Elementary Set Theory

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Coming up next

Language of Set Theory

- **Symbols:**

$$\mathcal{L} = \mathcal{L}_{\{\hat{\in}\}} \text{ with the equality predicate } \hat{=}.$$

- **Variables:** Lower case English letters [1]

$$x, y, z, u, v, w, x_i, y_j, z_k \text{ etc.}$$

  are variable symbols.

- **Connectives:**

$$\neg, \quad \wedge, \quad \vee, \quad \rightarrow, \quad \leftrightarrow$$

- **Quantifiers:**

$$\forall, \qquad \exists$$

[1] In informal cases, we may use capital letters, script fonts to denote sets of different types.

Formulas of Set Theory

Definition

- A *atomic formulas* of Set Theory is a sequence of the form $(x \mathrel{\hat{\in}} y),\ (x \mathrel{\hat{=}} y) \in \mathcal{L}_{\{\hat{\in}\}}$, where x, y are variable symbols.
- A *formulas* of Set Theory is a finite sequence of symbols in $\mathcal{L}_{\{\hat{\in}\}}$ built up on a finite set of atomic formulas by finite applications of operators below:
  - $\varphi \mapsto (\neg\varphi)$
  - $\varphi, \psi \mapsto (\varphi \wedge \psi)$
  - $\varphi, \psi \mapsto (\varphi \vee \psi)$
  - $\varphi, \psi \mapsto (\varphi \rightarrow \psi)$
  - $\varphi, \psi \mapsto (\varphi \leftrightarrow \psi)$
  - $\varphi \mapsto \forall x\, \varphi$
  - $\psi \mapsto \exists x\, \psi$

  where φ, ψ are formulas.

Conventions

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- Formulas with no free variables are called **sentences**.
- $\varphi(x, p)$ denotes a formula in the language $\mathcal{L}_{\{\hat{\in}, p\}}$, where p is a manually added constant symbol, called a parameter.
- Often use a bar notation such as $\bar{x}$ to denote a finite sequence $\langle x_1, \ldots, x_n \rangle$ (or $(x_1, \ldots, x_n)$) when the precise subscription is not important.

Coming up next

Sets & Classes

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Many of the set operators that we shall define later are applicable to classes as well.

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$$\exists \boldsymbol{z} \forall u \, [u \mathrel{\hat{\in}} z \leftrightarrow u \mathrel{\hat{=}} x \vee u \mathrel{\hat{=}} y]$$

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- **Infinity.** There exists an infinite set.

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Regularity and **Choice** will be discussed later in Chapter 5 and 6, respectively.

Coming up next

Sets exist.

Implicitly, we assume that

Axiom 0 (Set Existence)

$$\exists \boldsymbol{x} \, (x \doteq x).$$

This says that our universe of sets is non-void.

Extensionality

Axiom 1 (Extensionality)

$$x \mathrel{\hat{=}} y \leftrightarrow \forall z \, (z \mathrel{\hat{\in}} x \leftrightarrow z \mathrel{\hat{\in}} y)$$

This says that a set is determined by its elements.
"$\rightarrow$" is an axiom of first-order logic.
"$\leftarrow$" is what accounts for this axiom.

Comprehension

Axiom 2 (Comprehension Schema)

For each formula $\varphi(u, p)$ (without y free) and each x,

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$$\exists \boldsymbol{y} \forall u \, [u \mathrel{\hat{\in}} y \leftrightarrow u \mathrel{\hat{\in}} x \land \varphi(u, p)].$$

- This y is unique by **Extensionality**. And we denote this by

$$\{u \mid u \in x \land \varphi(u)\} \quad \text{or} \quad \{u \in x \mid \varphi(u)\}.$$

Comprehension, II

- The restriction on y not being free in φ eliminates self-referential definitions of sets, for example,

  $$\exists y \forall x \, [x \mathrel{\hat{\in}} y \leftrightarrow x \mathrel{\hat{\in}} z \land \neg(x \mathrel{\hat{\in}} y)]$$

  would be inconsistent with the existence of a $z \neq \varnothing$.

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  would be inconsistent with the existence of a $z \neq \varnothing$.
- Note that this schema yields an infinite collection of axioms — one for each φ.
- By AXIOM 0-2, we can define the notion of empty set, $\varnothing$.

Comprehension, III

Definition 2

$\varnothing$ is the unique set y such that $\forall x \, \neg(x \mathrel{\hat{\in}} y)$.

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Exercise

Justify the above definition, i.e. show that $\varnothing$ exists and is unique.

Comprehension, IV

- $\varnothing$ is the only set which can be proved to exist from AXIOM 0-2.

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<u>Proof</u>.

Consider the structure $(\{\varnothing\}, \in)$, where $\in$ is an empty binary relation. Axiom 0-2 hold in this structure, but so does $\psi \equiv \forall y\,(y \mathrel{\hat{=}} \varnothing)$. So Axiom 0-2 cannot refute ψ. □

Comprehension, V

- We can also prove that there is no universal set. In other word, the universal class $V = \{x \mid x \mathrel{\hat{=}} x\}$ is a proper class.

Theorem 3

$$\neg \exists z \forall x \, (x \mathrel{\hat{\in}} z).$$

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$$\neg \exists z \forall x \, (x \mathrel{\hat{\in}} z).$$

Suppose NOT. By **Comprehension**, form

$$\{x \in z \mid x \notin x\} = \{x \mid x \notin x\}.$$

This would lead to the Russell's paradox.

Comprehension, VI

- Let $A \subseteq B$ abbreviate

$$\forall x\,(x \mathrel{\hat{\in}} A \rightarrow x \mathrel{\hat{\in}} B),$$

  and say A is a *subclass* of B. So $A \subseteq A$ and $\varnothing \subseteq A$. If $A \subseteq B$ and $A \neq B$, then A is a *proper subclass* of B.

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- Let $(\forall x \mathrel{\hat{\in}} y) \, \varphi$ abbreviate $\forall x \, (x \mathrel{\hat{\in}} y \rightarrow \varphi)$, and $(\exists x \mathrel{\hat{\in}} y) \, \varphi$ abbreviate $\exists x \, (x \mathrel{\hat{\in}} y \wedge \varphi)$.

Comprehension, VII

- Another consequence of **Comprehension** is that the following two classes are sets,

$$\{u \in X \mid u \in Y\}$$
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  Then $\bigcap C$ is a set, and $X \cap Y = \bigcap\{X, Y\}$. If $X \cap Y = \varnothing$, we say X and Y are *disjoint*.

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AXIOM 3 (Pairing)

$$\forall x \forall y \exists \boldsymbol{z} \, (x \mathrel{\hat{\in}} z \wedge y \mathrel{\hat{\in}} z).$$

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- By **Extensionality**, the set whose only elements are precisely x and y is unique. We call this set $\{x, y\}$. Then $\{x\} = \{x, x\}$ is the set whose unique element is x.
- $\{x, y\}$ is the *unordered pair* of x and y. We use $(x, y) = \{\{x\}, \{x, y\}\}$ to denote the *ordered pair* of x and y.

Pairing, III

Exercise

Verify that

$$\forall x \forall y \forall x' \forall y' ((x, y) = (x', y') \to x = x' \land y = y').$$

By induction, *ordered n-tuples* can be defined as follows:

$$(x_1, \ldots, x_{n+1}) = ((x_1, \ldots, x_n), x_{n+1}).$$

Show that two ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equal iff $x_i = y_i$ for all $i = 1, \ldots, n$.

Pairing, IV

- Using n-tuples, **Comprehension** can be put in a more general form: Let $\psi(u, p_1, \ldots, p_n)$ be a formula. Then $\forall X \forall p_1 \ldots \forall p_n \exists Y \forall u\, [u \mathrel{\hat{\in}} Y \leftrightarrow u \mathrel{\hat{\in}} X \wedge \psi(u, p_1, \ldots, p_n)]$.

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  Idea: Let $\varphi(u, p)$ be the following formula

  $$\exists p_1 \ldots \exists p_n \, [p \mathrel{\hat{=}} (p_1, \ldots, p_n) \wedge \psi(u, p_1, \ldots, p_n)].$$

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Axiom 4

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- This is equivalent to the $\leftrightarrow$ version, by **Comprehension.**
- By Axiom 4, we can define the *union* of X as follows:

$$\bigcup X = \{u \mid (\exists x \in X)(u \in x)\}.$$

Union, II

- Consecutively, we can define

$$X \cup Y \equiv \bigcup\{X, Y\} \quad \text{and} \quad X \cup Y \cup Z \equiv (X \cup Y) \cup Z, \quad etc.$$

$$\{a, b, c\} \equiv \{a, b\} \cup \{c\} \quad \text{and} \quad \{a_1, \ldots, a_n\} \equiv \{a_1\} \cup \cdots \cup \{a_n\}.$$

Union, II

- Consequently, we can define

  $X \cup Y \equiv \bigcup\{X, Y\}$ and $X \cup Y \cup Z \equiv (X \cup Y) \cup Z, \quad etc.$

  $\{a, b, c\} \equiv \{a, b\} \cup \{c\}$ and $\{a_1, \ldots, a_n\} \equiv \{a_1\} \cup \cdots \cup \{a_n\}.$

- The symmetric difference of X and Y is

$$X \,\Delta\, Y \equiv (X - Y) \cup (Y - X).$$

Power Set

Axiom 5

$$\forall x \exists Y \forall u \left[u \mathrel{\hat{\in}} Y \leftrightarrow \forall v \left(v \mathrel{\hat{\in}} u \to v \mathrel{\hat{\in}} x \right) \right]$$

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- By Axiom 5, we can define the *power set* of X as follows:

$$\mathscr{P}(X) = \{u \mid u \subseteq X\}.$$

Power Set

AXIOM 5

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Using the **Power Set Axiom**, we can define: product, relation, function, etc.

Product

- The *product* of X and Y is the set

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The existence of $X \times Y$ is due to the fact that

$$X \times Y \subseteq \mathscr{P}\mathscr{P}(X \cup Y).$$

Product, II

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Also write $X^n \equiv X \times \cdots \times X$. ($n$ times)

Relations

- R is an *n-ary (or n-placed) relation* on X if $R \subseteq X^n$. So an n-ary relation is a set of n-tuples. The following notations are often used:

$$R(x_1, \ldots, x_n) \quad \text{and} \quad x\,R\,y \quad \text{(when } R \text{ is binary).}$$

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  $R(x_1, \ldots, x_n)$ and $x \, R \, y$ (when R is binary).

- If R is a binary relation, the *domain, range* and *field* of R are

$$\begin{aligned} \mathrm{dom}(R) &= \{u \mid \exists v \, (u, v) \in R\} \\ \mathrm{ran}(R) &= \{v \mid \exists u \, (u, v) \in R\} \\ \mathrm{field}(R) &= \mathrm{dom}(R) \cup \mathrm{ran}(R) \end{aligned}$$

Exercise

Show that if R is a set, then $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets.

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(In fact, $\mathrm{field}(R) = \bigcup\bigcup R$.)

Relation, II

Let R be a binary relation,

- The **image** of A under R is the set

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- Let S be a binary relation, the **composition** of R and S is

$$S \circ R \equiv \{(x,z) \mid \exists y\,((x,y) \in R \wedge (y,z) \in S\}$$

Function

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- The class of all functions from X to Y are denoted as ${}^{X}Y$.

Exercise

Show that if X, Y are sets then ${}^{X}Y$ is also a set.

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- If f, g are functions such that $\mathrm{ran}(g) \subseteq \mathrm{dom}(f)$, then the *composition* of f and g is the function $f \circ g$ such that $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and for every $x \in \mathrm{dom}(g)$,
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A function is often called a *mapping*, *correspondence*, and similarly a set is called a *family* or a *collection*.

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- In Chapter 2, we will define a more useful notion of product of sets in terms of functions.

Equivalence Relation & Partition

- A binary relation E on X is an *equivalence relation* if it satisfies

$$x \, E \, x, \qquad \text{reflexive}$$

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- A set $A \subseteq X$ is called a **set of representatives** for the equivalence relation E (or for the partition P of X) if for every equivalence class C (or for every $C \in P$),

$$A \cap C = \{a_c\}, \quad \text{for some } a_c \in C.$$

Equivalence Relation & Partition, III

Theorem 4

- *If* E *is an equivalence relation on* X*, then*
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is a partition P_E *on* X*. Conversely,*

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- *If E is an equivalence relation on X and $P = X/_E$, then $E_P = E$; and If P is a partition on X and E_P is the corresponding equivalence, then $X/_{E_P} = P$.*

REMARK. All the above notions on relations and functions are also applicable to classes.

◂ Axiom

Models of set theory axioms

We've seen a model of Axiom 0-2: $(\{\varnothing\}, \in)$.

$$V_0 = \varnothing;$$
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\begin{aligned}
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- Note that $n < m \implies V_n \subset V_m$.
- $(V_1, \in) \models$ Axiom 0-2. $\models$ is read as "is a model of".
- $(V_n, \in) \models$ ZFC − Pairing − Power − Infinite, i.e. Axiom 0-2 + Union + Replacement + Regularity + Choice, for all $n \in \mathbb{N}^+$.
- $(\bigcup_{n\in\mathbb{N}} V_n, \in) \models$ ZFC − Infinite.

Coming up next

Language of Set Theory

Axioms of Set Theory (ZFC)

Axiom 0-2

Axiom 3-5

Axiom 6, 7

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To give a precise formulation of the **Axiom of Infinity**, we need the notion of finiteness, which normally uses the notion of a natural number (see Chapter 2). Here we give an alternative approach which mentions no numbers.

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There exists an inductive set, i.e.

$$\exists \boldsymbol{x}\,[\varnothing \mathrel{\hat{\in}} x \wedge \forall u\,(u \mathrel{\hat{\in}} x \to u \cup \{u\} \mathrel{\hat{\in}} x)]$$

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- An inductive set is infinite.
- An inductive set exists if there exists an infinite set.

Replacement

Axiom 7 (Replacement Schema)

For each formula $\varphi(x, y, p)$, where p is a parameter,

$$\forall x \forall y \forall z \, [\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y \hat{=} z]$$
$$\rightarrow \forall u \exists v \forall z \, [z \hat{\in} v \leftrightarrow (\exists w \hat{\in} u) \, \varphi(w, z, p)]$$

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The first part of the formula is often abbreviated as

$$\forall x \exists ! y \, \varphi(x, y, p)$$

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Remark

Note that the **Replacement Schema** can take you “out of” the set u when forming the set v. The elements of v need not be elements of u. By contrast, the **Separation Schema** yields new sets consisting only of those elements of a given set u which satisfy a certain condition φ.

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4. Is it always true that $((a, b), c) = (a, (b, c))$? Can we use the second set to define ordered triples?

5. Give an alternative definition of ordered pairs. Compare the advantages and disadvantages of these definitions.

Questions, II

6. Is it true that $X \times Y = Y \times X$?
   What about $(X \times Y) \times Z = X \times (Y \times Z)$?

7. Let $S \neq \varnothing$ and A be sets.
   - $a.$ (Generalized Distributive Law)
     Set $T_1 = \{X \cap A \mid X \in S\}$, and prove
     $$A \cap \bigcup S = \bigcup T_1.$$
   - $b.$ (Generalized De Morgan Laws)
     Set $T_2 = \{A - X \mid X \in S\}$, and prove
     $$A - \bigcup S = \bigcap T_2, \qquad A - \bigcap S = \bigcup T_2$$

Homework

Exercise 1

1. Using only $\hat{\in}$ and $\hat{=}$ to express the following formulas:
   - $z \mathrel{\hat{=}} ((x, y), (u, v))$
   - $\forall x\, [\neg(x \mathrel{\hat{=}} \varnothing) \to (\exists y \mathrel{\hat{\in}} x)\, (x \cap y \mathrel{\hat{=}} \varnothing)]$
   - $\forall u\, [\forall x \exists y\, (x, y) \mathrel{\hat{\in}} u \to \exists f \forall x\, (x, f(x)) \mathrel{\hat{\in}} u]$.
2. Suppose that R, S are two binary relations (as sets). Show that R_{-1} and $S \circ R$ exist, where
   $$S \circ R = \{(x, z) \mid \exists y\, ((x, y) \in R \wedge (y, z) \in S\}$$

Exercises in Textbook

1.2. There is no set X such that $\mathscr{P}(X) \subseteq X$.

Homework

Let $N = \bigcap\{X \mid X \text{ is inductive}\}$. N is the smallest inductive set. Let us use the following notation:

$$0 = \varnothing, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}$$

If $n \in N$, let $n + 1 = n \cup \{n\}$. And for $n, m \in N$,

$$n < m \leftrightarrow n \in m$$

A set T is *transitive* if $x \in T$ implies $x \subseteq T$.

1.3. If X is inductive, then the set

$$\{x \in X \mid x \subseteq X\}$$

is inductive. Hence N is transitive, and for each $n \in N$, $n = \{m \in N \mid m < n\}$.

Homework

1.4. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive}\}$$

is inductive. Hence every $n \in N$ is transitive.

1.5. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive and } x \notin x\}$$

is inductive. Hence $n \notin n$ and $n \neq n+1$ for each $n \in N$.

Homework

1.6. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\text{-minimal element}\}$$

is inductive.
(t is *$\in$-minimal* in z if there is no $s \in z$ such that $s \in t$.)

1.7. Every nonempty $X \subseteq N$ has an $\in$-minimal element.

Homework

1.8. If X is inductive then so is

$$\{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}.$$

Hence each $n \neq \varnothing$ is $m + 1$ for some m.

1.9. (**Induction**). Let A be a subset of N such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = N$.

Hints for homework 1.2

Prove by contradiction.
Assume $\mathscr{P}(X) \subseteq X$ for some X.

- HINT 1: $X \in X$. This leads an infinite $\in$-descanding chain $\cdots \in X \in X \in X$. Thus the nonempty set $\{X\}$ contains no $\in$-minimal element, contradicting to Wellfoundedness Axiom.
- HINT 2: Consider $Z = \{x \in X \mid x \notin x\}$.
  $Z \subset X \to Z \in X$. But then
  $$Z \in Z \iff Z \notin Z.$$
  Contradiction!

Hints for homework 1.3

Let $E = \{x \in X \mid x \subseteq X\}$. Begin with that E is inductive.

- X is inductive, so $\varnothing \in X$. $\varnothing$ contains no element, so $\forall t(t \in \varnothing \to t \in X)$, i.e. $\varnothing \subseteq X$. Therefore, $\varnothing \in E$.
- For $x \in E$, $x \in X \wedge x \subseteq X$, thus $x \cup \{x\} \subseteq X$. Besides, $x \in X \to x \cup \{x\} \in X$. Hence $x \cup \{x\} \in E$.

Next is to show that N is transitive.

- Let $E_0 = \{x \in N \mid x \subseteq N\}$. Then $E_0 \subseteq N$.
- N is inductive, so is E_0, thus $N \subseteq E_0$. That means $E_0 = N$, thus N is transitive.

Last, we show that $n = \{m \in N \mid m < n\}$

- $\{m \in N \mid m < n\} =_{\text{def}} \{m \in N \mid m \in n\} \subseteq n$.
- As $n \in N$ and N is transitive, we have $n \subseteq N$. Then $m \in n \to m \in N$. Therefore, $n \subseteq \{m \in N \mid m \in n\}$. Hence $n = \{m \in N \mid m < n\}$.