X$, Y is **closed under** f iff $f[Y^n] \subset Y$ (or $f \in B$ when $n = 0$).
3. A **finitary operation** is an n-ary operation for some $n < \omega$.
4. If $\mathcal{E}$ is a set of finitary operations on X and $Y \subset X$, the closure of Y under $\mathcal{E}$, denoted as $\mathrm{cl}_{\mathcal{E}}(Y)$, is the least $Y^* \subset X$ such that $Y \subset Y^*$, and Y^* is closed under all the operations in $\mathcal{E}$.

An Application, Theorem

Theorem 21 (AC)

Let κ be an infinite cardinal. Suppose $Y \subset X$, $Y \leq \kappa$, and $\mathcal{E}$ is a set of $\leq \kappa$ finitary operations on X. Then $|\operatorname{cl}_{\mathcal{E}}(Y)| \leq \kappa$.

An Application, Theorem

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EXAMPLE. Every infinite group has a countably infinite subgroup.

An Application, Theorem

<u>PROOF</u>.

- Let $E_0 \subset \mathcal{E}$ be the set of all 0-ary operations in $\mathcal{E}$.
- Let $C_0 = Y \cup E_0$. We may assume that $\mathcal{E}$ has no 0-ary operations.
- By induction on $n < \omega$, define

$$C_{n+1} = C_n \cup (\bigcup \{ f[^k C_n] \mid f \in \mathcal{E},\ f \text{ is } k\text{-ary.} \})$$

- Take $C_\omega = \bigcup_n C_n$. Check that $C_\omega = \mathrm{cl}_{\mathcal{E}}(Y)$.

□

Homework (Midterm Quiz)

1. Prove the following statements.
   1.1 If $x \cap y = \varnothing$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.
   1.2 If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.
2. Ex.3.1-3.3 in textbook.
3. Prove that $\kappa^\kappa \leq 2^{\kappa \cdot \kappa}$.
4. If $A \preccurlyeq B$, then $A \preccurlyeq^* B$.
5. If $A \preccurlyeq^* B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.[2]
6. Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

[2] Don't forget the case $A = \varnothing$.

<u>REMARK</u>.

- Assuming AC, the converse of (4) is true (see Lemma 19).
- (6) is related to so called "Dedekind-infinite". (see textbook Ex.3.14-3.16)

Exercises*

1. α is called an **epsilon number** iff $\alpha = \omega^\alpha$ (ordinal exponentiation). Show that
   - the first epsilon number ε_0 is countable.
   - for each $\alpha \in \mathrm{Ord} - \{0\}$, $\aleph_\alpha$ is an epsilon number.
   - for each $\alpha \in \mathrm{Ord} - \{0\}$, the set of epsilon numbers is unbounded below $\aleph_\alpha$. Hence, there are $\aleph_\alpha$ epsilon numbers below $\aleph_\alpha$.
2. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

Continuum Hypothesis

Since Cantor could show (under AC) that $\aleph_1 \leq 2^{\aleph_0}$, and had no way producing cardinals between $\aleph_1$ and $2^{\aleph_0}$, he conjectured that

Continuum Hypothesis (CH)

$$\aleph_1 = 2^{\aleph_0}?$$

Continuum Hypothesis

More generally,

Generalized Continuum Hypothesis (GCH)

For every $\alpha \in \mathrm{Ord}$,

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}?$$

Continuum Hypothesis

More generally,

GENERALIZED CONTINUUM HYPOTHESIS (GCH)

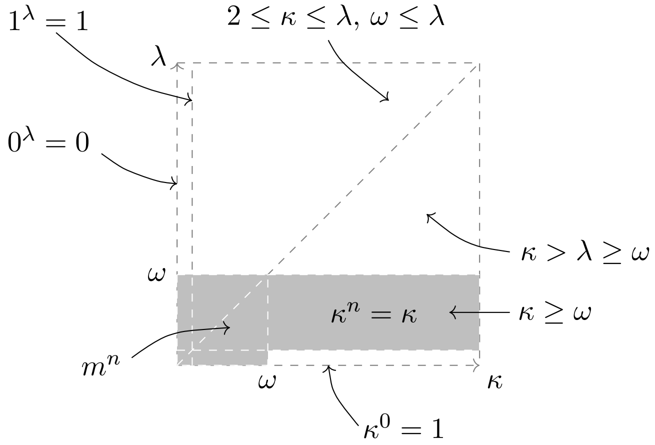
For every $\alpha \in \mathrm{Ord}$,

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}?$$

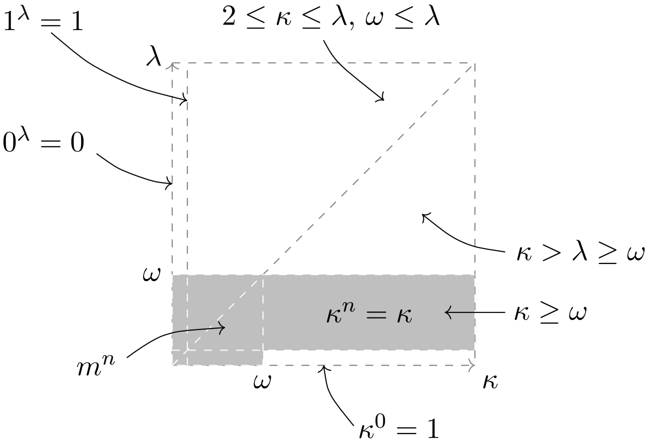
<u>REMARK</u>. Without AC, it is possible that $\aleph_1 \not\leq 2^{\aleph_0}$; however, one can still show that $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$, for every $\alpha \in \mathrm{Ord}$. (see textbook Ex.3.7-3.11)

Coming up next

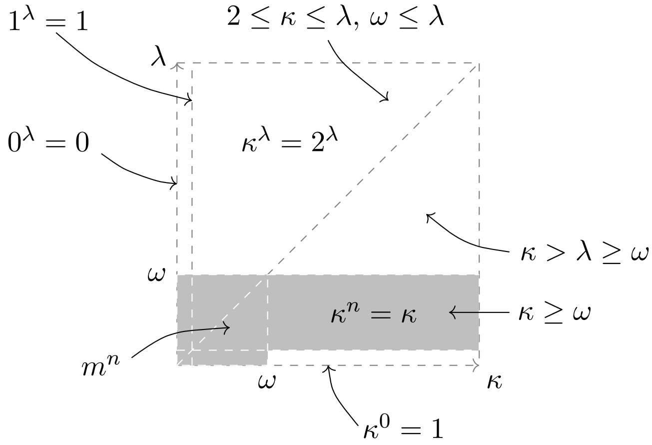
Exponentiation of Cardinals



Exponentiation of Cardinals



Exponentiation of Cardinals



Exponentiation of Cardinals

Lemma 22

If $\lambda \geq \omega$ *and* $2 \leq \kappa \leq \lambda$*, then* $\kappa^\lambda = 2^\lambda$*.*

Exponentiation of Cardinals

Lemma 22

If $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.

Under GCH, κ^λ can be easily computed, but the notion of **cofinality** is needed.

Cofinality

Definition 23

- If $f : \alpha \to \beta$, f maps α **cofinally (into β)** iff $\mathrm{ran}(f)$ is unbounded in β, i.e. $\forall b \in \beta, \exists a \in \alpha, f(a) \geq b$.
- The cofinality of β, $\mathrm{cf}(\beta)$, is the least α s.t. there is a map from α cofinally into β.

Cofinality

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- The cofinality of β, $\mathrm{cf}(\beta)$, is the least α s.t. there is a map from α cofinally into β.

Revise f to get a strictly increasing function $f|A : A \subset \alpha \to \beta$, and then $g_{f|A} : \mathrm{otp}(A) \to \beta$. Clearly $\mathrm{otp}(A) \leq \alpha$. Thus we have

Lemma 24

There is a cofinal map $f : \mathrm{cf}(\alpha) \to \alpha$ which is strictly increasing, i.e. $\xi < \eta \to f(\xi) < f(\eta)$.

In general, it is not true for $\gamma > \mathrm{cf}(\alpha)$.

Properties of $\mathrm{cf}(\cdot)$

Lemma 25

If α is a limit ordinal and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

PROOF.

"$\geq$": Let $\gamma = \mathrm{cf}(\alpha)$ and $g : \gamma \to \alpha$ be cofinal, then $f \circ g : \gamma \to \beta$ is cofinal. Thus $\gamma \geq \mathrm{cf}(\beta)$, as $\mathrm{cf}(\beta)$ is minimal.

"$\leq$": Let $\gamma = \mathrm{cf}(\beta)$ and $g : \gamma \to \beta$. A map $h : \gamma \to \alpha$ is defined as follows: for $a \in \gamma$,

$$h(a) = \min\{b \in \alpha \mid g(b) > f(a)\}.$$

h is well defined by the strictly-increasing-ness of f.
Verify that h is strictly increasing and cofinal.

Properties of $\mathrm{cf}(\cdot)$

Corollary 26

1. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
2. *If* α *is a limit ordinal, then* $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.

Properties of $\mathrm{cf}(\cdot)$

Corollary 26

1. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
2. *If α is a limit ordinal, then $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.*

Clearly,

- $\mathrm{cf}(\alpha) \leq \alpha$,
- if α is a successor, $\mathrm{cf}(\alpha) = 1$.
- if α is a limit ordinal, $\mathrm{cf}(\alpha)$ is a limit ordinal $\geq \omega$.

<u>EXAMPLE</u>. $\mathrm{cf}(\omega^n) = \mathrm{cf}(\aleph_\omega) = \omega$.

Regular Cardinal

Definition 27

α is **regular** iff α is a limit ordinal and $\mathrm{cf}(\alpha) = \alpha$. Otherwise, α is **singular**.

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Lemma 28

1. *For every limit ordinal α, $\mathrm{cf}(\alpha)$ is regular.*
   In particular, ω is regular.
2. *If α is regular, then α is a cardinal.*

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Lemma 28

1. *For every limit ordinal α, $\mathrm{cf}(\alpha)$ is regular.*
   In particular, ω is regular.
2. *If α is regular, then α is a cardinal.*

Proof of (2): Suppose $\gamma < \alpha$ and $\pi : \gamma \to \alpha$ were bijective. π would be unbounded, thus $\gamma \geq \mathrm{cf}(\alpha) = \alpha$. Contradiction!

Singular Cardinal

Lemma 29

Suppose $\kappa = \aleph_\alpha$ *for some* $\alpha \in \mathrm{Ord}$. κ *is singular iff there exists a cardinal* $\lambda < \kappa$ *and a family* $\{S_\xi \mid \xi < \lambda\}$ *of subsets of* κ *with each* $|S_\xi| < \kappa$, $\xi < \kappa$, *such that* $\kappa = \bigcup_{\xi<\lambda} S_\xi$. *The least cardinal* λ *that satisfies the condition is* $\mathrm{cf}(\kappa)$.

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PROOF.

"⇒": Suppose $\lambda < \kappa$ and $f : \lambda \to \kappa$ is cofinal. For each $\xi < \lambda$, let $S_\xi = f(\xi)$ (as subset of κ). Then $\kappa = \sup_{\xi<\lambda} S_\xi = \bigcup_{\xi<\lambda} S_\xi$. Moreover, least such $\lambda \leq \mathrm{cf}(\kappa)$.

"⇐": Let λ be least such. For $\delta < \lambda$, let $f(\delta) = \mathrm{otp}\big(\bigcup_{\xi<\delta} S_\xi\big)$. Each $f(\delta) \leq \kappa$. f is nondecreasing. By the minimality of λ, $f(\delta) < \kappa$, for $\delta < \lambda$. Clearly, $\kappa_f := \sup_{\delta<\lambda} f(\delta) \leq \kappa$.

$$\kappa = \left|\bigcup_{\xi<\lambda} S_\xi\right| \leq \lambda \times \kappa_f = \max\{\lambda, \kappa_f\}.$$

Since $\lambda < \kappa$, $\kappa = \kappa_f$. f is cofinal in κ, so $\lambda \geq \mathrm{cf}(\kappa)$.

Singular Cardinal

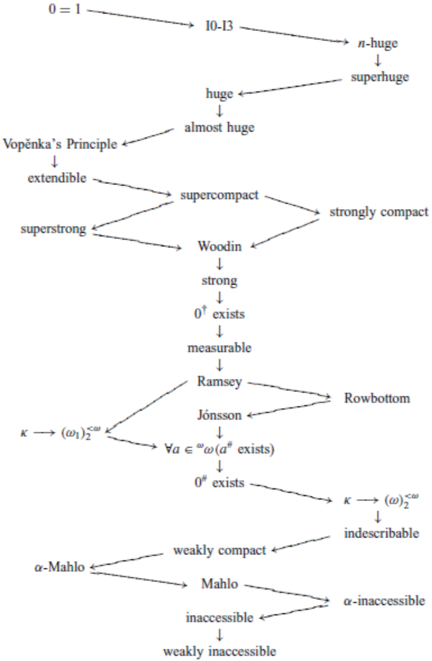
Corollary 30 (AC)

For each α, $\aleph_{\alpha+1}$ is regular.

<u>Remark</u>. Without AC, it is consistent that $\mathrm{cf}(\omega_1) = \omega$, i.e. ω_1 is a countable union of countable sets. In contrast, in ZF one can show that ω_2 cannot be a countable union of countable sets.

Large cardinals

- There are arbitrarily large singular cardinals.
  For each α, $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$.
- It is unknown whether one can prove in ZF that there exists a cardinal κ with $\mathrm{cf}(\kappa) > \omega$.
- (Hausdroff, 1908) κ is **weakly inaccessible** if κ is a regular limit cardinal ($\forall \lambda < \kappa$, $\lambda^+ < \kappa$). Every weak inaccessible is a fix point of the $\aleph$-sequence ($\aleph_\alpha = \alpha$). The first weakly inaccessible cardinal is rather large. And its existence is independent of ZFC.
- (Sierpiński-Tarski, Zermelo, 1930). κ is **strongly inaccessible** iff $\kappa > \omega$, κ is regular and $\forall \lambda < \kappa \, (2^\lambda < \kappa)$. Strong inaccessibles are weak inaccessibles. Under GCH, these two notions coincide.



From
The Higher Infinite,
by A. Kanamori

König's Theorem

Theorem 31 (König)

If κ *is an infinite cardinal then* $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.

König's Theorem

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Proof. Key: "No injection is surjective".

- Let $\{f_\alpha \mid \alpha < \kappa\}$ be an arbitrary subset of ${}^{\mathrm{cf}(\kappa)}\kappa$ of size κ.
- Construct an $f : \mathrm{cf}(\kappa) \to \kappa$ different from all f_α, $\alpha < \kappa$.
- Suppose $\kappa = \lim_{\xi < \mathrm{cf}(\kappa)} \alpha_\xi$. For each $\xi < \mathrm{cf}(\kappa)$, $f(\xi)$ is selected to ensure that at ξ, $f \neq f_\alpha$ for all $\alpha < \alpha_\xi$. □

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Corollary 32 (AC)

If $\lambda \geq \omega$, then $\mathrm{cf}(2^\lambda) > \lambda$.

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Corollary 32 (AC)

If $\lambda \geq \omega$, then $\mathrm{cf}(2^\lambda) > \lambda$.

Hint: Otherwise, $2^\lambda < (2^\lambda)^{\mathrm{cf}(2^\lambda)} \leq (2^\lambda)^\lambda = 2^\lambda$.

Further results in cardinal arithmetics will appear in Chapter 5.