Large cardinals and Higher Degree Theory

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This is a survey paper on some recent developments in the study of higher degree theory, the theory of degree structure of generalized degrees at uncountable cardinals, in particular at those with countable cofinality.

The study of generalized degree notions, known as α-recursion theory, was initiated by Sacks in the early 70s. But the early works were mainly from recursion theoretical perspective, and the results are mostly limited to Gödel's constructive universe. Some recent works (see Refs. 1,2) in set theory indicate that there is a profound connection between the complexity of degree structures at singular cardinals of countable cofinality and the large cardinal properties at the vicinity. This paper shall exam degree structures cross through L-like models for various mild large cardinals, or under large cardinal assumptions, by which hopefully a new line of research will be depicted.

Keywords: Turing degree, hyperarithmetic degree, inner model operator, jump, incomparable degrees, minimal degree, Posner-Robinson Theorem, Degree Determinacy, higher degree theory, Covering Lemma, large cardinal, core model, Axiom I_0

1. Organization of the paper

In this article, we survey some recent developments in degree theory, in particular, the study of degree structures of generalized degrees at singular cardinals. The paper consists of three parts. In §2, we give a brief account of various generalizations of classical recursion theory, in particular the two directions – one up in degree notion hierarchy and the other lifting to larger ordinals. In this part, we investigate these structures focusing on a particular set of structural questions. In §3, we present the latest discovery of the connection between the complexity of degree structures at countable cofinality singular cardinals and the large cardinal strength of relevant cardinals. The structure of Zermelo degrees at countable cofinality singular cardinals in various core models of large cardinals, or under strong large cardinal principles, are compared, according to that particular list of structural questions. In the last part (§4) we put down some remarks, proposing directions for further investigation.

Notations in this paper are very set theoretic, if not given explicitly, mostly follow Jech[3] and Kanamori[4].

2. Generalizing the classical recursion theory

Early in 1960s, efforts had already been made to generalize classical recursion theory to higher level or in broader context. In this section, we briefly recall various generalizations of classical recursion theory, in particular the two directions – one up in degree notion hierarchy and the other lifting to larger ordinals.

2.1. *Inner model operators*

In classical recursion theory, two subsets $A, B \subseteq \omega$, A is *Turing reducible to* B, denoted as $A \leq_T B$, if both A and the complement of A can be recursively computed given B as an oracle, or equivalently, A is Δ^0_1 definable in B (both A and $\omega - A$ are Σ^0_1 definable in B, using B as a predicate symbol). This partial ordering induces an equivalence relation – the notion of Turing degrees – on subsets of ω. Turing degrees may as well be called Δ^0_1-degrees. For $A \subseteq \omega$, let $[A]_T$[a] denote the degree represented by A, and $[A]'_T$ denote the Turing jump of $[A]_T$, the degree of the Σ^0_1-theory of $(\mathbb{N}, +, \times, 0, 1, A)$.

By replacing Δ^0_1 with larger collection of sets, one can define degree notions for higher levels of definability. For instance, a natural landmark in this hierarchy of definability degree notions is the hyperarithmetic degree. The hyperarithmetic degrees are the Δ^1_1-degrees, obtained by considering the class of Δ^1_1 subsets of ω; the hyperarithmetic jump of $[\emptyset]$ is the degree of Kleene's $\mathcal{O}$, a complete Π^1_1-set.

In Ref. 5, Hodes applied Jensen's fine structure theory to iterate the jump through transfinite up to $\aleph_1^L$.

Theorem 2.1 (Hodes[5]). *Use Jensen's J_α hierarchy for L.*

(1) Whenever $(\Delta_{n+1}(J_\alpha) - \Delta_n(J_\alpha)) \cap \mathscr{P}(\omega) \neq \varnothing$, within this set there is a largest Turing degree, which contains $\Delta_{n+1}(J_\alpha)$-master code.

[a] In recursion theory, people normally use $\boldsymbol{a}$ or $\boldsymbol{a}_T$. Due to our excessive use of subscripts, we use the bracket form $[a]_T$, treating it as equivalence class.

(2) *The master code degrees are wellordered by the order of construction of their members in the J-hierarchy, and this order coincides with Turing reducibility on these degrees: for $\alpha < \aleph_1^L$, the Turing jump of the α-th master code is an $(\alpha+1)$-th master code.*

Master code is a terminology from fine structure theory. We omit its definition. The degree of α-th master codes is now widely accepted as the α-th iterate of the Turing jump, denoted as $\emptyset^{(\alpha)}$. The hyperarithmetic degree of $\emptyset$ consists of exactly those $x \subseteq \omega$ such that $x \leq_T \emptyset^{(\alpha)}$ for some $\alpha < \omega_1^{\mathrm{CK}}$, and Kleene's $\mathcal{O} \equiv_T \omega_1^{\mathrm{CK}}$-th master code. But master code is only helpful for degrees below the constructible degrees (see the proposition on p.4).

In Ref. 6, Steel made the notion of degree operator precise.

Definition 2.1 (Steel[6]). *Let $M : 2^\omega \to \mathscr{P}(2^\omega)$ and*

(1) $\forall x, y\, (x \equiv_T y \Rightarrow M(x) = M(y))$,
(2) $\forall x\, (M(x)$ *is closed under join, Turing jump and Turing reducibility*$)$,
(3) $\forall x, y\, (y \in M(x) \Rightarrow M(y) \subseteq M(x))$,
(4) *there is a relation $W(x,y,z)$ so that $\forall x\, (W_x = \{(y,z) \mid W(x,y,z)\}$ is a wellorder of $M(x))$,*
(5) *for $\alpha < \mathrm{otp}(W_x)$ and $e \in \omega$, let $z_e = z$ if there is a y such that $y \equiv_T x$ via e and z is the α-th element of W_y; and $z_e = \emptyset$ otherwise. Then there is a real in $M(x)$ coding the sequence $\langle z_e : e < \omega\rangle$.*[b]

Then we say that M is an inner model operator *(IMO).*

The intuition behind IMO is to consider $M(x)$ as $M_x \cap \mathscr{P}(\omega)$, where M_x is a transitive (set) model of ZFC, or natural fragment of ZFC. For the hyperarithmetic degree, the associated inner model operator is the map $x \mapsto L_{\omega_1^{\mathrm{CK}}}[x]$, the smallest Kripke-Platek model containing x. The constructible degree is given by $x \mapsto L[x]$. The readers can find more examples of "natural" inner model operators in[7].

For the next result, we assume AD. As we only work with (Turing) degree invariant sets and functions, we shall not distinguish $d \subseteq \omega$ and its degree $[d]$. Let μ be the cone measure on Turing degrees. Given two IMOs M, N, we say $M \leq N$ if $M(d) \subseteq N(d)$ for μ-a.e. d. A *jump operator* is a function $f : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ such that $f(d) \geq_T d$ for μ-a.e. d and f is

[b]This item is a uniformity requirement. $\mathrm{otp}(W_x)$ stands for the ordertype of W_x.

uniformly (Turing) degree invariant, i.e. there is a $\pi : \omega \to \omega$ such that for all $e < \omega$, $x \equiv_T y$ via $e \Rightarrow f(x) \equiv_T f(y)$ via $\pi(e)$.

Theorem 2.2 (Steel[6]). [c]

(1) $\leq$ prewellorders inner model operators.
(2) If $f(d) \in L[d]$ for μ-a.e. d, then f is a jump operator iff $f(d)$ is a d-master code for μ-a.e. d.

In some sense, this says that degree notions defined via inner models of the form $L_{\gamma_d}[d] \cap \mathscr{P}(\omega)$ (for some $\gamma_d < \aleph_1^L$), where d is a master code, forms the initial segment of the hierarchies of degree notions. For IMOs of this form, the jump of the associated degree notions is the least d-master code not in $M(d)$ (for $d \subset \omega$), and this master code codes the relevant theory of $M(d)$:

Proposition 2.1 (Hodes[5]). *If $L_{\gamma_d}[d] \not\models \Delta_{n+1}\text{-}\mathsf{CA}$, then $(\omega\alpha + n)$-th master code for d codes the n-quantifier theory of $(L_{\gamma_d}[d], \in, d)$.*

The theories (in the language of second-order arithmetic) related to these IMOs include Δ^1_n-CA $(n < \omega)$, full CA, Σ^0_α-det $(\alpha < \omega_1^{\mathrm{CK}})$, Δ^1_1-det. Π^1_1-det is the least theory whose associated IMO is not of aforementioned form, as this theory is equivalent to the existence of sharps. For IMOs whose associated theories include full comprehension, the associated jump of $d \subset \omega$ naturally has to be the sharp of M_d, namely, the set coding the full (first-order) theory of $(M_d, \in, d)$.

Finding the "right" analogue of degree notions to higher levels, in particular generalizing Δ^1_1-degree to the second-order pointclasses in the context of projective determinacy, once had been in the focus of the interest of descriptive set theorists for some time around 1980s. Though many of the results are folklores among Cabal people, the reader can still find a good account on the development at the time in [7]. Although this is a fascinating topic, in this paper we would like to focus on the structure of these degree structures, which did not get much attention of descriptive set theorists, at least judging by the literature.

[c] Steel's results on jump operators is part of his investigation on Martin's conjecture, and further developments on this topic can be found in [8] and [7].

2.2. *Degree structures I: The Partial Order*

In this part, we compile some of known facts about various degree structures. Most results in the literature are statements in $L(\mathbb{R})$, however, $L(\mathbb{R})^V$ may vary if V is different, thus the degree structures differ drastically in these models. Later in this part, we will also briefly discuss certain degree structures in some inner models of set theory which do not have all the reals.

Given a definable reducibility notion $\leq$, let $\equiv$ denote the induced equivalence relation: $x \equiv y \Leftrightarrow x \leq y \wedge y \leq x$, and $[a]$ denote the equivalence class of a, i.e. the degree of a. For different degree notions, we use subscripts to distinguish them.

First, it's easy to see that the partial ordering $(\mathscr{D}, \leq)$, together with the join operator, is an upper semi-lattice. And secondly, as the set $\{y \subset \omega \mid y \leq x\}$ is countable for any $x \subset \omega$, $(\mathscr{D}, \leq)$ must have height ω_1. Then the next question is naturally about its width.

Post Problem is the question whether there are incomparable degrees, i.e. $[a], [b]$ such that $\neg([a] \leq [b] \vee [b] \leq [a])$. If yes, then the further question is how large the maximal antichains could be.

Another question regarding the order type of $(\mathscr{D}, \leq)$ is the *density question*: Is it always true that given two degrees $[a] < [b]$, there is always a $[c]$ such that $[a] < [c] < [b]$? The negation of this question is whether there is a gap, namely, a pair of degrees $[a] < [b]$ such that there is no c such that $[a] < [c] < [b]$. Such $[b]$ is called a *minimal cover* of $[a]$. If $[a] = [\emptyset]$, $[b]$ is called a *minimal degree*. One can also further ask the question whether the relation $\leq$ is well-founded, namely, if there is an infinite strictly $\leq$-decreasing sequence.

Let $(\mathscr{D}_T, \leq_T)$ denote the structure of Turing degrees and $(\mathscr{R}, \leq_T)$ the structure of recursively enumerable (r.e.) Turing degrees. Here are some of their properties:

(1) Friedberg-Muchnik[9,10] showed that there are incomparable degrees in $(\mathscr{R}, \leq_T)$. The same is true in $(\mathscr{D}_T, \leq_T)$ (Kleene-Post[11]).
(2) $(\mathscr{R}, \leq_T)$ is dense (Sacks[12]), while $(\mathscr{D}_T, \leq_T)$ has minimal degrees (Spector[13], Sacks[14]). In fact, there are 2^ω many minimal degrees, which are also pairwise incomparable degrees.
(3) $(\mathscr{R}, \leq_T)$ is ill-founded as it is dense. Furthermore, Harrison[15] (see also Ref. 16, III.3.6) showed that in $(\mathscr{D}_T, \leq_T)$ there is an infinite sequence of degrees $\langle [a_i]_T : i < \omega\rangle$ such that $[a_{i+1}]_T' \leq [a_i]_T$, for all $i < \omega$.

There are abundant results on the structural properties of $(\mathscr{R}, \leq_T)$ and $(\mathscr{D}_T, \leq_T)$, for instance, minimal upper bound, exact pairs, poset-embedding problem, high/low hierarchies, decidability problems, etc. It's beyond our ability to discuss all of them here, we only mention some basic ones to make our points.

By a result of Spector [17] (see also Ref. 16, II.7.2), restricted to Π^1_1 reals, there are only two $\equiv_h$-degrees, thus the analogue of $(\mathscr{R}, \leq_T)$ for hyperarithmetic degrees is a trivial poset, only has the degrees of $\emptyset$ and Kleene's $\mathcal{O}$. Let $(\mathscr{D}_h, \leq_h)$ denote the structure of hyperarithmetic degrees. Adding two mutually generic reals Cohen generic over $L_{\omega_1^{\mathrm{CK}}}$ produces two incomparable $\equiv_h$-degrees. And by adding ω Cohen reals that mutually generic over $L_{\omega_1^{\mathrm{CK}}}$, one can arrange a set of $<_h$-decreasing sequence of reals. Thus $(\mathscr{D}_h, \leq_h)$ is not wellfounded. With perfect set forcing, Sacks [18] showed that Sacks reals have minimal degrees in $(\mathscr{D}_h, \leq_h)$. Again, there are also 2^ω many minimal degrees, and therefore pairwise incomparable degrees in $(\mathscr{D}_h, \leq_h)$.

These forcing arguments all work for constructibility degrees $(\mathscr{D}_c, \leq_c)$ and lead to the same conclusion. Below are some other relevant results.

(1) (Friedman [19]) Assume $0^\sharp$ exists. There is a Π^1_2 singleton $a \subseteq \omega$ such that $0 <_c [a]_c <_c 0^\sharp$.[d] So the constructibility degrees restricted to Π^1_2 singletons is non-trivial.
(2) (Friedman [19]) Let X be the set of Π^1_2 singletons that is $\leq_c$-comparable with every Π^1_2-singleton. Then $\leq_c$ restricted to X is pre-wellordered, and for every $x \in X$, the immediate $<_c$-successor of $[x]_c$ is $[x^\sharp]_c$.
(3) (Harrington-Kechris [20]) If $d \subseteq \omega$ is a Π^1_2-singleton, then either $0^\sharp \leq_c a$ or $0^\sharp \equiv_c d^\sharp$. As a result of relativizing their argument, one can get that $[0^\sharp]_c, [0^{\sharp\sharp}]_c, [0^{\sharp\sharp\sharp}]_c, \ldots$ are the first ω degrees of sharps – $\leq_c$-jumps – of Π^1_2 singletons.

Next let us look at the structure of Δ^1_n-degrees. For that Projective Determinacy (PD) is always assumed for the sake of convenience, we leave it to the reader to figure out the necessary amount of "local" determinacy needed for each statement.

Theorem 2.3 (Kechris [21]). [e] *Suppose $n > 0$ is odd.*

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[d] $0^\sharp$ is a Π^1_2 singleton.
[e] According to Ref. 21, the three results below in the case $n = 1$ were also proved independently by D. Guaspari and G. Sacks [18].

(1) There exists a largest thin Π^1_n set, denoted as C_n;
(2) C_n is closed under Δ^1_n-jump, i.e. $[d]_n \subset C_n \Rightarrow [d]'_n \subset C_n$;
(3) The Δ^1_n-degrees of members of C_n are wellordered by $[a]_n \leq_n [b]_n \Leftrightarrow$ a is Δ^1_n in b, in particular, the immediate $<_n$-successor of every Δ^1_n-degree is its Δ^1_n-jump.

Let $(\mathscr{R}_n, \leq_n)$, $n > 0$ odd, be the analogue of $(\mathscr{R}, \leq_T)$ for Δ^1_n-degrees, namely the Δ^1_n-degrees for Π^1_n subset of ω. Then we have

Corollary 2.1. *$(\mathscr{R}_n, \leq_n)$, $n > 0$ odd, is a trivial poset, consisting of only the smallest and the largest elements – the degree of complete Π^1_n set of integers.*[f]

It is a well known fact in set theory that $\mathbb{R} \cap L$ is the largest countable Σ^1_2 set if $\aleph_1^L < \aleph_1$ (by Solovay[22]), and there exists a largest countable Σ^1_n set of reals for every even n (by Kechris-Moschovakis[23]). These largest countable sets are denoted as C_n for $n > 0$ even. Each C_{2n} ($n > 0$) is the set of reals in an inner model of ZFC (for instance $L(C_{2n})$). However, C_{2n+1} is not the set of reals of any transitive model of ZFC. In Ref. 24, the authors invented the Q-set in order to develop the "right" theory generalizing that of hyperarithmetic degrees to odd levels of second order arithmetics. We omit the definition of Q_{2n+1}, $n > 0$, but only the following relevant facts about Q_{2n+1}.

Theorem 2.4 (Kechris-Moschovakis-Solovay[24]). *Assume* PD.

(1) Q_{2n+1} is the maximal countable Π^1_{2n+1} set downward closed under Turing as well as Δ^1_{2n+1}-degrees;
(2) Q_{2n+1} is closed under the Δ^1_{2n+1}-jump;
(3) The Δ^1_{2n+1}-degrees restricted to Q_{2n+1} forms a proper initial segment of that restricted to C_{2n+1};
(4) $Q_{2n+1} = \mathbb{R} \cap L(Q_{2n+1})$.

Let $(\mathscr{D}_n, \leq_n)$ denote the poset of Δ^1_n-degrees. Then we have a rather simple structure of degree in an inner model of ZFC.

Corollary 2.2 (PD). *In $L(Q_{2n+1})$, $n > 0$, $(\mathscr{D}_{2n+1}, \leq_{2n+1})$ is a wellordering of ordertype $\omega_1^{L(Q_{2n+1})}$.*

[f]For the case $n = 1$, Spector[17] (see also Ref. 16, II.7.2) showed that every Π^1_1 set of integers is either $\leq_1 \emptyset$ or $\geq_1 \mathcal{O}$. The case $n > 1$ is Theorem (3B-1) of Ref. 21.

Of course, one can still do the forcing argument as before to produce complex degree structures, however, what interests us is the simplicity of these degree structures in inner models – notice that in $L(\mathbb{R})$ the poset $(\mathscr{D}, \leq_T)$ has all the properties discussed earlier using forcing arguments (see p.6). So the question is what causes, or when does, the change happen.

Very little is known about the degrees at the even levels at this point.

2.3. *Degree Structures II: Posner-Robinson Problem and Degree Determinacy*

Before moving on to the next topic, we would like to discuss two more properties of degree structures. The first one, we called it *Posner-Robinson problem.*

In classical recursion theory, a fundamental task is to understand the jump operator. In the literature, there are quite a number of jump inversion theorems for that purpose. The Posner-Robinson theorems to be discussed here belongs to jump inversion problems, the basic theme is that every nontrivial real can be viewed as a jump of some other real (modulo that real itself).

For $X, Y, Z \subset \omega$, $X \leq_T (Y, Z)$ if X is recursive in the pair (Y, Z), i.e. there is a recursive bijection $\pi : \omega \to \omega \times \omega$ such that $X \leq_T \pi^{-1}[Y \times Z]$. (Y, Z) here is essentially the join of Y and Z.

The classical Posner-Robinson theorem (see Refs. 25,26) asserts that for any $A \not\leq_T \emptyset$, there is a real G such that A appears to be the Turing jump of G modulo G, more precisely, $G' \equiv_T (A, G)$. Shore-Slaman[27] generalize this to any α-REA operators, $\alpha < \omega_1^{\mathrm{CK}}$. More precisely, for any $A \notin I_{<\alpha}$, there is a real G such that $G^{(\alpha)} \equiv_T (A, G)$, where $I_{<\alpha} = \{X \subset \omega \mid \exists \beta < \alpha\, (X \leq_T \emptyset^{(\beta)})\}$. Woodin later proved (unpublished) the Posner-Robinson Theorem for hyperarithmetic jump as well as for the sharp.

(1) For any real $A \notin L_{\omega_1^{\mathrm{CK}}}$, there is a $G \subset \omega$ such that $\mathcal{O}^G \equiv_T (A, G)$, where $\mathcal{O}^G$ is the complete Π^1_1-in-G set.
(2) Assume $\forall x \subset \omega\, (x^\sharp$ exists). Then for any real $A \notin L$, there is a $G \subset \omega$ such that $G^\sharp \equiv_T (A, G)$, where $G^\sharp$ is the real coding the theory of $L[G]$.

We are more interested in the following less specific statement:

(PR) There are co-countable many reals A such that the Posner Robinson equation $x^\sharp \equiv_T (A, x)$ has a solution.

So $(\mathscr{D}_T, \leq_T)$, $(\mathscr{D}_h, \leq_h)$ and $(\mathscr{D}_c, \leq_c)$ all satisfies (PR). Note that in inner model $L(Q_{2n+1})$, the structure $(\mathscr{D}_{2n+1}, \leq_{2n+1})$, $n > 1$, is a wellordering, and the immediate $<_{2n+1}$-successor of every Δ^1_{2n+1}-degree is its Δ^1_{2n+1}-jump, so the Posner-Robinson equations fail to have solutions at limit degrees (i.e. the limit iterates of Δ^1_{2n+1}-jump of $\emptyset$), thus we have

Corollary 2.3. *Assume $V = L(Q_{2n+1})$. PR is false over $(\mathscr{D}_{2n+1}, \leq_{2n+1})$.*

The second one is the *Degree Determinacy Problem*. Given a degree structure $(\mathscr{D}, \leq)$. A set $A \subseteq \mathbb{R}$ is *degree invariant* if $x \equiv y \Rightarrow (x \in A \leftrightarrow y \in A)$. A function $f : \mathbb{R} \to \mathbb{R}$ is *degree invariant* if $x \equiv y \Rightarrow f(x) \equiv f(y)$. A *cone* of reals is a set of the form $C_x =_{\text{def}} \{u \in \mathbb{R} \mid x \leq u\}$. Let $C_{[x]} =_{\text{def}} \{[u] \mid u \in C_x\}$. *Degree Determinacy* is the assertion that $C_{[x]}$, $x \in \mathbb{R}$, generate an ultrafilter on the poset of degrees, in other word, every degree invariant set of reals either contains or is disjoint from a cone. *Turing Determinacy* (TD) is the Turing degree version of Degree Determinacy. And the Turing cone measure is often called *Martin measure*.

TD is a consequence of AD, many consequences of AD can also be derived from TD (see Ref. 28 for some examples). In fact, all versions of Degree Determinacy for reasonable definability degree notions all follow from AD. Unlike structure properties discussed before, the statement of Degree Determinacy connects second order objects (subsets of reals) of second order arithmetic to first order objects (bases of cones), it does not speaks about the partial ordering directly, however, it has fundamental impact on the global structure of degree functions. In response to Sacks' question regarding the existence of degree-invariant solution to Posts problem [29], Martin made a global conjecture that the only nontrivial definable Turing invariant functions are the Turing jump and its iterates through the transfinite. More precisely,

Conjecture 2.1 (Martin Conjecture, See Ref. 30, p.281). *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$.

(1) *If f is a Turing degree invariant function and $x \nleq_T f(x)$ for a cone of x, then $f(x) \equiv_T x$ for a cone of x.*
(2) *Degree invariant functions on $\mathbb{R}$ are pre-wellordered by the relation $\leq_M$, where $f \leq_M g$ iff $f(x) \leq_T g(x)$ on a cone of x. Let f' be such that $f'(x) \equiv_T f(x)'$. Then $\operatorname{rank}_{\leq_M}(f) = \alpha \Rightarrow \operatorname{rank}_{\leq_M}(f') = \alpha + 1$.*

Although this conjecture remains open, it has already been proven to be true when restricted to the class of uniformly Turing invariant functions.

The conjecture were stated with AD, we believe that if it is true, TD should suffice. We refer interested readers to Refs. 6,8 and more recent Ref. 31. Aforementioned Posner-Robinson theorems for iterated jumps of Turing degrees played an important role in Slaman-Steel's proof of (1) for uniformly Turing invariant functions.

Back to the posets discussed earlier, Degree Determinacy holds in $(\mathscr{D}_T, \leq_T)$, $(\mathscr{D}_{2n+1}, \leq_{2n+1})$ $(n < \omega)$ and $(\mathscr{D}_c, \leq_c)$, while it is false in $(\mathscr{D}_{2n+1}, \leq_{2n+1})^{L(Q_{2n+1})}$, $n > 0$, as it is wellordered and any two disjoint unbound subsets of this ordering witness the failure of Degree Determinacy.

2.4. *α-recursion theory*

Another direction for generalization is to lift the notion of degrees to subset of α where α is an arbitrary limit ordinal $> \omega$. This is so called α-recursion theory. In order to preserve a good collection of results in classical recursing theory, it's necessary to consider ordinals with sufficient closure properties, in particular those are Σ_1-admissible. Many of the classical results lift to such α by means of recursive approximations and fine structure techniques.

A set is *admissible* if it is transitive and models KP set theory. This is the same as Σ_1-admissible, which is about to be defined later. An ordinal α is admissible if L_α is admissible. ω and ω_1^{CK} are the first two admissible ordinals. Let α be an admissible ordinal. Call a set *α-finite* if it belongs to L_α, *α-r.e.* (or *α-recursive*) if it is Σ_1-definable (Δ_1-definable, respectively) over $(L_\alpha, \in)$ (allowing parameters). The *α-jump* of $\emptyset$ is accordingly given by the complete $\Sigma_1(L_\alpha)$-set.[g] For readers not familiar to theory of admissible ordinals, Ref. 16 (Part C) and Ref. 32 are good places to look, Refs. 33,34 are good sources for advanced techniques in this area. The admissible initial segments of L provide natural settings for generalizing classical recursion theory, as such L_α admits an L_α-recursive bijection between L_α and α. L provides an ideal structure for developing higher recursion theory. The results in α-recursion theory cited in this paper all assume $V = L$.

Note that, although we follow the tradition and use the terminology α-degree, it should be called something like α-Δ_1-degree, at least in this paper, as it is the analogue of Turing degrees for subsets of α.

[g]It should be pointed out that although $[\emptyset]'$ is defined to the degree of some complete $\Sigma_1(L_\alpha)$-set, but in general $[\emptyset]^{(2)}$, the double jump $([\emptyset]')'$, does not have the same α-degree as some complete $\Sigma_2(L_\alpha)$-set.

Let $\alpha \in \mathrm{Ord}$ be admissible. A set $X \subseteq \alpha$ is *regular*[h] if $A \cap \beta \in L_\alpha$ for all $\beta < \alpha$. For $A, B \subseteq \alpha$, write $A \leq_\alpha B$ if A is Δ_1-definable over $(L_{\alpha_1^B}[B], \in, B)$ (allowing parameters), where α_1^B is the least ordinal $\geq \alpha$ such that $L_\alpha[B]$ is admissible. A is regular iff $L_\alpha[A] = L_\alpha$. In recursion theory, amenability is equivalent to an important dynamic property in priority argument. In a 1966 paper, Sacks established the following basics of α-recursion theory.

Theorem 2.5 (Sacks[35]). *Let $\alpha \in \mathrm{Ord}$ be admissible.*

(1) Every α-r.e. degree can be represented by an regular subset of α.
(2) The poset of α-r.e. degrees, $(\mathscr{R}_\alpha, \leq_\alpha)$, is nontrivial, i.e. there exists a non-α-recursive, regular, α-r.e. set.

By exploiting the combinatoric power of admissibility and the techniques from fine structure theory, Sacks and his students manage to use Σ_1-admissibility to do the work of Σ_2. They lifted the classical finite injury argument to α-recursion theory and provided positive solutions to Post's problem in α-recursion theory

Theorem 2.6.

(1) (Sacks-Simpson[36]) There exist two $\leq_\alpha$-incomparable α-r.e. subsets of α. Furthermore,
(2) (Shore[37]) There is a uniform solution to Post's problem: There exist $m, n < \omega$ such that for all admissible α, the m-th and n-th lightface Σ_1 subsets of α are $\leq_\alpha$-incomparable.

With his Σ_2-blocking technique, Shore proved a splitting theorem at α, from which a positive solution to Post's problem also follows.

Theorem 2.7 (Shore[38]). *Let A be α-r.e. and regular. Then there exists α-r.e. B_0 and B_1 such that $A = B_0 \cup B_1$, $B_0 \cap B_1 = \varnothing$ and $A \not\leq_\alpha B_i$ $(i < 2)$.*

Soon after the splitting theorem, Shore proved the density theorem at α, which is also the first instance of α-infinite injury.

Theorem 2.8 (Shore[39]). *Let A and C be α-r.e. sets such that $A <_\alpha C$. Then there exists an α-r.e. B such that $A <_\alpha B <_\alpha C$.*

This pretty much gives us the picture of $(\mathscr{R}_\alpha, \leq_\alpha)$. The picture in $(\mathscr{D}_\alpha, \leq_\alpha)$, the global poset of α-degrees, is complicated – in general it is not dense about $[\emptyset]'_\alpha$.

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[h] This is Sacks' terminology. Jensen call it *amenable*.

For an ordinal α, its Σ_n-*cofinality*, $n < \omega$, is the least $\rho \leq \alpha$ such that there is a $\Sigma_n(L_\alpha)$ function f mapping ρ into cofinally into α. An ordinal is Σ_n-*admissible* if its Σ_2-cofinality equals to itself. Regular cardinals are Σ_n-admissible for all $n < \omega$.

Theorem 2.9.

(1) (*MacIntyre*[40]) *There exists a minimal α-degree for every countable admissible ordinal.*
(2) (*Shore*[41]) *Minimal α-degrees exist for Σ_2-admissible cardinal α.*

These leave out the case for Σ_1-admissible but not Σ_2-admissible cardinals,[i] in particular $\alpha = \aleph_\omega$, is still open. Although the cardinal $\aleph_{\omega_1}$ is also a such cardinal, inspired by Silver's work[43] on Singular Cardinal Problem, using the combinatorics of stationary set, Sy Friedman[44] showed that $\aleph_{\omega_1}$-degree is wellordered above $[0]'_{\aleph_{\omega_1}}$. Given a singular cardinal λ and a degree notion at λ, we call a degree represented by a cofinal subset of λ of ordertype $\mathrm{cf}(\lambda)$ *a singularizing degree* (at λ).

Theorem 2.10 (Sy Friedman[44]). *For any singular cardinal λ with $\mathrm{cf}(\lambda) > \omega$, the λ-degrees are wellordered above every singularizing degree, and the immediate successor is given by the jump. In particular, $\aleph_{\omega_1}$-degree is wellordered above $[0]'_\alpha$.*

Therefore, minimal $\aleph_{\omega_1}$-cover exists for every degree above $[0]'_{\aleph_{\omega_1}}$. But the rest of the picture is still not quite clear. Chong[45] recently gives a partial answer: Minimal $\aleph_{\omega_1}$-degree if exists must be $< [0]'_{\aleph_{\omega_1}}$.

Let Δ^α_1-*degree Determinacy* be the statement that any Δ^α_1-degree invariant subset of $\mathscr{P}(\alpha)$ either contains or is disjoint from of a cone of subset of α. From Sy Friedman's theorem, immediately we have

Corollary 2.4. *Δ^λ_1-degree Determinacy fails at singular cardinal λ with uncountable cofinality.*

Proof. By Sy Friedman's theorem, the λ-degrees are wellordered above $[0]'_\lambda$. Let A be the set of all the odd iterates of λ-jumps of $[\emptyset]$ and B for all the even iterates. Then A, B are disjoint and $\leq_\lambda$-unbounded subsets of $\mathscr{D}_\lambda$, and none of them contains a cone of λ-degrees. $\square$

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[i] Maass[42] improved Shore's result to a slight weaker assumption.

Remark 2.1. Sy Friedman's argument works for any degree notions with large equivalence classes. If the degree notion under consideration is Δ_n-degree at $\aleph_{\omega_1}$, $n > 1$, for instance, then the Δ_n-degree of $\emptyset$ already contains a cofinal subset of $\aleph_{\omega_1}$, thus Δ_n-degree at $\aleph_{\omega_1}$, $n > 1$, are completely wellordered. But the α-degrees (take $\alpha = \aleph_{\omega_1}$) below $[\emptyset]'_\alpha$ is illfounded by Shore's Density Theorem for α-r.e. sets (see p.11).

The situation at $\aleph_\omega$ is different. According to [46], Harrington and Solovay independently proved that there are incomparable $\aleph_\omega$-degrees above $[0]'_\alpha$.[j] Later, we will discuss recently developments in degree theories at countable cofinality singular cardinals, but focusing on what we called Zermelo degrees, rather than Δ_1-degrees, at $\aleph_\omega$.

There aren't any research on the determinacy of α-degrees to date. Although lack of anything like determinacy axioms for large ordinals in general, our study of consequences of strong axioms like I_0, whose impact at the associated countable cofinality cardinal resembles a great deal to that of AD at ω, in degree theory supports the view that there is a deep connection with the complexity of degree structures and the strength of large cardinals of the universe carries.

Next consider the Posner-Robinson Problem. There are no Posner-Robinson results for α-degrees in the literature. Here is something close to it. The Simpson jump theorem below is a lifting of the Friedberg jump theorem of classical recursion theory.

Theorem 2.11 (Simpson[48]). *The following are equivalent.*

(1) $\emptyset' \leq_\alpha D$ and D has the same α-degree as some regular set.
(2) $C' \equiv_\alpha D$ for some regular, hyperregular C.

Informally speaking, this says that every degree $[d]_\alpha \geq [0]'_\alpha$ is an α-jump of some generalized low degree. For our PR statement on p.8, it only make sense to generalize it to cardinals. Let λ be a cardinal. PR_λ is the following assertion:

(PR_α) There are co-λ many reals A such that the Posner Robinson equation $x' \equiv_\lambda (A, x)$ has a solution.

In other word, the Posner-Robinson equations fail at no more than λ many places. As a consequence of Sy Friedman and Simpson's theorems, we have

[j] Sy Friedman promised to give the proof in Ref. 47, but Ref. 47 seems never appeared.

Corollary 2.5. PR_λ *fails for Δ_1-degrees at singular cardinals λ with uncountable cofinality.*

Proof. In L, singular cardinals are strong limit, so every subset of λ that Δ_1 computes $\emptyset'$ is regular. By Simpson's theorem, every degree $[d]_\lambda \geq [\emptyset]'_\lambda$ is a λ-jump of some $[c]_\lambda$. Now suppose $[d]_\lambda$ is a limit iterate of λ-jump of $[\emptyset]_\lambda$, i.e. $d \equiv_\lambda c' \equiv_\lambda \emptyset^{(\alpha)}$, for some limit ordinal α.

We claim that the Posner-Robinson equation for $[c]_\lambda$ does not have a solution. Suppose NOT, $(c, g) \equiv_\lambda g'$ for some $g \subset \lambda$. Since $g' \geq_\lambda \emptyset'$, it must be that $g' \equiv_\lambda \emptyset^{(\beta)}$ for some β. Then

$$\emptyset^{(\beta+1)} \equiv_\lambda g'' \equiv_\lambda (c, g)' \equiv_\lambda (c', g') \equiv_\lambda (\emptyset^{(\alpha)}, \emptyset^{(\beta)}).$$

Then we would have $\alpha = \beta + 1$. Contradiction!

But there are at least λ^+ many such $[c]_\lambda$'s. This means that Posner-Robinson equations fails at more than λ many places, so PR_λ is false for Δ_1-degrees. $\square$

It is also worth mentioning that, using Simpson's jump theorem together with another result of Shore[49], Sy Friedman were able to give a quick proof that the α-jump operator is definable in $(\mathscr{D}_\alpha, \leq_\alpha)$ (See Ref. 46, Theorem 6), in contrast to the sophisticated machinery used to prove the definability of Turing jump in $(\mathscr{D}_T, \leq_T)$ (see Ref. 27).

There are also works on α-degrees at inadmissible ordinals, which we avoid in this paper, as it is hard to organize and fit them into the theme we are laying out here.

Sacks and Slaman also did some ground work (see Ref. 50) for generalized hyperarithmetic theory, namely generalizing the Δ_1-degree at α to the analogue of hyperarithmetic degree at α. This combines the aforementioned two directions for generalizing classical recursion theory. At this point, as far as the questions we are interested here, not much can we say about these generalized hyperarithmetic degree structures.

There are also so called E-recursion theory, which from a different perspective extends the notion of computation from hereditarily finite sets to sets of arbitrary rank. We will not discuss these generalization due to its little relevance to the later part of this paper (maybe we just don't see yet).

The interests of recursion theorists and (descriptive) set theorists seem shift away from generalizing recursion theory after mid 1990s. Until very recent, some applications of large cardinals to degree structures emerged. In the rest of this paper, we report the recent developments in, what we prefer to call, *higher degree theory*.

3. Higher Degree Theory

3.1. *Preparation*

This section surveys the results in Ref. 2. First are some definitions.

Definition 3.1 (ZFC). *Suppose* Γ *is a fragment of* ZFC *such that* ZFC *proves the consistency of* Γ. *Suppose* λ *is an uncountable cardinal satisfying* $2^{<\lambda} = \lambda$. *Let* H_λ *be the collection of sets whose transitive closure has cardinality* $< \lambda$. *Fix a* $\Delta \subset \lambda$ *which codes a well ordering of* H_λ *of ordertype* λ. *For each* $a \subseteq \lambda$, *let* α_a *be the least ordinal* $\alpha > \lambda$ *such that* $M_a =_{\text{def}} L_\alpha[\Delta, a]$ *models* Γ. *If there is a definable wellordering of* H_λ *(of ordertype* λ*) in* V, *then there is no need to mention* Δ *explicitly.*

For any two subsets $a, b \subseteq \lambda$, *set* $a \leq_\Gamma b$ *if and only if* $M_a \subseteq M_b$. *This gives rise to a degree notion, which we call* Γ-degree. *To each* $a \subseteq \lambda$, *the* Γ-jump of a, a'_Γ, *is the theory of* M_a, *which is identified as a subset of* γ.

For instance, hyperarithmetic degree is KP-degree, where KP is Kirpkeplatek set theory. For this section, we fix $\Gamma = \mathsf{Z}$, Zermelo set theory, i.e. ZF − Replacement. The point is that Z is sufficient for proving Covering lemmas for fine structure models. In this case, the ordinal α_a is called the *Zermelo ordinal for a*. The above definitions are given under ZFC. In general, suppose T_0 is our working theory, and T_1 is a consistently weaker fragment of T_0 (i.e. the existence of minimal models of T_1 can be derived from T_0), then one can define a degree notion for T_1 under T_0 as above. To illustrate our point, we shall only cross examine Z-degrees at countable cofinality singular cardinals in various large cardinal inner models.

Recall that in §2.2 and §2.3, we propose to consider the following structural properties of degree posets.

(1) (*Post Problem*). Are there two incomparable degrees, i.e. two sets $a, b \subseteq \lambda$ such that $a \not\leq b$ and $b \not\leq a$? One can also ask a relativized question, i.e. incomparable degrees above a given degree. A set of pairwise incomparable degrees form an antichain. A related question is the size(s) of maximal antichains, if exist.
(2) (*Minimal Cover*). Given a degree $[a]$, is there a minimal cover for $[a]$, i.e. a $[c] > [a]$ such that there is no $b \subseteq \lambda$ such that $[a] < [b] < [c]$?
(3) (*Posner-Robinson*). Is it true that for almost all (co-λ many) $x \subseteq \lambda$, the Posner-Robinson equation for x has a solution, i.e. $\exists g \subseteq \lambda\,[(x, g) \equiv g']$?

(4) (*Degree Determinacy*). Is it true that every degree invariant subset of $\mathscr{P}(\lambda)$ either contains or is disjoint from a cone?

Among the four questions, the first one is about antichains, therefore is related to the width of the degree poset, the second is about the organization of degrees, like whether the degrees are dense or discrete, the third is about the internal understanding of the degrees, such as what information do the degrees carry, and the last is more or less a question about the connection between the members of $\mathscr{P}(\lambda)$ and the subsets of $\mathscr{P}(\lambda)$, a bridge between these two types.

The first three problems are first order questions regarding $\mathscr{P}(\lambda)$ (more frequently $(V_{\lambda+1}, \in)$ in practice). However, for the degree determinacy problem, it makes more sense to state it for degree invariant subsets of $\mathscr{P}(\lambda)$ in $L(\mathscr{P}(\lambda))$, just like the situation of Turing Determinacy for sets of reals in $L(\mathbb{R})$. So it is more appropriate to think the degree determinacy problem as a question quantifying over second order sets in $L(\mathscr{P}(\lambda))$ (or $L(V_{\lambda+1})$ in practice). Since $V_{\lambda+1}$ varies in different universes, answers to degree theoretical questions, such as these four questions, often vary in different V's. We will give an example on this matter in §4.1.

These are certainly important degree theoretic questions. This list, however, is by no means meant to be comprehensive, it is merely a list of questions that at this point we are confident to answer.

At a strongly inaccessible cardinal λ (even regular cardinal satisfying $2^{<\lambda} = \lambda$), the degree notions in general are similar to their counterparts at ω, since most usual constructions for degrees at ω, priority argument, local forcing argument, et al, can be carried out at λ with very few changes. So the degree structures for an analogue degree notion at λ is very much like its counterpart at ω, not much new insight is obtained there. At these cardinals, the answers to the first three questions are all "Yes", as in the case of ω. However, for the Degree Determinacy question, the answer on the contrary is very likely to be "No". This will be discussed in §3.7. Also, as discussed in §2.4, Sy Friedman settled the situations at singular cardinals of uncountable cofinality, our focus will be mainly on degrees at singular cardinals of countable cofinality. We shall analyze Zermelo degree structures in some fine structure extender models. The reason for working with fine structure models will be discussed in §4.1.

Our main tool is Covering lemmas in various cases, Mitchell's handbook article (Ref. 51) is the main reference for that. Schimmering's introductory article (Ref. 52) is good enough for most fine structure contents in this paper, if the reader are not familiar with fine structures. We also assume familiarity of Prikry-type forcings, for which Gitik's handbook article (Ref. 53) is recommended.

3.2. *Zermelo degrees in L*

Let us start with Woodin's observation in L about Zermelo degrees at $\aleph_\omega$. The argument is a simple application of the Covering Lemma for L. First recall Jensen's Covering Lemma for L.

Lemma 3.1 (Covering Lemma for L, see Refs. 51,54). *Assume $0^\sharp$ does not exist. Then for every set x of ordinals, there is an $y \in L$ such that $x \subseteq y$ and $|x| = |y| + \omega_1$.*

Theorem 3.1 (Shi[2]). *Assume $V = L$. Let λ be a singular cardinal with* $\mathrm{cf}(\lambda) > \omega$. *Let $x \subset \lambda$ be cofinal in λ and* $\mathrm{otp}(x) = \mathrm{cf}(\lambda)$. *Then the Zermelo degrees at λ above $[x]_{\mathsf{Z}}$ are well ordered. In particular, Zermelo degrees at $\aleph_\omega$ are well ordered.*

Proof. The argument is a simple application of the Covering Lemma for L. Suppose $a \subseteq \lambda$ and $a \geq_{\mathsf{Z}} x$. Consider $M(a)$, the minimal Zermelo model containing a. Notice that $M(a)$ has the same reals as V and sharps are absolute between transitive models containing ω_1, therefore M_a contains no sharps. In M_a, as $x \in M_a$, every $z \subset \lambda$ can be identified with a countable subset of λ. By Covering in M_a, a is covered by a $b \in L^{M_a} = L_{\alpha_a}$ such that $|b| = |a| + \omega_1$. As M_a and L_{α_a} agree on $\mathscr{P}(\omega_1)$, $|b|^{L_{\alpha_a}} = \omega_1$. Let $\pi : b \to |b|^{L_{\alpha_a}}$ be a bijection in L_{α_a}. Then $\pi[a] \in \mathscr{P}(\omega_1) \subseteq L_{\alpha_a}$. It follows that a can be computed from b in L_{α_a}. (This is essentially the proof of Theorem V.5.4 in Devlin[55].) Thus $M_a = L_{\alpha_a}$. This means that the mapping $[a] \mapsto \alpha_a$ is injective. Therefore Zermelo degrees at λ are well ordered above $[x]$. Since the sequence $\{\aleph_n \mid n < \omega\}$ is Σ_1-definable over $L_{\aleph_\omega}$, Zermelo degrees at $\aleph_\omega$ are fully well ordered. $\square$

Immediately, we have the following answers to the four questions. Enumerate $\{\alpha_x \mid x \subset \lambda\}$ in increasing order, let α_η be its η-th member. Say $[x]$ is a *successor* (resp. *limit*) *degree* if $\alpha_x = \alpha_\eta$ for some successor (resp. limit) ordinal η.

Corollary 3.1. *Assume $V = L$. Let λ be a singular cardinal and* $\mathrm{cf}(\lambda) = \omega$. *Then*

(1) There are no incomparable Zermelo degrees above any singularizing degree. In particular, every two Zermelo degrees at $\aleph_\omega$ are comparable.

(2) Every Zermelo degree above any singularizing degree has a unique minimal cover. Again this holds for every Zermelo degree at $\aleph_\omega$.

(3) PR_λ is false for Zermelo degrees at λ.

(4) Degree Determinacy fails for Zermelo degrees at λ.

Proof. (1) and (2) are immediate from this wellordered structure, so the answers are "No" for the first question and "Yes" for the second. However, for the multi-minimal-cover question, the answer is "No".

In this wellordered structure, Posner-Robinson equation is equivalent to the jump inversion equation, namely, $(\exists G)(x \equiv_{\mathsf{Z}} J_{\mathsf{Z}}(G))$. Notice that whenever there is a new subset of λ is constructed, say in $L_{\alpha+1} \setminus L_\alpha$, we have $L_{\alpha+1} \models$ "$|L_\alpha| = \lambda$". Therefore, a successor degree knows that the minimal Zermelo model associated to its (immediate) predecessor degree has size λ, therefore can compute the jump of its predecessor. So the jump operator in the degree structure coincides with the successor operator in the well-order. Thus limit degrees can not be the jump of any degree. There are λ^+ many limit degrees, therefore the answer for (3), the Posner-Robinson question, is "No".

(4) follows from the fact that there are two disjoint sets of degrees that are unbounded in this wellorder, which witness the failure of degree determinacy. $\square$

Here L is viewed as the core model for the negation of the large cardinal axiom that $0^\sharp$ exists. The core of the argument is the Covering lemma for L. The same form of Covering Lemma holds for inner models between L and $L[\mu]$ (the inner model for one measurable cardinal), which include models like $L(0^\sharp)$, $L(0^{\sharp\sharp})$ etc., and Dodd-Jensen's core model K^{DJ}, the core model below a measurable cardinal. For instance,

Lemma 3.2 (Covering Lemma for $K^{\mathbf{DJ}}$, see Refs. 51,56). *Assume that there is no inner model for one measurable cardinal and the Dodd-Jensen core model K^{DJ} exists. Then for every set x of ordinals, there is a $y \in K^{\mathrm{DJ}}$ such that $x \subseteq y$ and $|x| = |y| + \omega_1$.*

These models can be obtained by (proper) partial measures using Steel's construction. For these models, the same covering argument works. The point is that in these inner models, the minimal model of the form M_a, $a \subseteq \lambda$, λ a countable cofinality singular cardinal, is always an initial segment of the core model. Thus Zermelo degrees for countable cofinality singular cardinal are always well ordered above the singularizing degree, the same as in L.

3.3. *Zermelo degrees in $\boldsymbol{L[\mu]}$*

Next, consider $L[\mu]$, the canonical model for one measurable cardinal. The Covering Lemma for $L[\mu]$ starts to be different.

Lemma 3.3 (Covering Lemma for $L[\mu]$, see Refs. 51,57).
Assume that $0^\dagger$ does not exist but there is an inner model with a measurable cardinal, and that the model $L[\mu]$ is chosen so that $\kappa = \mathrm{crit}(j_\mu)$, the least ordinal moved by the elementary embedding j_μ given by μ, is as small as possible. Here j_μ denote the canonical embedding associated to μ. Then one of the following two statements holds:

(1) For every set x of ordinals there is a set $y \in L[\mu]$ with $x \subseteq y$ and $|y| = |x| + \omega_1$.
(2) There is a sequence $C \subseteq \kappa$, which is Prikry generic over $L[\mu]$, such that for all sets x of ordinals there is a set $y \in L[\mu, C]$ such that $x \subseteq y$ and $|y| = |x| + \omega_1$. Furthermore, the sequence C is unique up to finite initial segments.

But the difference does not affect the structure of Zermelo degrees at cardinals other than κ.

Theorem 3.2 (Shi[2]). *Assume $V = L[\mu]$. Zermelo degrees at countable cofinality singular cardinals are well ordered above any singularizing degree. Moreover, the successor of a degree above any singularizing degree is its jump.*

Proof. Reorganize $L[\mu]$ as $L[E]$ using Steel's construction, where E is a sequence of (possibly partial) measures. Here we omit the predicate for the extender: When we say $L_\alpha[E]$ ($\alpha \in \mathrm{Ord}$ or $\alpha = \mathrm{Ord}$), we often refer to the structure $\langle L_\alpha[E{\restriction}\alpha], \in, E{\restriction}\alpha\rangle$ or $\langle L_\alpha[E{\restriction}\alpha], \in, E{\restriction}\alpha, E(\alpha)\rangle$. A crucial point of using Steel construction is the acceptability condition, which says for any $\gamma < \alpha$,

$$(L_{\alpha+1}[E] \setminus L_\alpha[E]) \cap \mathscr{P}(\gamma) \neq \varnothing \quad\implies\quad L_{\alpha+1}[E] \models |\alpha| = \gamma.$$

Here are some benefits of having the acceptability condition:

(1) $\mathscr{P}(\gamma) \cap L[E] = \mathscr{P}(\gamma) \cap L_{\gamma^+}[E]$ for any cardinal γ,

(2) Suppose λ is a cardinal, a, b are unbounded subsets of λ. If $b \leq_{\mathsf{Z}} a$ and $\alpha_b < \alpha_a$, then $[b]'_{\mathsf{Z}} \in M_a$, hence $[b]'_{\mathsf{Z}} \leq_{\mathsf{Z}} a$.

Fix an $a \subset \lambda$ in $L[E]$, consider M_a in $L[E]$. First suppose $\lambda > \kappa$. As M_a and $L[E]$ agree up to λ, arguing as in L, we get that M_a, $a \subseteq \lambda$, are initial segments of $L[E]$. Now suppose $\lambda < \kappa$. As M_a has the same reals as V, M_a does not have $0^\dagger$, so M_a could have at most one full measure.

Case 1. M_a has no full measure. In this case, as the Covering Lemmas for inner models below one measurable have the same form as the Covering Lemma for L (for instance, see the Covering Lemma for K^{DJ} on page 19), applying the corresponding covering lemma, we get that the minimal models M_a, $a \subset \lambda$, are the same as its own core model, namely $K^{M_a} = M_a$. Run the comparison process for M_a against $K^V = L[E]$. M_a is iterable and since M_a agrees with $L[E]$ up to λ, iteration maps used during the comparison do not move M_a, thus M_a is an initial segment of $K^V = L[E]$.

Case 2. M_a has one full measure, say

$$M_a \models \mu' \text{ is a measure at some } \gamma > \lambda.$$

M_a satisfies the hypothesis of Covering Lemma for one measurable. Apply the Covering in M_a, then there are two cases, either

(1) a is covered by a set $y \in (L[\mu'])^{M_a}$ with $|y| = |a| + \aleph_1$, or
(2) a is covered by a set $y \in (L[\mu', C])^{M_a}$ with $|y| = |a| + \aleph_1$, where C is a Prikry generic over $(L[\mu'])^{M_a}$. Such C is unique up to finite differences.

Notice that $\lambda < \gamma$ and Prikry generics do not add new bounded subsets. a is a bounded subset of γ, so it must be case 1 – the covering set y for a is in $(L[\mu'])^{M_a}$. Thus $M_a = L[\mu']^{M_a} = L_{\alpha_a}[\mu']$. Run the comparison for M_a against K^V. Arguing as before, M_a is an initial segment of $K^V = L[E]$.

So either case, we have that M_a, $a \subset \lambda$, are initial segments of $L[E]$, exactly the same picture as in L – Zermelo degrees at λ in $L[E]$ are well ordered above every singularizing degrees via their Zermelo ordinals. The "moreover" clause follows from the acceptability condition. This completes the proof. $\square$

It is not difficult to see that this argument can be adapted to show the result for core models of finitely many measurable cardinals. So Corollary 3.1 should also include large cardinal core models beyond L up to core models for finitely many measurable cardinals.

3.4. *Zermelo degrees in Mitchell models for an ω-sequence of measures.*

New picture starts to emerge in the canonical model for ω many measurable cardinals, $L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ and each μ_n is a measure on κ_n and $\kappa_n < \kappa_{n+1}$, $n < \omega$.

Consider $V = L[\bar{\mu}]$. Again we view $L[\bar{\mu}]$ as built with (partial) measures using Steel's construction. Let $\kappa_\omega = \sup_n \kappa_n$. Let λ be a countable singular cardinal. It is not difficult to see that, when $\lambda > \kappa_\omega$ or $\lambda < \kappa_\omega$, arguing as in $L[\mu]$, Zermelo degrees at λ is well ordered above the singularizing degree. The new picture appears at $\lambda = \kappa_\omega$. The Covering Lemma for $L[\bar{\mu}]$ is similar to that of $L[\mu]$, except that C in the second case now is a system of indiscernibles $C = \langle C_n : n < \omega \rangle$ with the following property:

(1) Each $C_n \subset \kappa_n$ is either finite or a Prikry sequence;
(2) C as a whole is a uniform system of indiscernibles, i.e.

$$(\forall \bar{x} \in L[\bar{\mu}])\,(\forall n < \omega)(x_n \in \mu_n) \quad \Longrightarrow \quad |\bigcup\{C_n - x_n \mid n < \omega\}| < \omega.$$

In fact, for any function $f : \omega \to \omega \cup \{\omega\}$ with infinite support, i.e. the set $\mathrm{supp}(f) =_{\mathrm{def}} \{i \in \omega \mid f(i) > 0\}$ is infinite, one can use the following variation of diagonal Prikry forcing $\mathbb{P}^f_{\bar{\mu}}$ to produce an indiscernible system such that $|C_n| = f(n)$:

- The conditions of $\mathbb{P}^f_{\bar{\mu}}$ are pairs $(\bar{a}, \bar{A})$ such that each $a_i \subset \kappa_i$, and $|a_i| \leq f(i)$, each $A_i \in \mu_i$, for $i < \omega$, moreover, $\bigcup_i a_i$ is finite.
- The order is defined by $(\bar{a}, \bar{A}) \leq (\bar{a}', \bar{A}')$ iff $a(i) \supseteq a'(i)$, $A'(i) \subset A(i)$, and $a_i - a'_i \in A_i$ for $i < \omega$.

These discussion about system of indiscernibles can be found in §4 of Mitchell's handbook article (see Ref. 51). The proof of classical Mathias condition for characterizing diagonal Prikry sequence for $\mathbb{P}_{\bar{\mu}}$ can be easily adapted to show the $\mathbb{P}^f_{\bar{\mu}}$-version of Mathias condition.

Proposition 3.1 (Mathias Condition for $\mathbb{P}^f_{\bar{\mu}}$). *Suppose M is an inner model of* ZC*, Zermelo set theory plus choice, $\bar{\mu} \in M$, f and $\mathbb{P}^f_{\bar{\mu}}$ are defined in M as above. Suppose $G \in \prod_{n\in\omega}(\kappa_n)^{f(n)}$. Then G is a generic sequence for $\mathbb{P}^f_{\bar{\mu}}$ over M if and only if for any sequence $\bar{A} \in M$ such that $A_n \in \mu_n$ for $n < \omega$, there is an $m < \omega$ such that $G(n) \subset A_n$, for $n \geq m$.*

To simplify the presentation of our next theorem, we use the standard diagonal Prikry poset, namely $\mathbb{P}_{\bar{\mu}} = \mathbb{P}^f_{\bar{\mu}}$ with the constant function $f(n) =$

1, for $n < \omega$. With this diagonal Prikry forcing, one can use a single diagonal Prikry sequence C in case (2) of the Covering Lemma for $L[\bar{\mu}]$.

Lemma 3.4 (Covering Lemma for $L[\bar{\mu}]$, 51). *Assume the sharp of $L[\bar{\mu}]$ does not exist and there is an inner model containing ω measurable cardinals. Let $L[\bar{\mu}]$ be such that every $\kappa_\omega = \sup_{n<\omega} \kappa_n$, where each $\kappa_n = \mathrm{crit}(j_{\mu_n})$, is as small as possible. Then one of the following two statements holds:*

(1) For every set x of ordinals there is a set $y \in L[\bar{\mu}]$ with $x \subseteq y$ and $|y| = |x| + \omega_1$.
(2) There is a sequence $C \subsetneq \kappa$, which is $\mathbb{P}_{\bar{\mu}}$-generic over $L[\bar{\mu}]$, such that for all sets x of ordinals there is a set $y \in L[\bar{\mu}, C]$ such that $x \subseteq y$ and $|y| = |x| + \omega_1$. Furthermore, the sequence C is unique up to finite differences.

Using generics for the standard diagonal Prikry forcing as the system of indiscernibles, we describe the structures of Zermelo degrees at countable cofinality singular cardinals in $L[\bar{\mu}]$ as follows. For this subsection, we say an ordinal $\alpha > \lambda$ is a *Zermelo ordinal for $a \subseteq \lambda$* if $L_\alpha[\bar{\mu}, a] \models$ Zermelo set theory. For each ordinal η, let β_η denote the η-th Zermelo ordinal (for $\varnothing$).

Theorem 3.3 (Shi[2]). *Assume $V = L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$, each μ_n is a measure on κ_n. Let $\kappa_\omega = \sup_n \kappa_n$. Suppose λ is a singular cardinal of countable cofinality.*

(1) If $\lambda \neq \kappa_\omega$, then the Zermelo degrees at λ are wellordered above any singularizing degree;
(2) If $\lambda = \kappa_\omega$, consider only Zermelo degrees at λ above the degree of $\bar{\mu}$, identifying $\bar{\mu}$ as a subset of λ. Then
  (a) $\{\alpha_a \mid a \subset \lambda\} = \{\beta_\eta \mid \eta < \lambda^+ \wedge \beta_\eta > \lim_{\xi<\eta} \beta_\xi\}$.
  Therefore Zermelo degrees at λ above the degree of $\bar{\mu}$ are prewellordered via their Zermelo ordinals, i.e. for $a, b \subset \lambda$, the ordering $\preccurlyeq$ given by $[a] \preccurlyeq [b] \Leftrightarrow \alpha_a \leq \alpha_b$ prewellorders the Zermelo degrees at λ above the degree of $\bar{\mu}$.
  (b) For each $\eta < \lambda^+$, let α_η be the η-th member of $\{\beta_\eta \mid \beta_\eta > \lim_{\xi<\eta} \beta_\xi\}$, let A_η be a subset of λ that codes the sequence $\langle \alpha_\xi : \xi < \eta \rangle$, and $\mathcal{C}_\eta$ be the set of diagonal Prikry generic sequences for $\mathbb{P}_{\bar{\mu}}$ that are $L_{\alpha_\eta}[\bar{\mu}]$-generic.

Then Zermelo degrees at λ (above the degree of $\bar{\mu}$) whose Zermelo ordinals equal to α_η are exactly the degrees given by

$$A_\eta \oplus \mathcal{C}_\eta = \{(A_\eta, C) \mid C \in \mathcal{C}_\eta \cup \{\varnothing\}\}.$$

Although we use the standard diagonal Prikry sequence to state this theorem, the argument works for every $\mathbb{P}^f_{\bar{\mu}}$. So for each f as on p.21, every Zermelo degree can be represented by a diagonal Prikry sequence for $\mathbb{P}^f_{\bar{\mu}}$.

Compared with previous pictures, though not eventually well ordered, this is still a rather simple structure. We have definite answers to the four questions.

Corollary 3.2. *Assume $V = L[\bar{\mu}]$, and $\bar{\mu}, \bar{\kappa}, \lambda$ be as in Theorem 3.3. Let $\lambda = \sup_n \kappa_n$. Consider the Zermelo degrees at λ above the degree of $\bar{\mu}$.*

(1) There are incomparable Zermelo degrees.
(2) No Zermelo degree has a minimal cover.
(3) Posner-Robinson Theorem for Zermelo degrees at λ is false.
(4) Degree determinacy for Zermelo degrees at λ is false.

The proof is rather sophisticated, we refer the reader to Ref. 2 Corollary 3.1 for the details.

For (1), a further question is whether there is a size λ^ω antichain of Zermelo degrees. As there are no minimal degrees, the usual way of getting 2^ω many incomparable degree at ω by constructing 2^ω many minimal degrees no longer works here.

Note that if $C_i \in \mathcal{C}_\eta$, $i = 0, 1$, are such that $C_0(n) \subset C_1(n)$ for all $n < \omega$, and $C_1(n) \setminus C_0(n) \neq \varnothing$ infinitely often, then $C_0 <_{\mathsf{Z}} C_1$. As all the models of the form $M_{A_\eta, C}$, $C \in \mathcal{C}_\eta$ have the same reals, it follows that the poset $(\mathscr{P}(\omega)/\mathrm{Fin}, \subseteq^*)$, where $[a] \subseteq^* [b]$ iff $a \setminus n \subseteq b \setminus n$ for some $n < \omega$, can be embedded into Zermelo degrees. In particular, there are infinite descending sequences of Zermelo degrees. However, as Zermelo-jump increases associated Zermelo ordinals, there is no Harrison-type (see Ref. 15, or Ref. 16 III.3.6) descending sequence, i.e.

Corollary 3.3 (Shi[2]). *Assume $V = L[\bar{\mu}]$, and $\bar{\mu}, \bar{\kappa}, \lambda$ be as in Theorem 3.3. At λ, there are infinite descending sequences of Zermelo degrees, but there is no infinite sequence $\langle [a_i]_{\mathsf{Z}} : i < \omega\rangle$ such that $[a_{i+1}]'_{\mathsf{Z}} \leq_{\mathsf{Z}} [a_i]_{\mathsf{Z}}$.*

The theorem below says that over the structure $(\mathscr{D}_{\mathsf{Z}}, \leq_{\mathsf{Z}})$, the set of degrees represented by the sets coding the Zermelo ordinals, and the relation that two sets share the same Zermelo ordinal, are definable.

Theorem 3.4 (Shi[2]). *Assume* $V = L[\bar{\mu}]$, *and* $\bar{\mu}, \bar{\kappa}, \lambda$ *be as in Theorem 3.3. The following are definable over the structure* $(\mathscr{D}_{\mathsf{Z}}, <_{\mathsf{Z}})$:

(1) $\mathcal{I} = \{[A_\eta]_{\mathsf{Z}} \mid A_\eta \subset \lambda$ *codes* $\langle \alpha_i : i < \eta \rangle, \eta < \lambda^+\}$.
(2) $\mathcal{R} = \{([a]_{\mathsf{Z}}, [b]_{\mathsf{Z}}) \mid a, b \subset \lambda, \alpha_a = \alpha_b\}$.

3.5. *Zermelo degrees in models beyond* ω *many measurable cardinals.*

Let us look at Mitchell models with more measurable cardinals. In Ref. 58 (Theorem 4.1) Mitchell showed that if there is no inner model with an inaccessible limit of measurable cardinals then, as in the Dodd-Jensen covering lemma, for each minimal Zermelo model M_a, there is a single *maximal* system of indiscernibles C which can be used to cover any set $x \subset \lambda$ in M_a. A fair amount of analyses above can be carried out at ω-limits of measurable cardinals below the least inaccessible limit of measurable cardinals, if there exists one. Therefore, the pictures at those places are rather similar to the one at κ_ω in $L[\bar{\mu}]$.

Once we past models with inaccessible limit of measurable cardinals, the systems of indiscernibles are no longer unique – may depend on the set of ordinals to be covered (see Ref. 51, p.1555) – and are extremely difficult to analyze. However, Yang proved the existence of minimal covers at ω-limit of certain measurable cardinals.

Theorem 3.5 (Yang[1]). *Suppose* $\langle \kappa_n : n < \omega \rangle$ *is an increasing sequence of measurable cardinals such that each* κ_{n+1} *carries* κ_n *different normal measures,* $n \in \omega$, *and* $\lambda = \sup_n \kappa_n$. *Let* $\mathcal{U}$ *denote this matrix of normal measures, and let* W *be any subset of* λ *that codes* $\langle V_\lambda, \in, \Delta, \{\kappa_i \mid i < \omega\}, \mathcal{U} \rangle$, *where* $\Delta \subset \lambda$ *and codes a well ordering of* V_λ *of ordertype* λ. *Then there is a minimal cover for the Zermelo degree of* W.

Yang's result holds for Δ_1-degrees and any larger degree notions at λ, here we only state it for Zermelo degrees. This result can be relativized to any degrees above that of W. This implies that for instance, in the Mitchell model for $o(\kappa) = \kappa$,[k] a new picture appears at the λ in the hypothesis – there are minimal covers (over almost every degree). Yang's forcing in fact produces a large perfect set (has size λ^ω) of subsets of λ that are minimal above W. As every degree contains only at most λ many of them, thus the

[k] This is not the minimal inner model for Yang's hypothesis, however it has the "shortest" o-expression.

size of antichains of Zermelo degrees at this λ can be as large as possible. So we have "Yes" to the first two questions. We don't know the answers to Posner-Robinson and Degree Determinacy at this λ, but speculate "No" for both of them.

3.6. *The picture from I_0*

The analyses above relies heavily on the fine structure theory, especially covering and comparison. Once past Mitchell models, we are out of comfort zone. Though there are still some variations of covering lemmas for inner models past Mitchell models, very little have we derived from them for the structures of Zermelo degrees in those models. But the emerging new pictures suggest that larger cardinals give us more power to create rich degree structures.

In the later part of this paper, we consider the degree structures at countable cofinality singular cardinals from the other extreme – looking at the strongest large cardinal, Axiom I_0. I_0 asserts the existence of an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point below λ. This λ is an ω-limit of very large cardinals, it satisfies Yang's hypothesis. Therefore at this λ, there are a large perfect set of minimal covers for every degree (above the degree of Yang's W set). Let $\mathcal{E}(L(V_{\lambda+1}))$ denote the set of all elementary embeddings that witness I_0 at λ.

Corollary 3.4. *Assume* ZFC *and $\mathcal{E}(L(V_{\lambda+1})) \neq \varnothing$. Let W be as in Theorem 3.5. Consider the Zermelo degrees at λ above the degree of W.*

(1) There are incomparable Zermelo degrees. In fact, there are antichains (of Zermelo degrees) of size λ^ω.
(2) No Zermelo degree has a minimal cover. In fact, every Zermelo degree has λ^ω many minimal covers.

Moreover, as applications of Generic Absoluteness Theorem in I_0 theory (see Refs. 59,60) we have the following following results regarding Posner Robinson problem and Degree determinacy at λ for Zermelo degrees.

Theorem 3.6 (Shi[2]). *Assume* ZFC *and $\mathcal{E}(L(V_{\lambda+1})) \neq \varnothing$. Then for every $A \in V_{\lambda+1}$, and for almost all (i.e. except at most λ many) $B \geq_{\mathsf{Z}} A$, the Posner-Robinson equation for B has a solution, i.e. there exists a $G \in V_{\lambda+1}$ such that $(B, G) \equiv_{\mathsf{Z}} G'$, where G' denote the Zermelo jump of G.*

The proof in fact shows something stronger, $\leq_{\mathsf{Z}}$ here can be replaced by Δ_1-reducibility for subsets of λ. The argument works if G' is replaced by any reasonable jump operator at λ. This theorem says that PR_λ holds for Zermelo degrees of subsets of λ above any $A \subset \lambda$.

For the degree determinacy problem, we have an almost negative answer.

Theorem 3.7 (Shi[2]). *Assume* ZFC *and* $j \in \mathcal{E}(L(V_{\lambda+1})) \neq \varnothing$. *Let* $\kappa = \mathrm{crit}(j)$ *and suppose* $V_\lambda \models$ *"the supercompactness of* κ *is indestructible by* κ*-directed closed posets". Then* $L(V_{\lambda+1}) \models$ *Degree Determinacy fails for Zermelo degrees at* λ.

Although the theorem uses an additional indestructibility requirement (see Ref. 61), the hypothesis of this theorem is equiconsistent with $\mathsf{ZFC} + I_0$ (see Ref. 2).

3.7. *A conjecture*

We continue the discussion on degree determinacy problem in this subsection. We have seen the structures of Zermelo degrees at countable cofinality strong limit singular cardinals in the early subsections. Now consider the situations for singular (strong limit) cardinals of uncountable cofinalities as well as for regular cardinals. The case that λ is a strong limit singular cardinal of uncountable cofinality follows from the following result of Shelah (see Ref. 62).

Theorem 3.8 (Shelah[62]). *Assume* ZFC*. Then for every strong limit singular cardinal* λ *of uncountable cofinality,* $L(\mathscr{P}(\lambda)) \models$ *Axiom of Choice.*

Using the choice, one can select in $L(\mathscr{P}(\lambda))$ two disjoint unbounded set of degrees, which witness the failure of degree determinacy for Zermelo degrees at λ.

On page 16, we mentioned that at regular cardinals, answers to the degree determinacy questions are alway "No". Here we discuss this matter. For regular cardinals, we first look at the case that λ is regular and satisfies the weak power condition, i.e. $2^{<\lambda} = \lambda$. Jensen's lemma (see Ref. 63) can be generalized to such λ – Jensen's proof in the context of ω can be literally adapted for such λ. More precisely, we have the following lemma.

Lemma 3.5 (Jensen[63]). *Assume* $\mathsf{ZF} + \lambda^+$-DC*. Suppose* $\lambda > \omega$ *is a regular cardinal and* $2^{<\lambda} = \lambda$*. Suppose* $a \subset \lambda$ *and* $A \subset (\lambda, \lambda^+)$ *is a scattered set (i.e.* $\alpha > \sup(A \cap \alpha)$ *for every* $\alpha \in A$*) such that every* $\alpha \in A$ *is a Zermelo ordinal for* a*. Suppose* $\mathrm{otp}(A) \leq \lambda$*. Suppose* $B \subseteq A$ *and* $\mathrm{otp}(B) = \mathrm{otp}(A)$*. Then there is a* $b \subset \lambda$ *such that* $b \geq_{\mathsf{Z}} a$ *and* B *is the set of the* α*-th Zermelo ordinal for* b*,* $\alpha < \mathrm{otp}(A)$*.*

Let $\mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})$ denote the statement of degree determinacy for Zermelo degrees. If $\mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})$ were true in $L(\mathscr{P}(\lambda))$, then applying this lemma to $A \subset \lambda$ that is scattered and $\mathrm{otp}(A) = \omega_1$, one would get a countably additive coherent measure on $[\lambda^+]^{\omega_1}$. This implies the determinacy for sets of reals that are ω_1-Suslin, hence there is no sequence of distinct reals of length ω_1, contradicting to the assumption that V_λ is wellordered. So in ZFC models, $\mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})$ fails in $L(\mathscr{P}(\lambda))$ for regular cardinal λ such that $2^{<\lambda} = \lambda$. Let $\mathsf{ZFC}^{-\epsilon}$ be a fragment of ZFC sufficient for proving this.

Now consider the case that λ is regular but $2^{<\lambda} > \lambda$. If degree determinacy for Zermelo degrees at λ were true in $L(\mathscr{P}(\lambda))$, then there would be a (in fact a cone of) $u \subset \lambda$ and an $\eta_u < \lambda^+$ such that

$$L_{\eta_u}[u] \models \mathsf{ZFC}^{-\epsilon} + \text{“}L(\mathscr{P}(\lambda)) \models \mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})\text{”}.$$

However $L_{\eta_u}[u]$ “thinks” $2^{<\lambda} = \lambda$. This is because if $x \subset \delta < \lambda$, then there are $\alpha, \beta < \lambda$ such that $x \in L_\alpha[u \cap \beta]$; then it follows that $|\mathscr{P}(\delta)| \leq \lambda$. Therefore $L_{\eta_u}[u] \models$ “$2^{<\lambda} = \lambda$”. But according to the discussion in the last paragraph, it must be that $L_{\eta_u}[u] \models$ “$L(\mathscr{P}(\lambda)) \models \neg\,\mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})$”. Contradiction!

This concludes the case that λ is regular. So at least, we know that

Theorem 3.9. *If $\lambda > \omega$ is a regular cardinal or a strong limit singular cardinal of uncountable cofinality, then $L(\mathscr{P}(\lambda)) \models \neg\,\mathrm{Det}_\lambda(\mathscr{D}_\mathsf{Z})$.*

In light of Shelah's result that $L(\mathscr{P}(\lambda))$ is a model of choice if λ is a strong limit singular cardinal and $\mathrm{cf}(\lambda) > \omega$, together with evidences for degree structures at other cardinals, the author (see Ref. 2) makes the following conjecture

Conjecture 3.1 (ZFC)**.** *Let λ be any uncountable cardinal. Then Degree Determinacy for Zermelo degrees at λ is false in $L(\mathscr{P}(\lambda))$.*

At this point, very little is known about singular cardinals that are not strong limit.

4. Remarks

Now is the time for final comments.

4.1. *Why inner models?*

The first remark is regarding the question why we focus on degree structures in inner models.

Yang's theorem and those I_0 results are stated under large cardinal assumption, it seems to be natural to study the consistency strength of those degree theoretical properties. For instance,

- What is the consistency strength of having minimal covers as in Yang's Theorem (see page 24)?
- What is the consistency strength of having Posner-Robinson result as in Theorem 3.6?

These are interesting questions in its own, especially for set theorists. Properties regarding generalized recursive degree are subjects of α-recursion theory, which concerns only degrees in L (see Ref. 16). While we investigate structural properties of higher level degree notions, it also makes more sense to consider them in canonical settings such as fine structure extender models. This is because ZFC alone, even plus large cardinal assumption, though may decide certain individual properties, can hardly determine the structure of degree posets. For instance, consider the structure of Zermelo degrees at $\aleph_\omega$.

Example 4.1. Assume $\mathsf{ZFC} + \mathsf{GCH}$ and plus some large cardinal assumption, say a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ plus a measurable cardinal $\kappa' > \kappa$. With a small forcing, one can arrange that in the generic extension $\kappa = \aleph_\omega$, GCH remains true below $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ while the measurability of κ' is preserved (This combines results of Woodin and Gitik, see Ref. 64). But then the Zermelo degree posets at $\aleph_\omega$ can not be well ordered (even prewellordered) in the generic extension, as every degree has only $\aleph_\omega$ many predecessors in the degree partial ordering. This is in contrast to the pictures in $L[\mu]$ (see Theorem 3.2, p.19).

It is the well organized structure of $L[\mu]$ (organized using Steel's construction) that forces the degrees to line up in a well ordered fashion. The existence of measurable cardinals alone (more precisely, without appealing to forcings) is not strong enough to create "untamed" degrees – incomparable degrees, unless we go up to the ω-limit of measurable cardinals (see Corollary §3.2) and beyond.

4.2. *Degree structures in canonical models*

In §3, we analyzed Zermelo degree structures in several canonical models or under some stronger large cardinals. An immediate conclusion is that larger cardinals create more complicated Zermelo degree structures at some critical cardinals (more precisely, ω-limit of certain large cardinals). In other words, in these models the complexity of the Zermelo degree structure at these critical cardinals reflects the strength of the relevant cardinals.

The next natural step is to look into larger cardinal axioms and hope to find more complicated degree structures. For instance, what degree structures can one see at an ω-limit of strong cardinals, or Woodin cardinals, or supercompact cardinals, etc.? During the process, it would be interesting in itself to extend the question list on page 15 to differentiate these degree structures, in a way that the natural order of large cardinals sorts these structural properties into layers.

At the mean time, the pictures of Zermelo degree structures in L and through up to the core models for finitely many measurable cardinals strongly suggest that in any reasonable inner model, at every singular cardinal λ with $\mathrm{cf}(\lambda) = \omega$ and below the least measurable, the Zermelo degrees are wellordered above some degree. In particular,

Conjecture 4.1. *In all fine structure extender models the Zermelo degree structures at (their) $\aleph_\omega$ are all (eventually) wellordered and the immediate successor is given by Zermelo jump.*

Combining these remarks, one can see that the complexity of a particular degree structure does not necessarily gives the large cardinal strength of the core model, but it does indicate the levels of the associated cardinals that the structure resides. In other words, from the variety of the types of degree structures that appear in a core model one can tell the lower bound of the large cardinals the given core model carries. This is a complete new perspective for looking into large cardinal axioms.

4.3. *New techniques are needed*

Our proofs for L and up to $L[\bar{\mu}]$ use heavily one particular form of covering lemmas, we expect that that analysis will work as far as that form of Covering Lemma holds, namely at least up to Mitchell models for sequences of measures.

Next key step is to check whether in M_1, the minimal iterable class model for one Woodin cardinal, the scenarios described above continue.

Conjecture 4.2. *In M_1, the Zermelo degrees at $\aleph_\omega$ are well ordered.*

Moreover, we expect that this to be true for singular countable cofinality λ's that are above or in-between critical large cardinals, as this fits well with the intuition that universes of small large cardinals are initial segment of universes of larger cardinals.

Climbing up the cardinal ladder, although new pictures may appear at certain cardinals, as well as degree structures in between these special cardinals, as what we have just discussed about $\aleph_\omega$, it seems reasonable to conjecture that the structure at a particular cardinal once appear in a core model for certain large cardinal axiom, will stay unchanged as we move up to core models for larger cardinals, assuming they exist.

However, as the classical form of Covering is not available for M_1 and larger core models, deeper understanding of their structures[1] and new techniques are necessary for the investigation of degree structures in these models.

Besides the classical fine structure models, recent developments in descriptive inner model theory (see for example Sargsyan's survey paper 66) suggest a much advanced and daring path of investigation – looking into higher degrees in the HODs of determinacy models, as determinacy gives a whole family of canonical models – the ones given by Solovay hierarchy. Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$, it's believed that HOD is a canonical model. Although it's still an open question whether $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$ proves that HOD is a fine structure model, HOD of $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$ models are believed to be fine structure models at least all the way up to $\Theta =_{\text{def}} \sup\{\alpha \mid \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}$ (see Steel 67). Based on this understanding, the first test question would be

Question 4.1. Assume $\mathsf{AD}^+ + V = L(\mathscr{P}(\mathbb{R}))$. Look at Zermelo degrees within HOD at $(\aleph_\omega)^{\mathsf{HOD}}$, are they (eventually) well ordered?

This is a *great* question! One can not expect to solve this problem with only Covering, one would need mouse analysis for arbitrary AD^+ models. But the mouse analysis technique is still in its development, there is very little on this matter that is valuable to say at this point.

4.4. *Evidences of the impact of large cardinals on structures of degrees*

Next we leave the canonnical models, look at the impact of large cardinal alone on the structures of degrees. The theme is that stronger large

[1] So far the best result on constructing core models is due to Neeman (see 65), who produces a core model for a Woodin limit of Woodin cardinals.

cardinal yields more complicated degree structures at certain strong limit, countabe cofinality, singular cardinals. We have seen two evidences, one is the prewellordered degree structure in Theorem 3.3, where you can find incomparable degrees (see Corollary 3.2), the other is Yang Sen's minimal cover result quoted on page 24.

In Ref. 2, it is shown that I_0, one of the strongest large cardinal hypotheses, entails a richer degree structures at a certain strong limit λ with countable cofinality – it gives positive answers to Post problem, minimal covver problem and Posner-Robinson problem. Furthermore, it proves that I_0 together with a mild indestructibility assumption imply the failure of degree determinacy in $L(\mathscr{P}(\lambda))$ for Zermelo degrees at a particular strong limit, countable cofinality, singular cardinal (see Theorem 3.7) by exploiting the richness of the degree structure provided by the large cardinal axiom. As part of the global conjecture (see page 27), it is also conjectured that the failure of degree determinacy in $L(\mathscr{P}(\lambda))$ for Zermelo degrees at λ, for countable cofinality λ, is a theorem of ZFC. But as our analysis indicates, if one wants to prove this conjecture, the proof has to be very subtle: In early stages of canonical inner models, the degree structures are very simple, the degree determinacy fails due to that simplicity and our approach for proving the failure of degree determinacy by exploiting the richness of degree structures does not work there.

4.5. *Structures of other degrees*

In §3, we have been focusing on Zermelo degrees, only compare structures of Zermelo degrees crossing over inner models. But there is a whole spectrum of degree notions one can explore, as we have seen in §2. And certainly there are many more questions one can pursue if structures of different degree notions are compared.

In fact, the newly discovered connection between the complexity of degree structures and the large cardinal strength of relevant cardinals lead us to review some old results with new perspective.

Recall that there are incomparable $\aleph_\omega$-degrees (see p.13). Comparing it with the fact that the Zermelo degree at $\aleph_\omega$ is wellordered, one may draw a conclusion that smaller degree operator exhibit rich degree structures at early stage of inner models. This is not something exciting, as the larger degree operator often absorbs part of structure induced by the smaller degree operator. And this is well supported by examples discussed in §2 – structures of larger degree notions are always simpler than those given by smaller degree operators. For instance, while the poset $(\mathscr{R}, \leq_T)$ is dense and has incomparable degrees, it analogue for hyperarithmetic degrees – the poset of hyperdegrees restricted to Π^1_1 subsets of ω – is trivial, consisting of only $[\emptyset]_h$ and $[\mathcal{O}]_h = [\emptyset]'_h$. Here is a question on the spot:

Question 4.2. Assume $V = L$. Are there incomparable generalized hyperdegrees at $\aleph_\omega$?

Let us take a closer look. In $V = L$, although it is still open whether there are minimal $\aleph_\omega$-degrees, if move to a Σ_2-admissible ordinal α, one starts to see minimal α-degrees. Yang's argument gives minimal α-degrees at α which is the ω-limit of certain large cardinals there are, a little adaption gives minimal Zermelo degrees as well. So these large cardinals create not only minimal Δ_1-degrees at α, but any reasonable degree operator at α. This is beyond Σ_2-admissibility. Just as admissible ordinals are "recursively" regular, Δ_1-ly speaking, Σ_2-admissible ordinals behave like a very large "recursive" large cardinals. To extend this similarity, a sample question would be to find the ordinal for minimal hyperdegrees.

Question 4.3. Assume $V = L$. At what ordinal α can there be minimal generalized hyperdegrees at α?

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