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PROJECTIVE PREWELLORDERINGS VS PROJECTIVE WELLFOUNDED RELATIONS

XIANGHUI SHI

Abstract. We show that it is relatively consistent with ZFC that there is a projective wellfounded relation with rank higher than all projective prewellorderings.

§1. Introduction. Let $\omega = \{0, 1, 2, \dots\}$ be the set of natural numbers and $\mathbb{R}$ be the set of functions from ω to ω or simply the set of reals. *Product spaces* are spaces of the form

$$\mathfrak{X} = X_1 \times \cdots \times X_k,$$

where each X_i is ω or $\mathbb{R}$. Subsets of product spaces are called *pointsets*, and a *pointclass* is a class of pointsets, usually in all the product spaces.

Let X be a product space. A binary relation $\prec \subseteq X \times X$ is *wellfounded* if every nonempty subset $Y \subseteq X$ has a $\prec$-minimal element. Otherwise, we call $\prec$ *ill-founded*. For every wellfounded relation $\prec$, let

$$\text{field}(\prec) = \{x \mid \exists y\ (x \prec y) \text{ or } \exists y\ (y \prec x)\}.$$

In general, it is not necessary that $\text{field}(\prec) = X$. But in this paper, it makes no difference to assume that the equality holds for every wellfounded relation. For every wellfounded relation $\prec$, one can associate a rank function

$$\mu_\prec : \text{field}(\prec) \to \text{Ord}$$

as follows, for every $x \in \text{field}(\prec)$, $\mu_\prec(x) = \sup\{\mu_\prec(y) + 1 \mid y \prec x\}$, in particular $\mu(x) = 0$ if x is $\prec$-minimal. The rank of $\prec$ is given by

$$\text{rank}(\prec) = \sup\{\mu_\prec(x) + 1 \mid x \in \text{field}(\prec)\}.$$

A binary relation $\preccurlyeq \subseteq X \times X$ is a *prewellordering* if it is

- reflexive, i.e., $(\forall x \in X)(x \preccurlyeq x)$,
- connected, i.e., $(\forall x, y \in X)(x \preccurlyeq y \vee y \preccurlyeq x))$,
- transitive, i.e., $(\forall x, y, z \in X)(x \preccurlyeq y \wedge y \preccurlyeq z \to x \preccurlyeq z)$, and
- every nonempty subset of X has a $\preccurlyeq$-least element, or equivalently, the strict part $x \prec y \Leftrightarrow x \preccurlyeq y \wedge \neg(y \preccurlyeq x)$ is wellfounded.

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The rank of $\preccurlyeq$ is define to be the rank of its strict part. Let $\preccurlyeq$ be a prewellordering. Then it induces a wellordering $\prec^*$ on $X/\sim$, the quotient of X by the equivalence relation $x \sim y \Leftrightarrow x \preccurlyeq y \wedge y \preccurlyeq x$. The ordertype of $X/\sim$ gives rise to a rank function $f : X \to \text{ordertype}(\prec^*)$. Conversely, to each rank function $\mu : X \to \text{Ord}$, one can associate a prewellordering, $\preccurlyeq_\mu$, such that $x \preccurlyeq_\mu y \Leftrightarrow \mu(x) \leq \mu(y)$, for every $x, y \in X$.

Let Γ be a pointclass. We are interested in comparing two ordinals associated to Γ:

$$\delta_\Gamma = \sup\{\xi \mid \xi \text{ is the rank of a prewellordering in } \Delta_\Gamma\},$$

$$\lambda_\Gamma = \sup\{\xi \mid \xi \text{ is the rank of a wellfounded relation in } \Gamma\}.$$

These two types of ordinals were first introduced by Moschovakis [7] (δ-ordinal) and Kechris [4] (λ-ordinal) to study the "definable length" of the continuum. The following notations are widely used in the literature:

$$\underset{\sim}{\boldsymbol{\sigma}}^1_n = \lambda_{\underset{\sim}{\boldsymbol{\Sigma}}^1_n}, \quad \underset{\sim}{\boldsymbol{\pi}}^1_n = \lambda_{\underset{\sim}{\boldsymbol{\Pi}}^1_n}, \quad \text{and} \quad \underset{\sim}{\boldsymbol{\delta}}^1_n = \delta_{\underset{\sim}{\boldsymbol{\Sigma}}^1_n} = \delta_{\underset{\sim}{\boldsymbol{\Pi}}^1_n}.$$

By definition for any pointclass Γ, $\delta_\Gamma \leq \lambda_\Gamma$. Note that given a wellfounded relation, one can get a prewellordering of the same rank via its rank function. So $\delta_{\mathscr{P}(\mathbb{R})} = \lambda_{\mathscr{P}(\mathbb{R})}$. $\mathscr{P}(\mathbb{R})$ can be reduced to smaller pointclasses, for instance, the pointclass of all hyperprojective sets. In general, the equality holds for pointclasses Γ with the following closure property (see [9]):

If $Q \in \Gamma$ and R is hyperprojective in $(\mathfrak{R}, Q)$, then $R \in \Gamma$.

Here $\mathfrak{R}$ is the structure of the second order arithmetic. The pointclass of hyperprojective sets is the smallest pointclass with this property. The reader is referred to Moschovakis [8] for basic facts about hyperprojective pointclass.

Under determinacy assumptions, these two projective ordinals behave nicely. Kechris [4] showed that assuming all the projective sets are determined, we have the following picture below for projective hierarchy:

$$\underset{\sim}{\boldsymbol{\pi}}^1_0 = \underset{\sim}{\boldsymbol{\delta}}^1_1 = \underset{\sim}{\boldsymbol{\sigma}}^1_1 < \underset{\sim}{\boldsymbol{\pi}}^1_1 = \underset{\sim}{\boldsymbol{\delta}}^1_2 = \underset{\sim}{\boldsymbol{\sigma}}^1_2 < \underset{\sim}{\boldsymbol{\pi}}^1_2 = \underset{\sim}{\boldsymbol{\delta}}^1_3 = \underset{\sim}{\boldsymbol{\sigma}}^1_3 < \underset{\sim}{\boldsymbol{\pi}}^1_3 = \cdots.$$

Consequently, we have $\delta_{\mathsf{P}} = \lambda_{\mathsf{P}}$, where P denotes the pointclass of all projective sets. This in fact follows from a more general result. A (boldface) pointclass Γ is *strongly closed* if it is closed under finite unions and intersections, complements and projection along ω ($\exists^\omega$) and existential quantification over $\mathbb{R}$ ($\exists^\mathbb{R}$). The projective pointclass P is strongly closed. Kechris-Solovay-Steel [5] showed that if Γ is a strongly closed pointclass and every set in Γ is determined, then $\delta_\Gamma = \lambda_\Gamma$.

Note that for a sufficiently closed Γ, if every pointset in Γ is determined, no sets in Γ wellorders the reals. But if Γ contains a wellordering of the reals, the equality still holds for Γ. More precisely, if Γ is a Δ-like pointclass (i.e., closed under complement) and contains a wellordering of the reals, then $\delta_\Gamma = \lambda_\Gamma$. This is because for every wellfounded relation in Γ, the tree ordering of its associated wellfounded tree has the same rank, and the Kleene-Brower ordering extending the tree ordering can be extended to a prewellordering in Γ, given that Γ is Δ-like. As corollaries, $\delta_{\mathsf{P}} = \lambda_{\mathsf{P}}$ holds in many well-known inner models of set theory, for instance, Gödel's L, Silver's $L[\mu]$, Steel's core model $\mathbf{K}_i$ for i Woodin cardinals, etc.

These seem to be strong evidences for $\delta_{\mathsf{P}} = \lambda_{\mathsf{P}}$. In this paper, we show that it is relatively consistent with ZFC that the equality fails.

MAIN THEOREM. *If* ZFC *is consistent, then it is consistent with* ZFC *that* $\delta_{\mathsf{P}} < \lambda_{\mathsf{P}}$.

Our proof consists of two steps: (1) Starting with the constructible universe L, we first add a "tall" wellfounded relation by adding κ many Cohen reals for some sufficiently large cardinal κ, and then code this wellfounded relation projectively with almost disjoint forcings. These are discussed in Section 2. (2) Let $\mathbb{P}$ and $\dot{\mathbb{Q}}$ denote the two forcing partial ordered sets used to add a projective tall wellfounded relation, and $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}}$ the largest element of $\mathbb{P} * \dot{\mathbb{Q}}$ (i.e., the empty condition). Then in Section 3, we prove the following theorem of $\mathsf{ZFC} + V = L$.

THEOREM ($\mathsf{ZFC} + V = L$). *Let* κ *be any cardinal* $\geq \omega_{\omega_1}$. *Then*

$$\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta_{\mathsf{P}} \leq \omega_{\omega_1}.$$

So starting with a $\kappa \geq \omega_{\omega_1}$ in L, in the final forcing extension we have $\delta_{\mathsf{P}} \leq \omega_{\omega_1} \leq \kappa < \lambda_{\mathsf{P}}$. The last section discusses some limitations of our technique.

Our notations are more or less standard. The reader is referred to Kunen [6] or Jech [3] for background knowledge and explanation of notations.

§2. Step I: Add a tall projective wellfounded relation.

Now we begin to prove the main theorem. This section shows how to add a "tall" projective wellfounded relation.

DEFINITION 2.1. Suppose A, B are two sets of reals. We say B is *projective in* A if there is a formula φ and a real r such that

$$B = \{x \in \mathbb{R} \mid (\mathfrak{R}, A) \models \varphi[A, r, x]\}$$

where $\mathfrak{R}$ is the structure of second order arithmetic. P_A is the pointclass of all the sets that are projective in A.

NOTATION. For a pointset $A \subseteq \mathbb{R}^k$, for convenience, we always write δ_A for δ_{P_A} and λ_A for λ_{P_A}. When $A = \varnothing$, we omit the subscript.

2.1. Add a tall projective wellfounded relation.

Let I be an arbitrary set, $\leq_I$ be a binary relation on I. Consider $\mathrm{Fn}(I \times \omega, 2)$.

Suppose that $G \subseteq \mathrm{Fn}(I \times \omega, 2)$ is a generic filter, then $\mathrm{Fn}(I \times \omega, 2)$ adds a collection of Cohen reals indexed by I, $\{(\dot{x}_i)^G \mid i \in I\}$, where each $\dot{x}_i$ is the canonical $\mathrm{Fn}(I \times \omega, 2)$-name for the generic real with index i, and $(\dot{x}_i)^G$ denotes the interpretation of $\dot{x}_i$ in $V[G]$. Let

$$\mathcal{R}^G_{\mathscr{I}} = \{\langle (\dot{x}_i)^G, (\dot{x}_j)^G \rangle \mid i, j \in I \wedge i \leq_I j\},$$

and X^G_I be the set of $a \in \mathbb{R}^{V[G]}$ such that a codes a pair $(c, d) \in \mathcal{R}^G_I$. We may omit the subscripts or subscripts when they are clear from the context.

Let κ be an ordinal. Let $\langle I_\kappa, \leq_{I_\kappa} \rangle$ denote the following partial ordering:

- $I_\kappa = [\kappa]^{<\omega\downarrow}$, the set of finite descending sequences of ordinals $< \kappa$, and
- for every $s, t \in I_\kappa$, $t \leq_I s$ if and only if $t|\mathrm{dom}(s) = s$.

Let $\mathbb{P} = \mathrm{Fn}(I_\kappa \times \omega, 2)$, and $G \subseteq \mathbb{P}$ an M-generic filter. $\mathcal{R}^G_{I_\kappa}$ is not a prewellordering as it is not total. $\mathcal{R}^G_{I_\kappa}$ is a wellfounded relation and has rank κ.

PROPOSITION 2.2. I_κ *is a wellfounded relation and* $\mathrm{rank}(I_\kappa) = \kappa$.

PROOF. $\langle I_\kappa, \leq_I \rangle$ is in fact the relation obtained by reversing the wellfounded tree associated to the wellfounded relation $\langle \kappa, \in \rangle$. ⊣

Next we need a projective definition for $\mathcal{R}_{I_\kappa}$. This can be done by applying Harrington's coding forcing.

THEOREM 2.3 (Harrington [2]) (ZFC). *Suppose* $\omega_1^V = \omega_1^L$ *and* X (*in* V) *is a set of reals. Then there is a generic extension that preserves all cardinals and in which* X *is projective* (*in fact*, $\utilde{\Pi}^1_2$).

However, in order to prove a property needed in the second part of our proof, we made a slight change to Harrington's coding forcing.

Assume $\omega_1 = (\omega_1)^L$. Fix an enumeration of $\omega^{<\omega}$, say $\langle s_n \mid n < \omega\rangle$. For each $x \in \omega^\omega$, let $\sigma_x = \{2n \mid s_n \subset x\}$. Then $\{\sigma_x \mid x \in \omega^\omega\}$ forms an almost disjoint family of subsets of ω. Fix a sequence $\langle \sigma_{\alpha,i} : (\alpha, i) \in \omega_1 \times \omega\rangle \in L$ such that for all $(\alpha, i), (\beta, j) \in \omega_1 \times \omega$,

1. $\sigma_{\alpha,i} \subset \{2k \mid k < \omega\}$ and $\sigma_{\alpha,i}$ is infinite,
2. $(\alpha, i) \neq (\beta, j)$ implies $\sigma_{\alpha,i} \cap \sigma_{\beta,j}$ is finite,
3. $\langle \sigma_{\alpha,i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ is Δ_1-definable (no parameters) in $H(\omega_1)$.

Define $\mathbb{Q}_X$ as follows. Conditions are triples (A, t, b) such that

- $A \subset X$, A is finite;
- t is a finite partial function, $t : \omega_1 \to [\omega]^{<\omega}$; and
- $b \in [\omega]^{<\omega}$, where $[\omega]^{<\omega}$ is the set of finite subsets of ω.

The order is defined as follows: $(A_1, t_1, b_1) \leq (A_2, t_2, b_2)$ if

1. $\mathrm{dom}(t_2) \subseteq \mathrm{dom}(t_1)$, $A_2 \subseteq A_1$, $b_2 \subseteq b_1$, and for all $\alpha \in \mathrm{dom}(t_2)$, $t_2(\alpha) \subseteq t_1(\alpha)$ and $b_2 = b_1 \cap (m + 1)$, where $m = \max(b_2)$.
2. if $\alpha \in \mathrm{dom}(t_2)$, then for all $x \in A_2$, $t_1(\alpha) \cap \sigma_x = t_2(\alpha) \cap \sigma_x$.
3. if $\alpha \in \mathrm{dom}(t_2)$ and $i \in t_2(\alpha)$ then $b_2 \cap \sigma_{\alpha,i} = b_1 \cap \sigma_{\alpha,i}$.

By a standard Δ-system argument, we have

PROPOSITION 2.4. $\mathbb{Q}_X$ *is c.c.c.*

Let $h \subset \mathbb{Q}_X$ be a V-generic filter. Define $b_h = \bigcup\{b \mid (A, t, b) \in h\}$. b_h uniquely determines h. For each $\alpha < \omega_1$, let $t_h(\alpha) = \bigcup\{t(\alpha) \mid \alpha \in \mathrm{dom}(t) \wedge (A, t, b) \in h\}$. Next are two lemmas from Harrington [2].

LEMMA. *For every* $x \in \mathbb{R}^{V[h]}$, $x \in X$ *iff* $\forall \alpha < \omega_1$, $|\sigma_x \cap t_h(\alpha)| < \omega$.

The left-to-right direction is clear from the forcing. The right-to-left direction is also clear, for x in the ground model V. It is in establishing this direction for x which are not in the ground model that one uses the fact that the right-hand-side of the equivalence holds for uncountably many ordinals α.

LEMMA. *For every* $(\alpha, i) \in \omega_1 \times \omega$, $i \in t_h(\alpha)$ *iff* $|\sigma_{\alpha,i} \cap b_h| < \omega$.

From these, we get X is Π_1 over $H(\omega_1)$. Hence in $V[h]$, X is $\Pi^1_2(b_h)$. Now force over $L[G]$ with $\mathbb{Q}_X$, where $X = X^G$ and G is a L-generic filter over $\mathbb{P}$. Let $h \subseteq \mathbb{Q}_X$ be a $L[G]$-generic filter. $\omega_1^{L[G]} = \omega_1^L$. So by Harrington's Theorem, $\mathcal{R}^G$ is $\Pi^1_2(b_h)$ in $L[G][h]$.

2.2. Property (H)**.** To ensure the second part of the proof for the main theorem runs smoothly, we need the two forcing partial orders satisfy an additional property. Let (H) denote the following property:

For every formula φ, every condition p and any $x_1, \ldots, x_n \in V$,

$$\text{if } p \Vdash \varphi(\check{x}_1, \cdots, \check{x}_n), \text{ then } \mathbb{1} \Vdash \varphi(\check{x}_1, \cdots, \check{x}_n).$$

It is easy to see that $\mathrm{Fn}(I, J)$ has the property (H). We show that $\mathbb{Q}_X$ also has the property (H).

PROPOSITION 2.5. *$\mathbb{Q}_X$ has the property (H).*

PROOF. Suppose $q_0 = (A_0, t_0, b_0)$ is a condition. Choose an odd integer k such that

1. for all $\alpha \in \mathrm{dom}(t_0)$, $t_0(\alpha) \subset k$, and
2. $b_0 \subset k$.

Let $p = (A, t, \{k\})$ where $A_0 \subset A$, $\mathrm{dom}(t_0) \subset \mathrm{dom}(t)$, and where $t(\alpha) = \{k\}$ for all $\alpha \in \mathrm{dom}(t)$.

CLAIM. Suppose G is V-generic with $p \in G$. Then there exists a V-generic filter G_0 such that $V[G] = V[G_0]$ and such that $q_0 \in G_0$.

PROOF OF CLAIM. Let

$$Z = \bigcup \{\sigma_{\alpha,i} \backslash k \mid \alpha \in \mathrm{dom}(t_0),\ i \in t_0(\alpha)\} \subset \{2n \mid n < \omega\}.$$

Let $b_G = \bigcup \{b \mid (A, t, b) \in G\}$. This set uniquely determines G: G is the set of all $(A, t, b) \in \mathbb{Q}_X$ such that

- $b = b_G \cap n$, for some n (i.e., $n = \max(b) + 1$),
- if $\alpha \in \mathrm{dom}(t)$ and $i \in t(\alpha)$ then $b \cap \sigma_{\alpha,i} = b_G \cap \sigma_{\alpha,i}$.

The set $(b_G \backslash Z) \cup b_0$ defines a V-generic filter G^* such that $q_0 \in G^*$, i.e., such that $b_{G^*} = (b_G \backslash Z) \cup b_0$. This is straightforward to verify.

The problem is that $V[G] \neq V[G^*]$, in fact $V[G] = V[G^*][b_G \cap Z]$ (trivially). Define G_0 by defining b_{G_0}. Fix a bijection (in V)

$$\pi : \{2n + 1 \mid n < \omega\} \backslash k \to Z \cup \{2n + 1 \mid n < \omega\} \backslash k.$$

Define a set $B \subset \omega$ by:

- $B \cap k = b_0$.
- For every even n such that $n > k$, $n \in B$ iff $n \in b_G \backslash Z$.
- For every odd n such that $n > k$, $n \in B$ iff $\pi(n) \in b_G$.

It is straightforward to verify that there is a V-generic filter $G_0 \subset Q_X$ such that $b_{G_0} = B$ (and of course G_0 is uniquely specified by B). Further $q_0 \in G_0$.

Finally $V[G_0] = V[B] = V[b_{G^*}][b_G \cap Z] = V[b_G] = V[G]$. This proves the claim. $\dashv_{\text{CLAIM}}$

Now we prove that $\mathbb{Q}_X$ has the property (H) using the claim. Suppose toward a contradiction that $q_0 \Vdash \varphi[a_1, \dots, a_n]$ and $\mathbb{1} \nVdash \varphi[a_1, \dots, a_n]$, where $a_1, \dots, a_n$ are in V. Choose q_1 such that $q_1 \Vdash (\neg\varphi)[a_1, \dots, a_n]$. Choose a large enough odd number k such that conditions 1. and 2. at the beginning of the proof hold for k relative to q_0 and q_1.

Let $p = (A, t, \{k\})$, where $A = A_0 \cup A_1$, $\mathrm{dom}(t) = \mathrm{dom}(t_0) \cup \mathrm{dom}(t_1)$, and where $t(\alpha) = \{k\}$ for all $\alpha \in \mathrm{dom}(t)$. Let $G \subset \mathbb{Q}_X$ be V-generic with $p \in G$. Then we have V-generic filters G_0 and G_1 such that

1. $q_0 \in G_0$, $q_1 \in G_1$.
2. $V[G] = V[G_0] = V[G_1]$.

But then $V[G] \models \varphi[a_1 ... a_n]$ and $V[G] \models (\neg\varphi)[a_1 ... a_n]$. Contradiction! $\dashv$

§3. Step II: δ is small. In the previous section, we have shown that given any cardinal κ, there is a two-step forcing that adds a projective wellfounded relation of rank κ, thus $\lambda > \kappa$ in the final forcing extension. Our goal in this section is to prove that for a sufficiently large cardinal κ, these two steps of forcings do not add projective prewellorderings of rank $\geq \kappa$, hence, $\delta < \lambda$ in the final extension.

THEOREM 3.1 ($\mathsf{ZFC} + V = L$). *Let κ be any cardinal $\geq \omega_{\omega_1}$. Then*

$$\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta \leq \omega_{\omega_1},$$

where $\mathbb{P} = \mathrm{Fn}(I_\kappa \times \omega, 2)$ *and* $\mathbb{Q} = \mathbb{Q}_{X^G_{I_\kappa}}$.

3.1. Basic framework. The basic framework of our approach is the following. Towards a contradiction, we assume the statement fails at some κ in a model (M, E). Let $\mathbb{P}$, $\mathbb{Q}$ be the two forcings associated to κ. Let $(p, \dot{q}), s, \tau \in M$ be such that for some Σ^1_n formula $\varphi(t_0, t_1, t_2)$:

1. $(M, E) \models$ "$(p, \dot{q}) \Vdash \tau \in \mathbb{R}$".
2. $(M, E) \models$ "$(p, \dot{q}) \Vdash$ '$\{(x, y) \in \mathbb{R} \mid \varphi[x, y, \tau]\}$ is the strict part of a total pre-order'".
3. $(M, E) \models$ " s is a function with domain ω_{ω_1}".
4. $(M, E) \models$ " for all $b < \omega_{\omega_1}$, $s(b)$ is a term in $V^{\mathbb{P}*\mathbb{Q}}$".
5. $(M, E) \models$ " for all $b_1 < b_2 < \omega_{\omega_1}$, $(p, \dot{q}) \Vdash \varphi[s(b_1), s(b_2), \tau]$".
6. s and τ are each definable in (M, E) (no parameters).

Suppose $G \subset \mathbb{P}$ is an (M, E)-generic filter, $h \subset \mathbb{Q} = \mathbb{Q}_{X^G}$ is a $(M, E)[G]$-generic filter such that $(p, \dot{q}) \in H$, where $H \subset \mathbb{P} * \dot{\mathbb{Q}}$ is the (M, E)-generic filter given by (G, h). We wish to construct a G^* from G such that:

1. G^* is (M, E)-generic over $\mathbb{P}$ and $p \in G^*$. H^* is (M, E)-generic over $\mathbb{P} * \dot{\mathbb{Q}}$ and $(p, \dot{q}) \in H^*$, where H^* is the filter given by (G^*, h).
2. $\mathcal{R}^{G^*} = \mathcal{R}^G$, hence $\mathbb{Q}_{X^{G^*}} = \mathbb{Q}_{X^G}$.
3. $\mathbb{R}^{(M,E)[G^*]} = \mathbb{R}^{(M,E)[G]}$, hence $\mathbb{R}^{(M,E)[H^*]} = \mathbb{R}^{(M,E)[H]}$.
4. $\tau^{H^*} = \tau^H$.
5. $(s(b_0))^{H^*} = (s(b_1))^H$ and $(s(b_1))^{H^*} = (s(b_0))^H$, for some $b_0, b_1 < \omega_{\omega_1}$.

If this can be done, then in $(M, E)[H]$ we have

$$\mathbb{R}^{(M,E)[H]} \models \varphi[(s(b_0))^H, (s(b_1))^H, \tau^H].$$

Meanwhile in $(M, E)[H^*]$,

$$\mathbb{R}^{(M,E)[H^*]} \models \varphi[(s(b_0))^{H^*}, (s(b_1))^{H^*}, \tau^{H^*}].$$

But by the above properties, we have

$$\mathbb{R}^{(M,E)[H]} \models \varphi[(s(b_1))^H, (s(b_0))^H, \tau^H].$$

Contradiction!

Our first attempt was to work in $(M, E) = (L, \in)$ directly. Unfortunately this approach doesn't work: H is uniquely determined by the pair (G, h). If (M, E) is wellfounded, then

$$(M, E)[H] = (M, E)[G][h] = (M, E)[\mathcal{R}^G][h].$$

The key point is that $\mathcal{R}^G$ uniquely determines G if and only if $\kappa^{(M,E)}$ is wellfounded. If we work in a wellfounded (M, E) and require that $\mathcal{R}^{G^*} = \mathcal{R}^G$, then $H = H^*$

and the Requirement (5) above can never happen. So we must work with non-wellfounded models.

3.2. Theory of indiscernibles. In order to work with and have control of non-wellfounded models, we "wrap up" the previous approach with an additional lemma on a certain type of theories of indiscernibles.

Let $\mathcal{L}$ denote the language of set theory. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i \mid i < \omega\}$, where c_i's are constant symbols. For every formula φ in $\mathcal{L}$, we use t_φ to denote its skolem term. Since we only work with models elementarily equivalent to L_δ for some δ, all the skolem functions are considered to be definable.

LEMMA 3.2 ($\mathsf{ZFC} + V = L$). *Let λ be a limit ordinal $> \omega_{\omega_1}$ such that*

$$L_\lambda \models \mathsf{ZFC}\backslash\textit{Replacement } + \Sigma_1\text{-}\textit{Replacement}\,.$$

Let $T_0 = \mathrm{Th}(L_\lambda)$. Then there exists a theory T such that

1. *T is an extension of T_0 in the language $\mathcal{L}^*$.*
2. *T is a complete theory for which c_i, $i < \omega$, are indiscernibles.*
3. *T satisfies the following properties*:
   - *"$c_0 < \omega_{\omega_1}$" is in T.*
   - (ω-completeness) *For every formula $\varphi(x_0, \dots, x_n)$ in $\mathcal{L}$, if*

     $$\text{“}t_\varphi(c_0, \dots, c_n) \in \omega\text{” is in } T$$

     then for some $k < \omega$,

     $$\text{“}t_\varphi(c_0, \dots, c_n) = k\text{” is in } T.$$

By indiscernibility, "$c_0 < \omega_{\omega_1}$" is in T is equivalent to

$$\text{for every } i < \omega, \text{“}c_i < \omega_{\omega_1}\text{” is in } T.$$

We say a theory T as in the above lemma, satisfying the conditions (1)–(3), is an *ω-complete extension of* T_0. A big advantage of using ω-complete theories is that we can work with ω-models, in which natural numbers are all standard. This makes the presentation of our proof much easier. However, the ω-completeness condition makes ω_ω a lower bound for all the indiscernibles, which makes it difficult to see that there is in fact an upper bound for δ in our model (see Section 4).

Later we want to consider forcing over models of T_0, so we require that

$$L_\lambda \models \mathsf{ZFC}\backslash\text{Replacement } + \Sigma_1\text{-Replacement}.$$

Generic extensions of models satisfying "$\mathsf{ZFC}\backslash$Replacement $+\ \Sigma_1$-Replacement" also satisfy this theory so this is a reasonable theory for forcing. A sufficient condition for such λ is that $L \models \lambda$ is a limit cardinal $> \omega$.

Let $\{x_0 < \dots < x_n\}$ abbreviate $\{x_0, \cdots, x_n\}$ with $x_0 < \dots < x_n$.

FACT. Suppose T is an ω-complete theory extending T_0. Then for every total order $(Z, <_Z)$ there is a model $(M, E) \models T$ such that $\mathrm{Ord}^{(M,E)}$ contains a subset X such that

1. (X, E) is isomorphic to $(Z, <_Z)$.
2. for all formulas $\varphi(x_0, \dots, x_n)$, for all $\{a_0\ E \cdots E\ a_n\}$ in X,

   $$(M, E) \models \varphi[a_0, \dots, a_n] \quad \text{iff} \quad \varphi(c_0, \dots, c_n) \in T.$$

3. Every element of M is definable in (M, E) with parameters from X.

Further (M, E) and X are unique up to isomorphism in the sense that if (M^*, E^*) and X^* satisfy (1)–(3) and $\pi : (X, E) \to (X^*, E^*)$ is an order isomorphism then π extends uniquely to an isomorphism of (M, E) and (M^*, E^*).

Such an M is called a $(Z, <_Z)$-*model for* T. We often write Z-*model*, when $<_Z$ is clear from the context. Let $(\mathbb{Z}, <_{\mathbb{Z}})$ denote the linear ordering of all integers.

LEMMA 3.3. *Let T_0 be as in Lemma 3.2, and T an ω-complete extension of T_0. Let M be the $\mathbb{Z}$-model of T. Then in M, for any $\kappa \geq \omega_{\omega_1}$, $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta \leq \omega_{\omega_1}$, where $\mathbb{P} = \mathrm{Fn}(I_\kappa \times \omega, 2)$ and $\mathbb{Q} = \mathbb{Q}_{X^G_{I_\kappa}}$.*

Lemmas 3.2 and 3.3 will be proved in subsequent subsections. Granting these two lemmas, we prove Theorem 3.1 as follows.

Assume the statement in Theorem 3.1 is false. Select a sufficiently large λ such that $L \models$ "λ is a limit cardinal $> \omega_{\omega_1}$" and

$$L_\lambda \models \text{" for some } \kappa \geq \omega_{\omega_1},\ \mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta > \omega_{\omega_1}\text{"},$$

where $\mathbb{P}$, $\mathbb{Q}$ are the two forcings associated to κ.

Let $T_0 = \mathrm{Th}(L_\lambda)$. Let T be a ω-complete extension of T_0 given by Lemma 3.2. Let (M, E) be the $\mathbb{Z}$-model for T. By Lemma 3.3, in M, $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta \leq \omega_{\omega_1}$, for every $\kappa \geq \omega_{\omega_1}$. But this contradicts that $(M, E) \models \mathrm{Th}(L_\lambda)$.

3.3. Proof of Lemma 3.2. In this subsection, we give the proof of

LEMMA 3.2. *Let λ be a limit ordinal $> \omega_{\omega_1}$ such that*

$$L_\lambda \models \mathsf{ZFC}\backslash\textit{Replacement } + \Sigma_1\textit{-Replacement}.$$

Let $T_0 = \mathrm{Th}(L_\lambda)$. Then there exists ω-complete extension of T_0.

Work in L. Let λ and T_0 be as in the hypothesis. We shall get this ω-complete extension T from a nonstandard model.

Force with $(\mathscr{P}(\omega_1)\backslash I_0, \leq_0)$, where $I_0 = \{X \subset \omega_1 \mid |X| < \omega_1\}$, and $p \leq_0 q$ iff $p \setminus q \in I_0$. Let $G \subseteq \mathscr{P}(\omega_1)\backslash I_0$ be an L-generic filter. Since I_0 is ω_1-complete and contains all the singletons, G is a nonprincipal ω_1-complete ultrafilter over κ. Let $j : (L, \in) \to (M, E) = \mathrm{Ult}(L, G)$ be the induced generic elementary embedding. $\mathrm{crit}(j) = \omega_1^L$. This implies that (M, E) is an ω-model. However, the wellfoundedness breaks down at some countable ordinals in (M, E). In particular, $[\mathrm{id}]_G$ is in the nonstandard part of (M, E).

Work in $L[G]$. We shall construct a sequence $\langle (X_i, \alpha_i) : i < \omega \rangle$ such that for every $i < \omega$,

1. $X_i \in M$.
2. $X_i \subseteq j(\omega_{\omega_1})$.
3. $\alpha_i \in M$, $(M, E) \models$ "$\alpha_i < (\omega_1)^M$".
4. for all $\alpha < (\omega_1)^L$, $(M, E) \models$ "$\alpha < \alpha_i$".
5. $X_i \supseteq X_{i+1}$ and $\alpha_i > \alpha_{i+1}$.
6. $(M, E) \models |X_i| \geq \omega_{\alpha_i}$.
7. For $i > 0$, all the i-element subsets of X_i are indiscernible for $j(V_\lambda)$ (from the view of (M, E)), i.e., for every formula φ in $\mathcal{L}$, for every $\{x_1 < \cdots < x_i\}$ and $\{y_1 < \cdots < y_i\}$ in X_i,
$$(M, E) \models \text{"} V_{j(\lambda)} \models \varphi[x_1, \ldots, x_i] \leftrightarrow \varphi[y_1, \ldots, y_i]\text{"}.$$

Our construction uses the following well-known Erdős-Rado Theorem.

THEOREM (Erdős-Rado [1]). *For every* $\alpha > 1$ *and* $n < \omega$, $\omega_{\alpha+n} \to (\omega_\alpha)^{n+1}_{\omega_1}$.

Let α_0 be any nonstandard countable ordinal in (M, E) (for instance, $[\mathrm{id}]_G$) and $X_0 = j(\omega_{\omega_1})$. Suppose α_i, X_i are given, we describe how to get α_{i+1}, X_{i+1}.

Choose α_{i+1}, a nonstandard countable ordinal such that $\alpha_{i+1} + i \leq \alpha_i$. Such an ordinal exists, since α_i is nonstandard. Color $(i+1)$-tuples from X_i by their theories in $j(V_\lambda)$, i.e., for $\{x_0 < \cdots < x_i\} \subseteq X_i$, let

$$F(x_0, \dots, x_i) = \{\ulcorner\varphi\urcorner \mid (M, E) \models \text{“}V_{j(\lambda)} \models \varphi[x_0, \dots, x_i]\text{”}\},$$

where $\ulcorner\varphi\urcorner$ is the Gödel number of φ. $F \in M$ and for every $\{x_0 < \cdots < x_i\} \subseteq X_i$, $F(x_0, \dots, x_i) \subseteq \omega$. By CH, we can view F as an ω_1-coloring of $[X_i]^{i+1}$. Since $\alpha_i \geq \alpha_{i+1} + i$, applying Erdős-Rado in (M, E), we get a homogeneous $X_{i+1} \in M$ such that $(M, E) \models |X_{i+1}| \geq \omega_{\alpha_{i+1}}$.

This defines in $L[G]$, using $V^{(M,E)}_{j(\lambda)}$, the sequence $\langle (X_i, \alpha_i) : i < \omega \rangle$. For each $i < \omega$, let T_i be the theory satisfied (in (M, E)) by some (or all) i-tuples from X_i, i.e.,

$$T_i = \{\varphi(c_0, \dots, c_{i-1}) \mid V^{(M,E)}_{j(\lambda)} \models \varphi(x_0, \dots, x_{i-1}),$$
$$\text{for some } \{x_0 < \cdots < x_{i-1}\} \in [X_i]^i\}.$$

Let $T = \bigcup_i T_i$. By compactness, T is consistent. Clearly, T extends T_0. The construction runs through all formulas in $\mathcal{L}$, thus T is complete in $\mathcal{L}^*$. Again by the construction, T contains “$c_i < \omega_{\omega_1}$” for all $i < \omega$; and since (M, E) is an ω-model, the ω-completeness condition is satisfied automatically. T is obtained in $V[G] = L[G]$. By absoluteness, such a theory T must exist in L: the existence of T is a Σ^1_1 statement about T_0 and so the existence is absolute.

3.4. Proof of Lemma 3.3. Before proving Lemma 3.3, we give some preliminaries on forcing over ω-models.

3.4.1. *Forcing over ω-models.* Suppose (M, E) is an ω-model, and

$$(M, E) \models \mathsf{ZFC}\backslash\text{Replacement} + \Sigma_1\text{-Replacement}.$$

We identify $(V_{\omega+1})^{(M,E)}$ with its transitive collapse. We also identify (M, E)-generic filters with the corresponding subsets of M. Thus if $(P, <_P) \in M$ and

$$(M, E) \models \text{“}(P, <_P) \text{ is a partial order”}$$

then a set $G \subset \{a \in M \mid a\ E\ P\}$ is (M, E)-generic for P if and only if for each $D \in M$ such that $(M, E) \models$ “$D \subseteq P$ and D is dense in $(P, <_P)$”,

$$\{b \in M \mid b\ E\ D\} \cap G \neq \varnothing.$$

The corresponding extension $(M, E)[G]$ is an ω-model and (M, E) is a submodel of $(M, E)[G]$.

Suppose $(M_0, E_0) \prec (M, E)$. Suppose $\mathbb{P}$ is a partial order in (M, E) such that $\mathbb{P}$ is c.c.c. in (M, E). Suppose $G \subset \mathbb{P}$ is (M, E)-generic and that $\mathbb{P} \in M_0$. Then $G \cap M_0$ is (M_0, E_0)-generic for $\mathbb{P}$ (in the sense defined above). Let $G_0 = G \cap M_0$. This gives a canonical interpretation of $(M_0, E_0)[G_0]$ as a submodel of $(M, E)[G]$: If $a \in (M_0, E_0)[G_0]$ then $a = \tau^{G_0}$ for $\tau \in M_0$ such that

$$(M_0, E_0) \models \text{“}\tau \text{ is a term in } V^{\mathbb{P}}\text{”}.$$

Define $I(a) = \tau^G$. This is well-defined and moreover,

$$I : (M_0, E_0)[G_0] \to (M, E)[G]$$

is an elementary embedding, $I|M_0$ is the identity, and $I(G_0) = G$.

Thus we can naturally denote the range of I by $(M_0, E_0)[G]$ and identify it with $(M_0, E_0)[G_0]$. Our use for this is the following:

> Suppose $\sigma \in M_0$, $(M, E) \models$ "$\mathbb{1} \Vdash \sigma \in V_{\omega+1}$". Then σ^G is uniquely determined by G_0 and $\sigma^G = \sigma^{G_0}$.

3.4.2. *Proof of Lemma* 3.3*, a warm-up.* As a warm-up, we prove a version of Lemma 3.3 for $\mathrm{Fn}(I_\kappa \times \omega, 2)$ to illustrate the idea.

LEMMA 3.3*. *Let T_0 be as in Lemma* 3.2*, and T an ω-complete extension of T_0. Let (M, E) be the $\mathbb{Z}$-model of T. Then in (M, E), for any cardinal $\kappa > \omega_{\omega_1}$,*

$$\mathbb{1}_{\mathrm{Fn}(I_\kappa \times \omega, 2)} \Vdash \delta_{X^G} \leq \omega_{\omega_1}.$$

Let T be a theory as in the hypothesis. Let (M, E) be the $\mathbb{Z}$-model for T. By the ω-completeness condition, (M, E) is an ω-model. So we do not distinguish between n and $n^{(M,E)}$ and we do not distinguish between ω and $\omega^{(M,E)}$. Let $\langle \rho_k : k \in \mathbb{Z} \rangle$ be the generating indiscernibles in ascending order. Notice that $\rho_k < \omega_{\omega_1}$, for all $k \in \mathbb{Z}$.

Let M_0 be the skolem hull of ρ_0, M_1 the skolem hull of ρ_1, inside (M, E). Then $(M_i, E) \prec (M, E)$ and $(M_0 \cap M_1, E) \prec (M_i, E)$, for $i = 0, 1$. Clearly $M_0 \cap M_1$ contains all the elements in (M, E) that are definable (without parameters) in (M, E).

We are trying to show if κ is a cardinal of (M, E) with $(M, E) \models$"$\omega_{\omega_1} < \kappa$", and if G is (M, E)-generic for adding I-Cohen reals then in $(M, E)[G]$, there is no prewellordering of rank ω_{ω_1} that is projective in $\mathcal{R}^G$.

Assume that the statement is false in (M, E). Let $\kappa, s, \tau \in M$ be such that for some Σ^1_n formula $\varphi(t_0, t_1, t_2)$:

1. $(M, E) \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash \tau \in \mathbb{R}$" .
2. $(M, E) \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash$ '$\{(x, y) \in \mathbb{R} \mid \varphi[x, y, \tau, \mathcal{R}]\}$ is the strict part of a total pre-order'".
3. $(M, E) \models$ "s is a function with domain ω_{ω_1}".
4. $(M, E) \models$ "for all $b < \omega_{\omega_1}$, $s(b)$ is a term in $V^{\mathbb{P}}$".
5. $(M, E) \models$ "for all $b_1 < b_2 < \omega_{\omega_1}$, $\mathbb{1}_{\mathbb{P}} \Vdash \varphi[s(b_1), s(b_2), \tau, \mathcal{R}]$".
6. κ, s and τ are each definable in (M, E) (no parameters).

Here $\mathbb{P} = \mathrm{Fn}(I \times \omega, 2)$ and $I = I_\kappa$.

The partial order $\mathbb{P}$ is definable and has the property (H) on page 582, so we can choose $\kappa, s, \tau \in M$ as required. Thus $s, \tau \in M_0 \cap M_1$.

Let $\langle \tau_b : b < (\omega_{\omega_1})^{(M,E)} \rangle$ be the sequence of elements of M given by s, so for each $b \in M$, if $(M, E) \models$ "$b < \omega_{\omega_1}$", then $(M, E) \models$ "$\tau_b = s(b)$". Therefore $\tau_{\rho_0} \in M_0$ and $\tau_{\rho_1} \in M_1$.

Let $G \subseteq \mathbb{P}$ be an (M, E)-generic filter. We want to define a function $f : I \to I$ which transforms G to G^* such that the following are satisfied:

1. G^* is an (M, E)-generic filter.
2. $\mathbb{R}^{M[G^*]} = \mathbb{R}^{M[G]}$.
3. $\mathcal{R}^{G^*} = \mathcal{R}^G$.

4. $(\tau_{\rho_0})^{G^*} = (\tau_{\rho_1})^G$.
5. $(\tau_{\rho_1})^{G^*} = (\tau_{\rho_0})^G$.
6. $\tau^{G^*} = \tau^G$.

Let $\pi : M \to M$ be the automorphism given by sending ρ_k to ρ_{k+1}, $k \in \mathbb{Z}$. As (M, E) is an ω-model, $\pi(n) = n$, for every $n < \omega (= \omega^{(M,E)})$. Moreover, π has the following local property.

Claim. π is *locally countable* in (M, E), i.e., for any $X \in M$ such that X is countable in (M, E), $\pi|X \in M$.

The key is that π is an automorphism and $\pi|\omega = \mathrm{id} \in M$. Suppose $X \in M$ and $\sigma : X \to \omega$ is a 1-1 function in (M, E). $\pi(\sigma)$ is in (M, E) as well. Note that for any $a \in X$, $\pi(\sigma)(\pi(a)) = \pi(\sigma(a))$, $\pi(\sigma) \circ \pi = (\pi|\omega) \circ \sigma$. Hence

$$\pi|X = (\pi(\sigma))^{-1} \circ (\pi|\omega) \circ \sigma \in M.$$

This proves the claim.

Now we define the function $f : I \to I$ as follows:
Suppose $s \in M_0 \cap M_1$. Then $f(s) = s$. For $s \notin M_0 \cap M_1$, define $f(s)$ as follows, there are three cases. Let i be least such that $s(i) \notin M_0 \cap M_1$,

- if $s(i) \in M_0$, let $f(s) = \pi(s)$;
- if $s(i) \in M_1$, let $f(s) = \pi^{-1}(s)$;
- if $s(i) \notin M_0 \cup M_1$, let $f(s) = s$.

f has the following properties:

Proposition 3.4. 1. *$f : I \to I$ is an automorphism on $\langle I, \leq_I \rangle$.*
2. *For $s \in I$,*
   - *if $s \in M_0 \cap M_1$, then $f(s) = s$.*
   - *if $s \in M_0$, then $f(s) = \pi(s)$.*
   - *if $s \in M_1$, then $f(s) = \pi^{-1}(s)$.*
3. *f is locally countable in (M, E).*

Proof of Proposition 3.4. (2) is because π and π^{-1} are the identity on $I \cap M_0 \cap M_1$. (3) follows from the local countability of π and π^{-1}. We verify (1).

It is not difficult to see that $f(s) \in I$ for every $s \in I$ and $f = f^{-1}$, thus f is a bijection on I. Now we show that f is order preserving.

Suppose $s \leq_I t$, we show that $f(s) \leq_I f(t)$. Note that t is an initial segment of s. If $s \in M_0 \cap M_1$, then $t \in M_0 \cap M_1$, thus $f(s) = s \leq_I t = f(t)$. Suppose $s \notin M_0 \cap M_1$, and i least such that $s(i) \notin M_0 \cap M_1$.

If $i < \mathrm{lh}(t)$, then $s(i) = t(i)$. No matter which set $s(i)$ belongs to, $f(s) \leq_I f(t)$. If $i \geq \mathrm{lh}(t)$, then $t \in M_0 \cap M_1$ and $f(t) = t$, while

$$\begin{aligned} f(s) &= \pi(s|i) \cup \pi(s \setminus (s|i)) \\ &= s|i \cup \pi(s \setminus (s|i)) \supseteq s|i \supseteq t. \end{aligned}$$

Hence $f(s) \leq_I f(t)$. $\dashv$ Proposition

f induces a transformation from G to G^* as follows: We view conditions in $\mathbb{P}$ as finite sets of triples of the form $p = (\sigma, n, i)$, where $\sigma \in I$ is a finite descending sequence of ordinals $< \kappa$, $n < \omega$ and $i \in \{0, 1\}$. For every $p = (\sigma, n, i) \in \mathbb{P}$, define

$F(p) = \{(f(\sigma), n, i) \mid (\sigma, n, i) \in p\}$. Let G^* be the image of G under F, $F''G$. For later use, we recursively define

$$F_*(\sigma) = \{(F_*(\tau), F(p)) \mid (\tau, p) \in \sigma\},$$

for $\sigma \in (M, E)^{\mathbb{P}}$.

The following is a property of locally countable isomorphisms on c.c.c. partial orders. Let ZFC^* denote a sufficient fragment of ZFC.

Proposition 3.5. *Suppose $(M, E) \models \mathsf{ZFC}^*$. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are two c.c.c. posets in (M, E). Suppose $F : \mathbb{P} \to \mathbb{Q}$ has the following properties:*

- *F is an isomorphism;*
- *F is locally countable in (M, E);*
- *For every $p_0, p_1 \in \mathbb{P}$, if $F(p_0), F(p_1) \in M$ and $(M, E) \models$ "p_0, p_1 are $\leq_{\mathbb{P}}$-incompatible", then $(M, E) \models$ "$F(p_0), F(p_1)$ are $\leq_{\mathbb{Q}}$-incompatible".*

Let $G \subseteq \mathbb{P}$. Then

1. *G is $\mathbb{P}$-generic over (M, E) iff $F''G$ is $\mathbb{Q}$-generic over (M, E).*
2. *If G is $\mathbb{P}$-generic over (M, E), then $\mathbb{R}^{(M,E)[G]} = \mathbb{R}^{(M,E)[F''G]}$.*

Proof. 1. Notice that if F has the above three properties then so does F^{-1} (with $\mathbb{P}$, $\mathbb{Q}$ switched in the third property). We only need to show one direction. Let G be a $\mathbb{P}$-generic filter.

Suppose $\mathcal{A} \in M$ and $(M, E) \models$ "$\mathcal{A}$ is a maximal antichain in $\mathbb{Q}$". Since $\mathbb{Q}$ is c.c.c. in (M, E), $\mathcal{A} \in M$ is countable in (M, E); by the local countability of F^{-1}, $(F^{-1})|\mathcal{A}$ and $(F^{-1})''\mathcal{A}$ are in M as well. The third property of F^{-1} ensures that $(M, E) \models$ "$(F^{-1})''\mathcal{A}$ is a maximal antichain in $\mathbb{P}$". Since G is $\mathbb{P}$-generic, there is some $r \in G \cap (F^{-1})''\mathcal{A} \cap M$. Then $F(r) \in (F''G) \cap \mathcal{A} \cap M$. Hence, $F''G$ is $\mathbb{Q}$-generic.

2. We show one direction $\mathbb{R}^{(M,E)[F''G]} \supseteq \mathbb{R}^{(M,E)[G]}$, a similar argument works for the other direction. We use nice names. Viewing names for reals as subsets of $\omega \times \mathbb{P}$, a *nice $(M, \mathbb{P})$-name* for a real ω is of the form $\bigcup_{n<\omega}\{n\} \times \mathcal{A}_n$, where each $\mathcal{A}_n \in M$ and $(M, E) \models$ "$\mathcal{A}_n$ is a maximal antichain in $\mathbb{P}$". For reals in the generic extension, we may consider nice $\mathbb{P}$-names directly.

If τ is a nice $(M, \mathbb{P})$-name for a real, then so is $F_*(\tau)$. Moreover,

$$p \Vdash n \in \tau \qquad \textit{iff} \qquad F(p) \Vdash n \in F_*(\tau).$$

Since $p \in G$ iff $F(p) \in F''G$, it follows that $(F_*(\tau))^{F''G} = \tau^G$. ⊣

Since $f : I \to I$ is a bijection, the map F defined on page 589 is an automorphism on $\mathrm{Fn}(I \times \omega, 2)$. It is easy to check that M, F satisfy the conditions in Proposition 3.5, so we have

Claim. If $G \subseteq \mathbb{P}$ is an (M, E)-generic filter, then G^* is an (M, E)-generic filter.

Claim. $\mathbb{R}^{(M,E)[G]} = \mathbb{R}^{(M,E)[G^*]}$.

For each $s \in I$, let $\dot{x}_s$ be the set $\{(\check{n}, p) \mid n < \omega,\ p \in \mathbb{P}$ and $(s, n, 1) \in p\}$. $\dot{x}_s$ is the canonical $\mathbb{P}$-name for the generic Cohen real indexed by s. By the definition of F_*, for each $s \in I$, $F_*(\dot{x}_s) = \dot{x}_{f(s)}$. Thus $F_*(\dot{x}_s)$ is the generic Cohen real indexed by $f(s)$. Note that $(F_*(\dot{x}_s))^{G^*} = (\dot{x}_s)^G$, it follows that $\mathrm{field}(\mathcal{R}^{G^*}) = \mathrm{field}(\mathcal{R}^G)$. By Proposition 3.4, f is an automorphism on $\langle I, \leq_I\rangle$, so we have

Claim. $\mathcal{R}^{G^*} = \mathcal{R}^G$.

(M, E) is an ω-model, so automorphisms on (M, E) fix natural numbers in (M, E). The following properties follows from Proposition 3.4,

Proposition 3.6. *For every condition* $p \in \mathbb{P}$,

$$\begin{aligned} p \in G \cap M_0 &\Leftrightarrow \pi(p) \in G^* \cap M_1, \\ p \in G \cap M_1 &\Leftrightarrow \pi^{-1}(p) \in G^* \cap M_0, \\ G \cap M_0 \cap M_1 &= G^* \cap M_0 \cap M_1. \end{aligned}$$

Recall in the preliminary remarks on forcing over ω-models: Suppose $\mathbb{P}$ is a c.c.c. partial order in (M, E). Suppose $(M_0, E_0) \prec (M, E)$ and $\mathbb{P} \in M_0$. Suppose $G \subset \mathbb{P}$ is (M, E)-generic. If $\sigma \in M_0$ and $(M, E) \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash \sigma \in V_{\omega+1}$", then the interpretation of σ by G, σ^G, is uniquely determined by $G \cap M_0$ and $\sigma^G = \sigma^{G \cap M_0}$.

Now let (x, y, z) be the interpretation of $(\tau_{\rho_0}, \tau_{\rho_1}, \tau)$ by G and let (x^*, y^*, z^*) be the interpretation of $(\tau_{\rho_0}, \tau_{\rho_1}, \tau)$ by G^*. Then

- x is the interpretation of τ_{ρ_0} by $G \cap M_0$,
- y is the interpretation of τ_{ρ_1} by $G \cap M_1$,
- z is the interpretation of τ by $G \cap M_0 \cap M_1$,
- x^* is the interpretation of τ_{ρ_0} by $G^* \cap M_0$,
- y^* is the interpretation of τ_{ρ_1} by $G^* \cap M_1$,
- z^* is the interpretation of τ by $G^* \cap M_0 \cap M_1$.

By Proposition 3.6, we have

Claim. $x = y^*$, $y = x^*$ and $z = z^*$.

Now in $(M, E)[G]$, we have

$$(\mathbb{R}^{(M,E)[G]}, \mathcal{R}^G) \models \varphi[x, y, z, \mathcal{R}^G].$$

Since G is (M, E)-generic, G^* is (M, E)-generic. Hence in $(M, E)[G^*]$,

$$(\mathbb{R}^{(M,E)[G^*]}, \mathcal{R}^{G^*}) \models \varphi[x^*, y^*, z^*, \mathcal{R}^{G^*}].$$

By the properties (2)–(6) on page 588, established in the last three claims, we have

$$(\mathbb{R}^{(M,E)[G]}, \mathcal{R}^G) \models \varphi[y, x, z, \mathcal{R}^G].$$

This contradicts the assumption that $(\varphi, z, \mathcal{R})$ defines a total order.

3.4.3. *Proof of Lemma 3.3, the full proof.* Now we prove the full version of Lemma 3.3.

As we did in the warm-up, let T be a theory as in the hypothesis, (M, E) be the $\mathbb{Z}$-model for T. Let $\langle \rho_k : k \in \mathbb{Z} \rangle$ be the generating indiscernibles in ascending order. $\rho_k < \omega_{\omega_1}$, for all $k \in \mathbb{Z}$.

Assume that the statement is false in (M, E). Let $p_0, \kappa, s, \tau \in M$ be such that for some Σ^1_n formula $\varphi(t_0, t_1, t_2)$:

1. $(M, E) \models$ "$(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \Vdash \tau \in \mathbb{R}$",
2. $(M, E) \models$ "$(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \Vdash$ '$\{(x, y) \in \mathbb{R} \mid \varphi[x, y, \tau]\}$ is the strict part of a total preorder'",
3. $(M, E) \models$ " s is a function with domain ω_{ω_1}",
4. $(M, E) \models$ " for all $b < \omega_{\omega_1}$, $s(b)$ is a term in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$",

5. $(M, E) \models$ " for all $b_1 < b_2 < \omega_{\omega_1}$, $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \Vdash \varphi[s(b_1), s(b_2), \tau]$"
6. p_0, κ, s and τ are each definable in (M, E) (no parameters).

Here $\mathbb{P} = \mathrm{Fn}(I_\kappa \times \omega, 2)$, $\mathbb{Q} = \mathbb{Q}_{X^G}$, and $\dot{\mathbb{1}}_{\mathbb{Q}}$ is the term for the maximal element of $\mathbb{Q}$, i.e., the condition $(\varnothing, \varnothing, \varnothing)$.

Clearly $\mathbb{P} * \dot{\mathbb{Q}}$ is definable in (M, E). We showed early that in $V[G]$, $\mathbb{Q}_{X^G}$ has the property (H). So $p_0, \kappa, s, \tau \in M$ can be chosen as required.

Let ρ_0, ρ_1, π, f, (M_0, E) and (M_1, E) be the same as in the warm-up. Suppose $G \subset \mathbb{P}$ is a (M, E)-generic filter, and let G^* denote the (M, E)-filter given by transforming G via the function f. In the warm-up, we have shown the following, with respect to $\mathbb{P}$:

1. $\mathbb{R}^{M[G^*]} = \mathbb{R}^{M[G]}$.
2. $\mathcal{R}^{G^*} = \mathcal{R}^G$.
3. For every condition $p \in \mathbb{P}$,

$$\begin{aligned} p \in G \cap M_0 &\Leftrightarrow \pi(p) \in G^* \cap M_1, \\ p \in G \cap M_1 &\Leftrightarrow \pi^{-1}(p) \in G^* \cap M_0, \\ G \cap M_0 \cap M_1 &= G^* \cap M_0 \cap M_1. \end{aligned}$$

As for $\mathbb{P} * \dot{\mathbb{Q}}$, suppose $H \subset \mathbb{P} * \dot{\mathbb{Q}}$ is a (M, E)-generic filter with $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \in H$. Let

$$\begin{aligned} G &= \{p \in \mathbb{P} \mid (\exists \dot{q} \in \dot{\mathbb{Q}})(p, \dot{q}) \in H\}, \\ h &= \{\dot{q}^G \mid (\exists p \in G)(p, \dot{q}) \in H\}. \end{aligned}$$

Then $G \subset \mathbb{P}$ is a V-generic filter, $h \subset (\dot{\mathbb{Q}})^G$ is a $V[G]$ generic filter, $p_0 \in G$ and H is determined by (G, h). Let

$$H^* = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \in G^* \text{ and } \dot{q}^{G^*} \in h\}.$$

We show that:

1. H^* is an (M, E)-generic filter over $\mathbb{P} * \dot{\mathbb{Q}}$ and $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \in H^*$.
2. $\mathbb{R}^{(M,E)[H^*]} = \mathbb{R}^{(M,E)[H]}$.
3. $(\tau_{\rho_0})^{H^*} = (\tau_{\rho_1})^H$.
4. $(\tau_{\rho_1})^{H^*} = (\tau_{\rho_0})^H$.
5. $\tau^{H^*} = \tau^H$.

Since $\mathcal{R}^{G^*} = \mathcal{R}^G$, we have $(\dot{\mathbb{Q}})^{(M,E)[G]} = (\dot{\mathbb{Q}})^{(M,E)[G^*]}$. $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is c.c.c.", so

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{"antichains of } \dot{\mathbb{Q}} \text{ can be coded by reals in } V[G]\text{"}.$$

Since $\mathbb{R}^{(M,E)[G]} = \mathbb{R}^{(M,E)[G^*]}$, $(M, E)[G^*]$ and $(M, E)[G]$ have the same collection of antichains for $\mathbb{Q}$. This implies that h is $(M, E)[G^*]$-generic for $\mathbb{Q}$. Note that p_0 is in $M_0 \cap M_1$, so $p_0 \in G^* \cap M_0 \cap M_1$ and $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \in H^*$. Therefore, we have shown that

Claim. H^* is (M, E)-generic for $\mathbb{P} * \dot{\mathbb{Q}}$ and $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \in H^*$.

Let b_h be the subset of ω given by h. Recall that h can be recovered from b_h. So every $r \in \mathbb{R}^{(M,E)[G][h]}$ is definable from some $x \in \mathbb{R}^{(M,E)[G]}$ and b_h, i.e.,

$$\mathbb{R}^{(M,E)[G][h]} = \bigcup \{L_\omega(x, b_h) \mid x \in \mathbb{R}^{(M,E)[G]}\}.$$

Since $\mathbb{R}^{(M,E)[G]} = \mathbb{R}^{(M,E)[G^*]}$, it follows immediately that

CLAIM. $\mathbb{R}^{(M,E)[H^*]} = \mathbb{R}^{(M,E)[H]}$.

$\mathbb{P} * \dot{\mathbb{Q}}$ is in $M_0 \cap M_1$. τ_{ρ_0}, τ_{ρ_1}, τ are in M_0, M_1, $M_0 \cap M_1$ respectively, so their interpretation by H are uniquely determined by $H \cap M_0$, $H \cap M_1$, $H \cap M_0 \cap M_1$, similarly for H^*. Therefore, to see (3)–(5), it suffices to show that

PROPOSITION 3.7. *For every condition* $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$,

$$\begin{aligned}(p, \dot{q}) \in H \cap M_0 &\Leftrightarrow \pi((p, \dot{q})) \in H^* \cap M_1,\\ (p, \dot{q}) \in H \cap M_1 &\Leftrightarrow \pi^{-1}((p, \dot{q})) \in H^* \cap M_0,\\ H \cap M_0 \cap M_1 &= H^* \cap M_0 \cap M_1.\end{aligned}$$

PROOF OF PROPOSITION 3.7. We only show the first equivalence. The argument for the second one is similar, and the third identity follows from the first two equivalences. Since π and f are bijections, it suffices to show one direction.

Assume $(p, \dot{q}) \in H \cap M_0$. $\pi((p, \dot{q}))$ is in M_1. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is in $M_0 \cap M_1$,

$$\pi((p, \dot{q})) = (\pi(p), \pi(\dot{q}))$$

is also a condition in $\mathbb{P} * \dot{\mathbb{Q}}$. By the definition of $\mathbb{Q}_X$, the transitive closure of $\dot{q}$ is countable in M, and therefore contained in M_0. Note that $p \in G \cap M_0 \Leftrightarrow \pi(p) \in G^* \cap M_1$. Applying this inductively on ranks of elements in the transitive closure of $\dot{q}$, we have $\dot{q}^G = \pi(\dot{q})^{G^*}$. Thus if $\dot{q}^G \in h \cap (M_0, E)[G]$ then $\pi(\dot{q})^{G^*} \in h \cap (M_1, E)[G^*]$. This shows that $(\pi(p), \pi(\dot{q})) \in H^* \cap M_1$. ⊣

Now let (x, y, z) be the interpretation of $(\tau_{\rho_0}, \tau_{\rho_1}, \tau)$ by H and let (x^*, y^*, z^*) be the interpretation of $(\tau_{\rho_0}, \tau_{\rho_1}, \tau)$ by H^*. Then

- x is the interpretation of τ_{ρ_0} by $H \cap M_0$,
- y is the interpretation of τ_{ρ_1} by $H \cap M_1$,
- z is the interpretation of τ by $H \cap M_0 \cap M_1$,
- x^* is the interpretation of τ_{ρ_0} by $H^* \cap M_0$,
- y^* is the interpretation of τ_{ρ_1} by $H^* \cap M_1$,
- z^* is the interpretation of τ by $H^* \cap M_0 \cap M_1$.

We have

CLAIM. $x = y^*$, $y = x^*$ and $z = z^*$.

Finally, in $(M, E)[H]$, we have

$$\mathbb{R}^{(M,E)[H]} \models \varphi[x, y, z].$$

Since H is (M, E)-generic, H^* is (M, E)-generic. So in $(M, E)[H^*]$,

$$\mathbb{R}^{(M,E)[H^*]} \models \varphi[x^*, y^*, z^*].$$

By the properties (2)–(5) on page 592, we have

$$\mathbb{R}^{(M,E)[H]} \models \varphi[y, x, z].$$

This contradicts the assumption that (φ, z) defines a total order.

This completes the proof of Lemma 3.3 and the proof of Theorem 3.1.

§**4. Upper bounds for** δ**.** The ω-completeness condition is heavily used throughout our argument. A big advantage of doing so is that models for a ω-complete theory are ω-models, hence a large amount of agreement among models M_0, M_1 and $M_0 \cap M_1$ holds automatically. This makes our presentation much easier. However, one drawback of using ω-completeness is that the ω-completeness condition makes ω_ω a lower bound for all the indiscernibles. This makes it hard to see that ω_ω is in fact an upper bound for δ in our model.

Fix a λ such that $L \models$ "λ is a uncountable limit cardinal". Let T be an ω-complete extension of $T_0 = \mathrm{Th}(L_\lambda)$.

LEMMA 4.1. *"$c_0 > \omega_n$" is in T, for every $n < \omega$. Moreover, "$c_0 > \omega_\omega$" is also in T.*

PROOF. The key is the following claim:

CLAIM. "$c_0 \to (m)^n_\omega$" is in T, for every $m, n < \omega$.

Suppose not. Let (M, E) be the $\mathbb{Z}$-model of T. Let $\langle \rho_k : k \in \mathbb{Z}\rangle$ be the generating indiscernibles. Fix an (m, n) such that

$$\text{“ } c_0 \nrightarrow (m)^n_\omega \text{ ” is in } T.$$

By indiscernibility, for every $k \in \mathbb{Z}$, $(M, E) \models \rho_k \nrightarrow (m)^n_\omega$. Pick any $i \in \mathbb{Z}$. Note that the set $\{k \in \mathbb{Z} \mid k <_{\mathbb{Z}} i\}$ is infinite. Let $F : [\rho_i]^n \to \omega$ be the $<_{(M,E)}$-least coloring function that witnesses $\rho_i \nrightarrow (m)^n_\omega$. By the ω-completeness and the indiscernibility of ρ_k's, $\{\rho_k \mid k <_{\mathbb{Z}} i\}$ is a F-homogeneous set of size $> m$. Contradiction!

According to Erdős-Hajnal-Rado [1]: For any cardinal $\kappa \geq \omega$ and $n < \omega$, $\kappa^{+n} \nrightarrow (n+2)^{n+1}_\kappa$. It must be that $T \models$ "$c_0 > \omega_n$", for all $n < \omega$, and by the indiscernibility, the same holds for every c_k $(k < \omega)$. And moreover, by the ω-completeness, $T \models$ "$c_0 > \omega_\omega$". ⊣

A condition weaker than ω-completeness enables us to obtain indiscernibles below ω_ω and hence to argue that $\delta \leq \omega_\omega$. Here is a version of Lemma 3.2 with ω-completeness replaced by what we call the *remarkability* condition:

LEMMA 3.2′. *Assume* ZFC $+ V = L$. *Let λ be a limit ordinal $> \omega_\omega$ such that*

$$L_\lambda \models \mathsf{ZFC}\backslash \textit{Replacement} + \Sigma_1\textit{-Replacement}.$$

Let $T_0 = \mathrm{Th}(L_\lambda)$. Then there exists a theory T such that

1. *T is an extension of T_0 in the language $\mathcal{L}^*$.*
2. *T is a complete theory for which c_i, $i < \omega$, are indiscernibles.*
3. *T satisfies the following properties*:
   - *"$c_0 < \omega_\omega$" is in T.*
   - (remarkability). *For every formula $\varphi(x_0, \dots, x_i)$ in the language of set theory, for any $\{c_{n_0} < \cdots < c_{n_i}\}$ and $\{c_{m_0} < \cdots < c_{m_i}\}$, if*

     $$\text{“}t_\varphi(c_{n_0}, \dots, c_{n_i}) \in \omega\text{” is in } T$$

     then

     $$\text{“}t_\varphi(c_{n_0}, \dots, c_{n_i}) = t_\varphi(c_{m_0}, \dots, c_{m_i})\text{” is in } T.$$

Such a theory can be obtained in an ultrapower of L by a nonprincipal ultrafilter μ on ω. Let j denote the elementary embedding from L to (M, E), the transitive collapse of $\mathrm{Ult}(L, \mu)$. Applying the Erdős-Rado Theorem inductively as in the proof of Lemma 3.2, but below $j(\omega_\omega)$, we get $\{T_i \mid i < \omega\}$, where each T_i is the theory of $(c_0, \dots, c_i)$ in $j(L_\lambda)$. Then the union $T = \bigcup_i T_i$ satisfies the above requirements. Since $\omega^{\mathrm{Ult}(L,\mu)}$ is not standard, T is not ω-complete.

With a complete, remarkable theory of indiscernibles T, it is still true that

$$\text{“}c_0 > \omega_n\text{” is in } T,$$

for every standard natural number n. But we can no longer conclude that

$$\text{“}c_0 > \omega_\omega\text{” is in } T,$$

as in the case of ω-complete theories. The point is that, letting (M, E) be a model for T, the identity $\omega_{\omega^{(M,E)}} = \sup\{\omega_n \mid n \in \omega^L\}$ is not always true.

Now let (M, E) be the $\mathbb{Z}$-model for T. Following the argument in Subsection 3.4, carefully working with non-standard natural numbers in (M, E) while keeping the agreement among models M_0, M_1 and $M_0 \cap M_1$, one can prove

LEMMA 3.3′. *Suppose T is a theory as in Lemma 3.2′. Let (M, E) be the $\mathbb{Z}$-model of T. Then in (M, E), for any cardinal $\kappa > \omega_\omega$,*

$$\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \delta \leq \omega_\omega,$$

where $\mathbb{P} = \mathrm{Fn}(I_\kappa \times \omega, 2)$ and $\mathbb{Q} = \mathbb{Q}_{X^G}$.

Exactly as Theorem 3.1 follows from Lemma 3.2 and 3.3, from Lemma 3.2′ and 3.3′ we can get: There is a c.c.c. forcing $\mathbb{P} \in L$ such that for any L-generic filter $G \subset \mathbb{P}$, $L[G] \models \delta < \lambda$.

Although the modified argument manages to “press” the upper bound of δ in the final extension down to ω_ω, this bound is probably the best one can get with the method of indiscernibles.

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BEIJING NORMAL UNIVERSITY
SCHOOL OF MATHEMATICAL SCIENCES
LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS
MINISTRY OF EDUCATION
BEIJING 100875, PEOPLE'S REPUBLIC OF CHINA
E-mail: shi.bnu@gmail.com