DISCRETE-TIME MODELS IN MATHEMATICAL FINANCE

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Preface

These notes originated from a short undergraduate course I gave in the spring terms of 2005 in Beijing Normal University. The materials were mainly taken from the following book:

> "*Introduction to Stochastic Calculus Applied to Finance*" by D. Lamberton and B. Lapeyre (Chapman and Hall, London, 1996).

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Contents

Chapter 0

Mathematical Preliminaries

In this chapter, we summarize some basic concepts and results from general probability theory which will be used in this course. Most of them can be found in standard text books; see e.g. Chow and Teicher (1988).

0.1 Measurable spaces and functions

Let Ω be a non-empty set. A family $\mathscr{F}$ of subsets of Ω is called a *σ-algebra* if

(i) $\emptyset \in \mathscr{F}$;

(ii) $A \in \mathscr{F}$ implies $A^c \in \mathscr{F}$, where $A^c := \Omega \setminus A$ denotes the complement of A;

(iii) $\{A_1, A_2, \cdots\} \subseteq \mathscr{F}$ implies $\cup_{j=1}^{\infty} A_j \in \mathscr{F}$.

If $\mathscr{F}$ is a σ-algebra on Ω, the pair $(\Omega, \mathscr{F})$ is called a *measurable space*. The sets in $\mathscr{F}$ are called *measurable sets*.

If $\mathscr{C}$ be a family of subsets of Ω, then $\sigma(\mathscr{C}) := \cap\{\mathscr{G} : \mathscr{G} \supseteq \mathscr{C} \text{ is a } \sigma\text{-algebra}\}$ is a σ-algebra, which is called the *σ-algebra generated by $\mathscr{C}$*. The σ-algebra generated by the family $\mathscr{O}(\mathbb{R}^d)$ of open sets in $\mathbb{R}^d$ is called the *Borel σ-algebra* of $\mathbb{R}^d$ and is denoted by $\mathscr{B}(\mathbb{R}^d)$. Clearly, $\mathscr{B}(\mathbb{R}^d)$ contains all open sets, all closed sets, all countable unions of closed sets, and so on.

Lemma 0.1.1 *If $\mathscr{E}$ is a σ-algebra on E, for any $f : \Omega \to E$ the family $f^{-1}(\mathscr{E}) := \{f^{-1}(B) : B \in \mathscr{E}\}$ is a σ-algebra on Ω.*

Proof. (Exercise.) $\square$

Given measurable spaces $(\Omega, \mathscr{F})$ and $(E, \mathscr{E})$, we say $f : \Omega \to E$ is $\mathscr{F}/\mathscr{E}$-*measurable*, or simply $\mathscr{F}$-*measurable*, provided $f^{-1}(\mathscr{E}) \subset \mathscr{F}$, that is, $f^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{E}$. We call $f : \mathbb{R}^d \to \mathbb{R}$ a *Borel function* if it is $\mathscr{B}(\mathbb{R}^d)$-measurable.

Lemma 0.1.2 *For any mapping $f : \Omega \to E$ and any family $\mathscr{U}$ of subsets of E we have $\sigma(f^{-1}(\mathscr{U})) = f^{-1}(\sigma(\mathscr{U}))$.*

Proof. By Lemma 0.1.1, the family $f^{-1}(\sigma(\mathscr{U}))$ is a σ-algebra on Ω. Since $f^{-1}(\mathscr{U}) \subseteq f^{-1}(\sigma(\mathscr{U}))$, we have $\sigma(f^{-1}(\mathscr{U})) \subseteq f^{-1}(\sigma(\mathscr{U}))$. On the other hand, let $\mathscr{A} = \{A \subseteq E : f^{-1}(A) \in \sigma(f^{-1}(\mathscr{U}))\}$. Then $\mathscr{U} \subseteq \mathscr{A}$ and $f^{-1}(\mathscr{A}) \subseteq \sigma(f^{-1}(\mathscr{U}))$. It is not hard to check that $\mathscr{A}$ is a σ-algebra. (Exercise.) Therefore we have $\sigma(\mathscr{U}) \subseteq \mathscr{A}$. This yields $f^{-1}(\sigma(\mathscr{U})) \subseteq f^{-1}(\mathscr{A}) \subseteq \sigma(f^{-1}(\mathscr{U}))$. □

Theorem 0.1.1 *Let $(\Omega, \mathscr{F})$ be a measurable space. If $X : \Omega \to \mathbb{R}^n$ is $\mathscr{F}$-measurable and $f : \mathbb{R}^n \to \mathbb{R}^d$ is Borel, then the composition $f \circ X : \Omega \to \mathbb{R}^d$ is $\mathscr{F}$-measurable.*

Proof. For any $B \in \mathscr{B}(\mathbb{R}^d)$, we have $f^{-1}(B) \in \mathscr{B}(\mathbb{R}^n)$. It follows that $(f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \in \mathscr{F}$. □

Theorem 0.1.2 *A continuous function $f : \mathbb{R}^n \to \mathbb{R}^d$ is Borel.*

Proof. If $f : \mathbb{R}^n \to \mathbb{R}^d$ is continuous, then $f^{-1}(\mathscr{O}(\mathbb{R}^d)) \subseteq \mathscr{O}(\mathbb{R}^n)$. By Lemma 0.1.2,

$$f^{-1}(\mathscr{B}(\mathbb{R}^d)) = f^{-1}(\sigma(\mathscr{O}(\mathbb{R}^d))) = \sigma(f^{-1}(\mathscr{O}(\mathbb{R}^d))) \subseteq \sigma(\mathscr{O}(\mathbb{R}^n)) = \mathscr{B}(\mathbb{R}^n).$$

That is, f is a Borel function. □

For a family $\mathscr{X} = \{X_i : i \in I\}$ of functions $X_i : \Omega \to \mathbb{R}^d$, the *$\sigma$-algebra generated by $\mathscr{X}$* is defined as

$$\sigma(\mathscr{X}) := \sigma(\{X_i^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^d), i \in I\}). \tag{1.1}$$

For a single function $X : \Omega \to \mathbb{R}^d$, we write $\sigma(X)$ instead of $\sigma(\{X\})$.

Theorem 0.1.3 *For any function $X : \Omega \to \mathbb{R}^d$, we have $\sigma(X) = X^{-1}(\mathscr{B}(\mathbb{R}^d))$.*

Proof. By Lemma 0.1.1, the family $X^{-1}(\mathscr{B}(\mathbb{R}^d)) = \{X^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^d)\}$ is a σ-algebra on Ω, so $X^{-1}(\mathscr{B}(\mathbb{R}^d)) = \sigma(X^{-1}(\mathscr{B}(\mathbb{R}^d)))$. By (1.1), we have $\sigma(X) = \sigma(\{X^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^d)\}) = X^{-1}(\mathscr{B}(\mathbb{R}^d))$. □

Theorem 0.1.4 *Let $(\Omega, \mathscr{F})$ be a measurable space. Then $X : \Omega \to \mathbb{R}^d$ is $\mathscr{F}$-measurable if and only if one of the following holds:*

(i) $X^{-1}((-\infty, x)) \in \mathscr{F}$ for every $x \in \mathbb{R}^d$;

(ii) $X^{-1}((x, \infty)) \in \mathscr{F}$ for every $x \in \mathbb{R}^d$.

Proof. Suppose that X is $\mathscr{F}$-measurable. Since $(-\infty, x) \in \mathscr{B}(\mathbb{R}^d)$, we have $X^{-1}((-\infty, x)) \in \mathscr{F}$, that is, (i) holds. Conversely, suppose that (i) holds. Let $\mathscr{G} = \{(-\infty, x) : x \in \mathbb{R}^d\}$. Then

$$X^{-1}(\mathscr{G}) = \{X^{-1}((-\infty, x)) : x \in \mathbb{R}^d\} \subseteq \mathscr{F}.$$

It is not hard to show that $\mathscr{B}(\mathbb{R}^d) = \sigma(\mathscr{G})$. (Exercise.) By Lemma 0.1.2 we have

$$X^{-1}(\mathscr{B}(\mathbb{R}^d)) = \sigma(X^{-1}(\mathscr{G})) \subseteq \sigma(\mathscr{F}) = \mathscr{F}.$$

Thus X is $\mathscr{F}$-measurable. The second assertion follows by similar arguments. $\square$

Theorem 0.1.5 *Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of $\mathscr{F}$-measurable real functions on $(\Omega, \mathscr{F})$. Then any of the functions*

$$\sup_{n\geq 1} X_n, \ \inf_{n\geq 1} X_n, \ \limsup_{n\to\infty} X_n \ \text{ and } \ \liminf_{n\to\infty} X_n \tag{1.2}$$

is $\mathscr{F}$-measurable if it is finite real-valued.

Proof. The measurability of the first two functions follows respectively from the relations

$$\{\omega : \sup_{n\geq 1} X_n(\omega) > x\} = \bigcup_{n\geq 1} \{\omega : \sup_{n\geq 1} X_n(\omega) > x\}$$

and

$$\{\omega : \inf_{n\geq 1} X_n(\omega) < x\} = \bigcup_{n\geq 1} \{\omega : \sup_{n\geq 1} X_n(\omega) < x\}.$$

Then $\limsup_{n\to\infty} X_n = \inf_{n\geq 1} \sup_{k\geq n} X_k$ and $\liminf_{n\to\infty} X_n = \sup_{n\geq 1} \inf_{k\geq n} X_k$ are measurable. $\square$

Example 0.1.1 A finite or countable family $\mathscr{U} = \{U_i : i \in I\}$ of disjoint subsets of Ω satisfying $\cup_{i\in I} U_i = \Omega$ is called a *partition of* Ω. If $\{U_i : i \in I\}$ is a partition of Ω, then $\sigma(\mathscr{U}) = \{\cup_{j\in J} U_j : J \subseteq I\}$. (Exercise.) In this case, each U_i is called an atom of $\sigma(\mathscr{U})$. (Here $\cup_{j\in\emptyset} U_j = \emptyset$ by convention.)

Example 0.1.2 Let $\mathscr{G} = \sigma(\{U_i : i \in I\})$ for a partition $\{U_i : i \in I\}$ of Ω. Then a function $X : \Omega \to \mathbb{R}$ is $\mathscr{G}$-measurable if and only if there are real constants $\{c_i : i \in I\}$ such that

$$X(\omega) = \sum_{i \in I} c_i \mathbf{1}_{U_i}(\omega), \qquad \omega \in \Omega. \tag{1.3}$$

(Exercise.)

Example 0.1.3 Let $\{c_i : i \in I\}$ be distinct real numbers and let $\{U_i : i \in I\}$ be a partition of Ω. If the function $X : \Omega \to \mathbb{R}$ has representation (1.3), then $\sigma(X) = \sigma(\{U_i : i \in I\})$. (Exercise.)

0.2 Probability spaces and random variables

A *finite measure* μ on a measurable space $(\Omega, \mathscr{F})$ is a function $\mu : \mathscr{F} \to [0, \infty)$ such that

(D) if $A_1, A_2, \cdots \in \mathscr{F}$ are disjoint, then $\mu(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$.

If $\mathbf{P}$ is a finite measure on $(\Omega, \mathscr{F})$ with $\mathbf{P}(\Omega) = 1$, we call it a *probability measure*. In this case, the triple $(\Omega, \mathscr{F}, \mathbf{P})$ is called a *probability space* and each $A \in \mathscr{F}$ is called an *event*. We call $N \subset \Omega$ a $\mathbf{P}$-*null set* if there is $B \in \mathscr{F}$ such that $N \subseteq B$ and $\mathbf{P}(B) = 0$. If $\mathscr{F}$ contains all $\mathbf{P}$-null sets, we say $(\Omega, \mathscr{F}, \mathbf{P})$ is a *complete probability space*. An event $A \in \mathscr{F}$ is said to *occur almost surely* if A^c is a $\mathbf{P}$-null set.

We say the events $\{A_i : i \in I\} \subseteq \mathscr{F}$ are *independent* if

$$\mathbf{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbf{P}(A_{i_1}) \cdots \mathbf{P}(A_{i_k}) \tag{2.1}$$

for every finite subset $\{i_1, \cdots, i_k\}$ of the index set I. A collection $\{\mathscr{H}_i : i \in I\}$ of families of events are *independent* if the family of events $\{H_i : i \in I\} \subseteq \mathscr{F}$ are independent for all possible choices of $H_i \in \mathscr{H}_i$ with $i \in I$.

Suppose that $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space. An $\mathscr{F}$-measurable function $X : \Omega \to \mathbb{R}^d$ is called a *d-dimensional random variable*. When the dimension number is unimportant or understood, we suppress the qualifier "d-dimensional". A random variable X induces a probability measure μ_X on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ by

$$\mu_X(B) = \mathbf{P}(X^{-1}(B)), \quad B \in \mathscr{B}(\mathbb{R}^d), \tag{2.2}$$

which is called the *distribution* of X. If

$$\int_{\Omega} |X(\omega)| \mathbf{P}(d\omega) = \int_{\mathbb{R}^d} |x| \mu_X(dx) < \infty,$$

the vector

$$\mathbf{E}[X] := \int_{\Omega} X(\omega)\mathbf{P}(d\omega) = \int_{\mathbb{R}^d} x\mu_X(dx). \tag{2.3}$$

is called the *expectation* of X.

A class of random variables $\{X_i : i \in I\}$ are said to be *independent* if the class of σ-algebras $\{\sigma(X_i) : i \in I\}$ are independent.

Theorem 0.2.1 *The two d-dimensional random variables X and Y are independent if and only if*

$$\mathbf{E}[f(X)g(Y)] = \mathbf{E}[f(X)]\mathbf{E}[g(Y)] \tag{2.4}$$

for all bounded Borel functions f and $g : \mathbb{R}^d \to \mathbb{R}$.

Proof. Clearly, if (2.4) holds for all bounded Borel functions f and g, then X and Y are independent. Conversely, suppose that X and Y are independent. If f and g are simple functions of the form

$$f = \sum_{j=1}^{m} f_j \mathbf{1}_{F_j} \quad \text{and} \quad g = \sum_{j=1}^{n} g_j \mathbf{1}_{G_j},$$

where $F_j, G_j \in \mathscr{B}(\mathbb{R})$, (2.4) follows from the definition of the independence. For a general f and g, the equality (2.4) follows by an approximation using simple functions. □

Let X_n and X be random variables taking values in $\mathbb{R}^d$. We say X_n *converges to X in probability* if $\mathbf{P}(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. Recall that X_n *converges to X almost surely* if there is a $\mathbf{P}$-null set N such that $X_n(\omega) \to X(\omega)$ for all $\omega \in N^c$.

Theorem 0.2.2 *If $X_n \to X$ almost surely, then $X_n \to X$ in probability.*

Proof. Observe that $X_n \to X$ almost surely if and only if

$$\begin{aligned} &\mathbf{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \{\omega : |X_j(\omega) - X(\omega)| \geq \varepsilon\} \right) \\ &\qquad = \lim_{n\to\infty} \mathbf{P}\left(\bigcup_{j=n}^{\infty} \{\omega : |X_j(\omega) - X(\omega)| \geq \varepsilon\} \right) = 0 \end{aligned}$$

for every $\varepsilon > 0$. This clearly implies $X_n \to X$ in probability. □

For $p \geq 1$ let $L^p(\Omega, \mathbf{P})$ be the totality of real-valued random variables X such that

$$\|X\|_p := \{\mathbf{E}[|X|^p]\}^{1/p} < \infty. \tag{2.5}$$

Given X_n and $X \in L^p(\Omega, \mathbf{P})$, we say X_n *converges to* X *in* $L^p(\Omega, \mathbf{P})$ if $\|X_n - X\|_p \to 0$.

Theorem 0.2.3 *If $X_n \to X$ in $L^p(\Omega, \mathbf{P})$, then $X_n \to X$ in probability.*

Proof. Suppose $X_n \to X$ in $L^p(\Omega, \mathbf{P})$. Then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\{|X_n - X| \geq \varepsilon\} \leq \lim_{n\to\infty} \varepsilon^{-p}\mathbf{E}[|X_n - X|^p] = 0.$$

by Chebyshev's inequality. □

0.3 Conditional expectations

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and $\mathscr{G} \subseteq \mathscr{F}$ a σ-algebra. Let $X : \Omega \to \mathbb{R}$ be a random variable such that $\mathbf{E}[|X|] < \infty$. It is known that there is an a.s. unique $\mathscr{G}$-measurable random variable $\xi : \Omega \to \mathbb{R}$ such that

$$\mathbf{E}[\mathbf{1}_G X] = \mathbf{E}[\mathbf{1}_G \xi], \qquad G \in \mathscr{G}; \tag{3.1}$$

see e.g. Chow and Teicher (1988, pp.202-203). We call ξ the *conditional expectation* of X given $\mathscr{G}$, and denote it by $\mathbf{E}[X|\mathscr{G}]$. The conditional expectation $\mathbf{E}[X|\mathscr{G}]$ reflects the change in the unconditional expectation $\mathbf{E}[X]$ due to the additional information provided by $\mathscr{G}$.

For any $A \in \mathscr{F}$, we call $\mathbf{E}[\mathbf{1}_A|\mathscr{G}]$ the *conditional probability* of A given $\mathscr{G}$, and denote it by $\mathbf{P}[A|\mathscr{G}]$. If Y is another random variable, we simply write $\mathbf{E}[X|Y]$ for $\mathbf{E}[X|\sigma(Y)]$. The conditional probability $\mathbf{P}[A|X]$ is defined in a similar way.

Theorem 0.3.1 *Let a and b be real constants and X and Y be real random variables with $\mathbf{E}[|X| + |Y|] < \infty$. Then we have the following properties*

(i) $\mathbf{E}[aX + bY|\mathscr{G}] = a\mathbf{E}[X|\mathscr{G}] + b\mathbf{E}[Y|\mathscr{G}]$*;*

(ii) $\mathbf{E}\{\mathbf{E}[X|\mathscr{G}]\} = \mathbf{E}[X]$*;*

(iii) $\mathbf{E}[X|\mathscr{G}] = X$ *if X is $\mathscr{G}$-measurable;*

(iv) $\mathbf{E}[X|\mathscr{G}] = \mathbf{E}[X]$ *if X is independent of $\mathscr{G}$.*

Proof. Those properties are immediate consequences of the definition of the conditional expectation. (Exercise.) □

Theorem 0.3.2 *Let X and Y be real random variables with $\mathbf{E}[|YX|] < \infty$. If X is $\mathscr{G}$-measurable, then $\mathbf{E}[XY|\mathscr{G}] = X\mathbf{E}[Y|\mathscr{G}]$.*

Proof. It suffices to prove

$$\int_G X\mathbf{E}[Y|\mathscr{G}]d\mathbf{P} = \int_G XY d\mathbf{P}, \qquad G \in \mathscr{G}. \tag{3.2}$$

If $X = \mathbf{1}_H$ for some $H \in \mathscr{G}$, we have

$$\int_G X\mathbf{E}[Y|\mathscr{G}]d\mathbf{P} = \int_{G\cap H} \mathbf{E}[Y|\mathscr{G}]d\mathbf{P} = \int_{G\cap H} Y d\mathbf{P} = \int_G XY d\mathbf{P}.$$

Similarly, (3.2) holds for a simple function

$$X = \sum_{j=1}^{m} c_j \mathbf{1}_{H_j},$$

where $H_j \in \mathscr{H}$ and $c_j \in \mathbb{R}$. For a general X the equation follows by an approximation using simple functions. □

Theorem 0.3.3 *Let X be a random variable such that $\mathbf{E}[|X|] < \infty$, and let $\mathscr{H}$ and $\mathscr{G}$ be σ-algebras such that $\mathscr{H} \subseteq \mathscr{G} \subseteq \mathscr{F}$. Then we have*

$$\mathbf{E}[X|\mathscr{H}] = \mathbf{E}\{\mathbf{E}[X|\mathscr{G}]|\mathscr{H}\} \tag{3.3}$$

Proof. Let $H \in \mathscr{H}$. By the definition of the conditional expectation,

$$\int_H X d\mathbf{P} = \int_H \mathbf{E}[X|\mathscr{H}]d\mathbf{P}.$$

Since $\mathscr{H} \subseteq \mathscr{G}$, we have $H \in \mathscr{G}$ and hence

$$\int_H X d\mathbf{P} = \int_H \mathbf{E}[X|\mathscr{G}]d\mathbf{P}.$$

It then follows that

$$\int_H \mathbf{E}[X|\mathscr{H}]d\mathbf{P} = \int_H \mathbf{E}[X|\mathscr{G}]d\mathbf{P}.$$

Certainly, $\mathbf{E}[X|\mathscr{H}]$ is $\mathscr{H}$-measurable, so (3.3) follows. □

Example 0.3.1 Suppose that $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space. Let $\mathscr{G} = \sigma(\{U_i : i \in I\})$ for a partition $\{U_i : i \in I\} \subseteq \mathscr{F}$ of Ω with $\mathbf{P}(U_i) > 0$ for each $i \in I$. Then for any $A \in \mathscr{F}$ we have

$$\mathbf{P}(A|\mathscr{G})(\omega) = \mathbf{P}(A|U_i), \qquad \omega \in U_i. \tag{3.4}$$

This gives an interpretation for the random variable $\mathbf{P}(A|\mathscr{G})(\omega)$. The σ-algebra $\mathscr{G}$ can be interpreted as the information obtained by observing a random system with different states $\{U_i : i \in I\}$ which has some influence on the event A. In this situation, (3.4) simply means that the probability of A varies according to the different status of the system. To show (3.4), define

$$\eta(\omega) = \mathbf{P}(A|U_i), \qquad \omega \in U_i.$$

From Example 0.1.2 we know that η is a $\mathscr{G}$-measurable random variable. By Example 0.1.1, each $B \in \mathscr{G}$ can be represented as $B = \cup_{j\in J} U_j$ for a (finite or countable) set $J \subseteq I$. It follows that

$$\begin{aligned}\mathbf{E}[\mathbf{1}_B \mathbf{1}_A] &= \sum_{j\in J} \mathbf{P}(AU_j) = \sum_{j\in J} \mathbf{P}(U_j)\mathbf{P}(A|U_j) \\ &= \int_{\cup_{j\in J} U_j} \eta(\omega)\mathbf{P}(d\omega) = \mathbf{E}[\mathbf{1}_B \eta].\end{aligned}$$

Then $\mathbf{P}(A|\mathscr{G}) = \eta$ by the definition of conditional probability.

Example 0.3.2 Consider a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Let $\mathscr{G} = \sigma(\{U_i : i \in I\})$ for a partition $\{U_i : i \in I\} \subseteq \mathscr{F}$ of Ω with $\mathbf{P}(U_i) > 0$ for each $i \in I$. Clearly, for each $i \in I$,

$$\mathbf{P}_i(A) := \mathbf{P}(A|U_i), \qquad A \in \mathscr{F} \tag{3.5}$$

defines a probability measure on $(\Omega, \mathscr{F})$. Suppose that X is a random variable such that $\mathbf{E}[|X|] < \infty$. Then we have

$$\mathbf{E}[X|\mathscr{G}](\omega) = \int_\Omega X d\mathbf{P}_i, \qquad \omega \in U_i, \tag{3.6}$$

which gives a representation for the conditional expectation. Indeed, by Example 0.1.2,

$$\xi(\omega) = \int_\Omega X d\mathbf{P}_i, \qquad \omega \in U_i$$

defines a $\mathscr{G}$-measurable random variable ξ. By (3.5), we have

$$\mathbf{P}_i(A) = \frac{\mathbf{P}(A \cap U_i)}{\mathbf{P}(U_i)}, \qquad A \in \mathscr{F}$$

and hence

$$\int_\Omega X(\omega) d\mathbf{P}_i(\omega) = \mathbf{P}(U_i)^{-1} \int_{U_i} X(\omega)\mathbf{P}(d\omega).$$

If $B \in \mathscr{G}$ has the representation $B = \cup_{j\in J} U_j$ for $J \subseteq I$, then

$$\begin{aligned} \mathbf{E}[\mathbf{1}_B X] &= \sum_{j\in J} \int_{U_j} X(\omega)\mathbf{P}(d\omega) = \sum_{j\in J} \mathbf{P}(U_j) \int_\Omega X(\omega)\mathbf{P}_j(d\omega) \\ &= \int_{\cup_{j\in J} U_j} \xi(\omega)\mathbf{P}(d\omega) = \mathbf{E}[\mathbf{1}_B \xi]. \end{aligned}$$

By the definition of conditional expectation we get $\mathbf{E}[X|\mathscr{G}](\omega) = \xi(\omega)$.

Chapter 1

Discrete-time models

1.1 Introduction

This chapter is devoted to the study of discrete time models. The objective is to present the main ideas related to option theory for the very simple models. The link between the mathematical concept of martingale and the economic notion of arbitrage is brought to light. The definition of complete markets and the pricing of options in these markets are given. The Cox-Ross-Rubinstein model is treated as an example. This chapter was adopted from Lamberton and Lapeyre (1996).

1.2 Markets and strategies

1.2.1 Financial markets

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a *finite* probability space, where $\mathscr{F}$ is the family of all subsets of the non-empty finite set Ω. We assume that $\mathbf{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. Suppose that the probability space is equipped with a *filtration* $(\mathscr{F}_n)_{0\le n\le N}$, which is an increasing sequence of σ-algebras included in $\mathscr{F}$. The σ-algebra $\mathscr{F}_n$ can be viewed as the information available at time n. A *stochastic process* means a collection of random variables $\{X_n : n = 0, 1, \cdots, N\}$. We say $\{X_n\}$ is *adapted* if X_n is $\mathscr{F}_n$-measurable for each $0 \le n \le N$. An adapted process $\{X_n\}$ is *predictable* if X_n is $\mathscr{F}_{n-1}$-measurable for each $1 \le n \le N$.

We consider a *market* consisting of $d+1$ financial assets, whose prices at time n are given by the non-negative random variables $S_n^0, S_n^1, \cdots, S_n^d$, which are measurable with respect to $\mathscr{F}_n$. Then $\{(S_n^0, S_n^1, \cdots, S_n^d) : n = 0, 1, \cdots, N\}$ form a $(d+1)$-dimensional adapted stochastic process. We call $S_n := (S_n^0, S_n^1, \cdots, S_n^d)$ the *vector* *of prices*. The asset indexed by 0 is the *riskless asset* and we have $S_0^0 = 1$. If the return of the riskless asset over one period is constant and equals to r, we have $S_n^0 = (1+r)^n$. The coefficient $\beta_n := 1/S_n^0$ is interpreted as the *discount factor*: if an amount β_n is investigated at time 0, then one dollar will be available at time n. The assets indexed by $1, \cdots, d$ are called *risky assets*. Naturally, we call $\tilde{S}_n := \beta_n S_n$ the *discounted prices*. Of course,

$$\tilde{S}_n = (\tilde{S}_n^0, \tilde{S}_n^1, \cdots, \tilde{S}_n^d) = (1, \beta_n S_n^1, \cdots, \beta_n S_n^d).$$

1.2.2 Self-financing strategies

A *trading strategy* is defined as a stochastic process $\phi = \{(\phi_n^0, \phi_n^1, \cdots, \phi_n^d) : 0 \leq n \leq N\}$ taking values from $\mathbb{R}^{d+1}$, where ϕ_n^i denotes the number of shares of asset i held in the portfolio at time n. We assume ϕ is predictable, i.e., ϕ_0 is $\mathscr{F}_0$-measurable and ϕ_n is $\mathscr{F}_{n-1}$-measurable for $1 \leq n \leq N$. This assumption means that the positions in the portfolio at time n are decided with respect to the information available at time $n-1$ and kept until time n when new quotations are availaible.

The *value of the portfolio* at time n is the scalar product

$$V_n(\phi) = \phi_n \cdot S_n = \sum_{i=0}^{d} \phi_n^i S_n^i. \tag{2.1}$$

The *discounted value* is

$$\tilde{V}_n(\phi) = \beta_n V_n(\phi) = \phi_n \cdot \tilde{S}_n. \tag{2.2}$$

In particular,

$$\tilde{V}_0(\phi) = \beta_0 V_0(\phi) = V_0(\phi)/S_0^0 = V_0(\phi).$$

Definition 1.2.1 *A strategy is called self-financing if*

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n \tag{2.3}$$

for each $n = 0, 1, \cdots, N-1$.

The interpretation of a self-financing strategy is at each time n the investigator adjust his position from ϕ_n to ϕ_{n+1} without bring or consuming any wealth. The equality (2.3) is obviously equivalent to

$$\phi_{n+1} \cdot S_{n+1} - \phi_n \cdot S_n = \phi_{n+1} \cdot S_{n+1} - \phi_{n+1} \cdot S_n$$

or to

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n). \tag{2.4}$$

Note that (2.4) means that the profit or loss lealized by following a self-financing strategy is only due to the price moves. The following proposition makes this clear in terms of discounted prices.

Proposition 1.2.1 *The following are equivalent*

(i) The strategy ϕ is self-financing.

(ii) For each $n = 1, \cdots, N$, we have

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j, \tag{2.5}$$

where $\Delta S_j = S_j - S_{j-1}$.

(iii) For each $n = 1, \cdots, N$, we have

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j, \tag{2.6}$$

where $\Delta \tilde{S}_j = \tilde{S}_j - \tilde{S}_{j-1}$.

Proof. Obviously, both (i) and (ii) are equivalent to equation (2.4). On the other hand, (2.3) is equivalent to

$$\phi_{n+1} \cdot \tilde{S}_{n+1} - \phi_n \cdot \tilde{S}_n = \phi_{n+1} \cdot \tilde{S}_{n+1} - \phi_{n+1} \cdot \tilde{S}_n$$

or to

$$\tilde{V}_{n+1}(\phi) - \tilde{V}_n(\phi) = \phi_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n), \tag{2.7}$$

which holds if and only (iii) holds. □

This proposition shows that, if an investor follows a self-financing strategy, the discounted value of his portfolio is completely defined by the initial wealth and the strategy $\{(\phi_n^1, \cdots, \phi_n^d) : 0 \le n \le N\}$. (This is only justified because $\Delta \tilde{S}_j^0 = 0$). More precisely, we can prove the following proposition.

Proposition 1.2.2 *For any predictable process $\{(\phi_n^1, \cdots, \phi_n^d) : 0 \le n \le N\}$ and any $\mathscr{F}_0$-measurable random variable V_0, there exists a unique predictable process $\{\phi_n^0 : 0 \le n \le N\}$ such that the strategy $\phi = \{(\phi_n^0, \phi_n^1, \cdots, \phi_n^d) : 0 \le n \le N\}$ is self-financing and its initial value is V_0.*

Proof. The self-financing strategy condition implies

$$\begin{aligned}\tilde{V}_n(\phi) &= \phi_n^0 + \phi_n^1 \tilde{S}_n^1 + \cdots + \phi_n^d \tilde{S}_n^d \\ &= V_0 + \sum_{j=1}^{n} \left(\phi_j^1 \Delta \tilde{S}_j^1 + \cdots + \phi_j^d \Delta \tilde{S}_j^d\right).\end{aligned}$$

Then we only need to set

$$\phi_n^0 = V_0 + \sum_{j=1}^{n-1} \left(\phi_j^1 \Delta \tilde{S}_j^1 + \cdots + \phi_j^d \Delta \tilde{S}_j^d\right) - (\phi_n^1 \tilde{S}_{n-1}^1 + \cdots + \phi_n^d \tilde{S}_{n-1}^d),$$

which defines a predictable process $\{\phi_n^0 : 0 \le n \le N\}$. □

1.2.3 Admissible strategies and arbitrage

We have not made any assumption on the sign of the quantity ϕ_n^0. If $\phi_n^0 < 0$, we have borrowed the amount $|\phi_n^0|$ in the riskless asset at time n. If $\phi_n^i < 0$ for $i \ge 1$, we say that we are *short* a number ϕ_n^i of asset i. In other words, short-selling and borrowing are allowed in the investment. However, the investor must be able to pay back his debts (in riskless or risky asset) at any time. This consideration leads to the following

Definition 1.2.2 *A strategy ϕ is admissible if it is self-financing and if $V_n(\phi) \ge 0$ for each $n = 0, 1, \cdots, N$.*

The notion of possibility of riskless profit can be formulated as follows.

Definition 1.2.3 *An arbitrage strategy is an admissible strategy with zero initial value and non-zero final value. More precisely, that is an admissible strategy ϕ such that $V_0(\phi) = 0$, $V_n(\phi) \ge 0$ for $n = 0, 1, \cdots, N$ and $\mathbf{P}\{V_N(\phi) > 0\} > 0$.*

Most models exclude any arbitrge opportunity and the objective of the next section is to characterize the models with the notion of martingale.

1.3 Martingales and arbitrage opportunities

1.3.1 Martingales and martingale transforms

We consider a finite probability space $(\Omega, \mathscr{F}, \mathbf{P})$ equipped with a filtration $(\mathscr{F}_n)_{0 \le n \le N}$. We assume that $\mathscr{F}$ is the family of all subsets of Ω and $\mathbf{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Definition 1.3.1 *An adapted process* $\{M_n : 0 \le n \le N\}$ *is called:*

- *a martingale if* $\mathbf{E}[M_{n+1}|\mathscr{F}_n] = M_n$ *for all* $n \le N-1$;
- *a supermartingale if* $\mathbf{E}[M_{n+1}|\mathscr{F}_n] \le M_n$ *for all* $n \le N-1$;
- *a submartingale if* $\mathbf{E}[M_{n+1}|\mathscr{F}_n] \ge M_n$ *for all* $n \le N-1$.

These definitions can be extended to the multi-dimensional case. For example, a process $\{M_n : 0 \le n \le N\}$ taking values from $\mathbb{R}^d$ is a martingale if each component is a real-valued martingale.

Remark 1.3.1 (i) The process $\{M_n : 0 \le n \le N\}$ is a martingale if and only if

$$\mathbf{E}[M_n|\mathscr{F}_m] = M_m, \qquad 0 \le m \le n \le N. \tag{3.1}$$

(ii) If $\{M_n : 0 \le n \le N\}$ is a martingale, then $\mathbf{E}[M_n] = \mathbf{E}[M_0]$ for each $0 \le n \le N$.

(iii) The sum of two martingales is also a martingale.

(iv) Similar properties hold for supermartingales and submartingales.

Proof. (Exercise.) □

Proposition 1.3.1 *Let* $\{M_n : 0 \le n \le N\}$ *be a martingale and* $\{H_n : 0 \le n \le N\}$ *a predictable process. Write* $\Delta M_n = M_n - M_{n-1}$ *and define the process* $\{X_n : 0 \le n \le N\}$ *by* $X_0 = H_0 M_0$ *and*

$$X_n = H_0 M_0 + H_1 \Delta M_1 + \cdots + H_n \Delta M_n, \qquad 1 \le n \le N. \tag{3.2}$$

Then $\{X_n : 0 \le n \le N\}$ *is a martingale.*

Proof. Clearly, $\{X_n\}$ is an adapted process. For $n \ge 0$ we have

$$\begin{aligned}\mathbf{E}[(X_{n+1} - X_n)|\mathscr{F}_n] &= \mathbf{E}[H_{n+1}(M_{n+1} - M_n)|\mathscr{F}_n] \\ &= H_{n+1}\mathbf{E}[(M_{n+1} - M_n)|\mathscr{F}_n] \\ &= H_{n+1}(\mathbf{E}[M_{n+1}|\mathscr{F}_n] - M_n) = 0.\end{aligned}$$

It follows that

$$\mathbf{E}[X_{n+1}|\mathscr{F}_n] = \mathbf{E}[X_n|\mathscr{F}_n] = X_n.$$

That shows that $\{X_n\}$ is a martingale. □

The process $\{X_n\}$ defined by (3.2) is sometimes called the *martingale transform* of $\{M_n\}$ by $\{H_n\}$.

Proposition 1.3.2 *An adapted process $\{M_n\}$ is a martingale if and only if for any predictable sequence $\{H_n\}$ we have*

$$\mathbf{E}\Big[\sum_{n=1}^{N} H_n \Delta M_n\Big] = 0. \tag{3.3}$$

Proof. If $\{M_n\}$ is a martingale and $\{H_n\}$ is a predictable process, the sequence $\{X_n\}$ defined by $X_0 = 0$ and

$$X_n = \sum_{j=1}^{n} H_j \Delta M_j, \qquad 1 \leq n \leq N,$$

is a martingale by Proposition 1.3.1. Then we have $\mathbf{E}[X_N] = \mathbf{E}[X_0] = 0$. Conversely, for $1 \leq j \leq N$ we can define the sequence $\{H_n\}$ as follows:

$$H_n = \begin{cases} 0 & \text{for } n \neq j+1 \\ 1_A & \text{for } n = j+1, \end{cases}$$

where A is any $\mathscr{F}_j$-measurable set. Clearly, $\{H_n\}$ is predictable and (3.3) becomes

$$\mathbf{E}\big[\mathbf{1}_A(M_{j+1} - M_j)\big] = 0.$$

Therefore, $\mathbf{E}[M_{j+1} - M_j | \mathscr{F}_j] = 0$ and hence $\{M_n\}$ is a martingale. □

From Proposition 1.2.1, we know

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j.$$

If the discounted prices of the assets are martingales, by Proposition 1.3.1 the discounted value $\tilde{V}_n$ is also a martingale.

1.3.2 Viable financial markets

Definition 1.3.2 *A market is called viable if there is no arbitrage opportunity.*

Lemma 1.3.1 *If the market is viable, for any predictable process $\{(\phi_n^1, \cdots, \phi_n^d)\}$ we have*

$$\tilde{G}_n(\phi) := \sum_{j=1}^{n} \big(\phi_j^1 \Delta \tilde{S}_j^1 + \cdots + \phi_j^d \Delta \tilde{S}_j^d)\big) \notin \Gamma,$$

where Γ is the convex cone of random variables $X \geq 0$ with $\mathbf{P}\{X > 0\} > 0$.

Proof. Let us assume that $\tilde{G}_N(\phi) \in \Gamma$. According to Proposition 1.2.2, there exists a unique predictable process $\{\phi_n^0\}$ such that the strategy $\{(\phi_n^0, \phi_n^1, \cdots, \phi_n^d)\}$ is self-financing and $V_0(\phi) = 0$. It is easy to see that $\tilde{V}_n(\phi) = \tilde{G}_n(\phi)$. We discuss the two cases as follows. First, if

$$\tilde{G}_n(\phi) \geq 0, \qquad n = 0, 1, \cdots, N, \tag{3.4}$$

the market is obviously not viable. Second, if (3.4) does not hold, we define

$$n = \sup\{k : \mathbf{P}\{\tilde{G}_k(\phi) < 0\} > 0\}.$$

It follows that $n \leq N - 1$ and $\mathbf{P}\{\tilde{G}_n(\phi) < 0\} > 0$. Moreover, $\tilde{G}_m(\phi) \geq 0$ for all $m > n$. We can now define a new process ψ by

$$\psi_j(\omega) = \begin{cases} 0 & \text{if } j \leq n, \\ 1_A(\omega)\phi_j(\omega) & \text{if } j > n, \end{cases}$$

where $A = \{\tilde{G}_n(\phi) < 0\}$. Because ϕ is predictable and A is $\mathscr{F}_n$-measurable, ψ is also predictable. Moreover

$$\tilde{G}_j(\psi) = \begin{cases} 0 & \text{if } j \leq n, \\ 1_A(\tilde{G}_j(\phi) - \tilde{G}_n(\phi)) & \text{if } j > n. \end{cases}$$

Thus $\tilde{G}_j(\psi) \geq 0$ for all $j \in \{0, \cdots, N\}$ and $\tilde{G}_N(\psi) > 0$ on A, i.e., $\mathbf{P}\{\tilde{G}_N(\psi) > 0\} > 0$. That contradicts the assumption that the market is viable and completes the proof. $\square$

We say two probability measures $\mathbf{P}_1$ and $\mathbf{P}_2$ on $(\Omega, \mathscr{F})$ are *equivalent* provided $\mathbf{P}_1(A) = 0$ if and only if $\mathbf{P}_2(A) = 0$ for every $A \in \mathscr{F}$. Under our assumption on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$, a probability measure $\mathbf{P}^*$ is equivalent to $\mathbf{P}$ if and only if $\mathbf{P}^*(\{\omega\}) > 0$ for every $\omega \in \Omega$. The following theorem gives a precise characterisation of viable financial markets.

Theorem 1.3.1 *The market is viable if and only if there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that all discounted prices of assets are martingales under $\mathbf{P}^*$.*

Proof. (a) Let us assume that there exists a probability $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which discounted prices are martingales. For any self-financing strategy $\{\phi_n\}$, the value process $\{\tilde{V}_n(\phi)\}$ is a $\mathbf{P}^*$-martingale. Therefore, $\mathbf{E}^*[\tilde{V}_N(\phi)] = \mathbf{E}^*[\tilde{V}_0(\phi)]$. If the strategy is admissible and its initial value is zero, then $\mathbf{E}^*[\tilde{V}_N(\phi)] = 0$ with $\tilde{V}_N(\phi) \geq 0$. Since $\mathbf{P}^*(\{\omega\}) > 0$ for all $\omega \in \Omega$, we have $\tilde{V}_N(\phi) = 0$. That is, the market is viable.

(b) Conversely, if the market is viable, according to the Lemma 1.3.1 we have $\tilde{G}_N(\phi) \notin \Gamma$ for any predictable process $\{(\phi_n^1, \cdots, \phi_n^d)\}$. We denote by $\mathscr{V}$ the family of random variables $\tilde{G}_N(\phi)$ with $\{(\phi_n^1, \cdots, \phi_n^d)\}$ predictable. Clearly, $\mathscr{V}$ is a vector subspace of $\mathbb{R}^\Omega$, which is the set of real random variables defined on Ω, and $\mathscr{V}$ does not intersect Γ. Therefore it does not intersect the convex compact set $K = \{X \in \Gamma : \sum_\omega X(\omega) = 1\}$ which is included in Γ. As a result of the convex sets separation theorem, there exists $\{\lambda(\omega) : \omega \in \Omega\}$ such that

$$\sum_\omega \lambda(\omega) X(\omega) > 0 \tag{3.5}$$

for all $X \in K$, and

$$\sum_\omega \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0 \tag{3.6}$$

for any predictable $\{(\phi_n^1, \cdots, \phi_n^d)\}$. From (3.5) we deduce that $\lambda(\omega) > 0$ for all $\omega \in \Omega$. We define the probability $\mathbf{P}^*$ by

$$\mathbf{P}^*(\{\omega\}) = \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')}.$$

Then $\mathbf{P}^*(\{\omega\}) > 0$ for all $\omega \in \Omega$, so $\mathbf{P}^*$ is equivalent to $\mathbf{P}$. Moreover, for any predictable process $\{(\phi_n^1, \cdots, \phi_n^d)\}$ we have

$$\begin{aligned}
\mathbf{E}^*\left(\tilde{G}_N(\phi)\right) &= \sum_{\omega \in \Omega} \tilde{G}_N(\phi)(\omega) \mathbf{P}^*(\{\omega\}) \\
&= \frac{1}{\sum_{\omega' \in \Omega} \lambda(\omega')} \sum_{\omega \in \Omega} \tilde{G}_N(\phi)(\omega) \lambda(\omega) \\
&= 0.
\end{aligned}$$

For any $i \in \{1, \cdots, d\}$ and any real-valued predictable process $\{\phi_n^i\}$, by considering the d-dimensional process $\psi := \{(0, \cdots, 0, \phi_n^i, 0, \cdots, 0)\}$ we get

$$\mathbf{E}^*\left(\tilde{G}_N(\psi)\right) = \mathbf{E}^*\left(\sum_{j=1}^N \phi_j^i \Delta \tilde{S}_j\right) = 0.$$

By Proposition 1.3.2, the discounted prices $\{\tilde{S}_n^1\}, \cdots, \{\tilde{S}_n^d\}$ are $\mathbf{P}^*$-martingales. □

Remark 1.3.2 By Theorem 1.3.1, if the market is viable and ϕ is any self-financing strategy, there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which the discounted prices are martingales. It follows that $\{\tilde{V}_n(\phi)\}$ is also a $\mathbf{P}^*$-martingale and hence

$$\tilde{V}_n(\phi) = \mathbf{E}^*[\tilde{V}_N(\phi) | \mathscr{F}_n], \qquad 0 \le n \le N.$$

1.4 Complete markets and option pricing

1.4.1 Complete markets

We define a *European option*, or more generally a *contingent claim*, of maturity N by giving its payoff h, which is an $\mathscr{F}_N$-measurable and non-negative random variable. For instance, a *call* on the underlying asset $\{S_n^1 : 0 \leq n \leq N\}$ with strike price K is defined by setting $h = (S_N^1 - K)_+$. A *put* on the same underlying asset with the same strike price K is defined by $h = (K - S_N^1)_+$. Those two examples are the most important ones in practice. Note that here h is only a function of S_N. There are some options dependent on the whole path of the underlying asset, i.e. h is a function of the whole cellection $\{S_0, S_1, \cdots, S_N\}$. That is the case of the so-called *Asian options* where the strike price is equal to the average of the stock prices observed during a certain period of time before maturity.

Definition 1.4.1 *The contingent claim defined by the payoff h is attainable if there exists an admissible strategy worth h at time N.*

Remark 1.4.1 In a viable financial market, we just need to find a self-financing strategy worth h at maturity to say that h is attainable. Actually, if ϕ is such a self-financing strategy, i.e., $V_N(\phi) = h \geq 0$. Then $\tilde{V}_N(\phi) \geq 0$ and, by Remark 1.3.2, we know

$$\tilde{V}_n(\phi) = \mathbf{E}^*[\tilde{V}_N(\phi)|\mathscr{F}_n] \geq 0, \qquad 0 \leq n \leq N.$$

Consequently, the strategy ϕ is admissible.

Definition 1.4.2 *The market is complete if every contingent claim is attainable.*

To assume that a financial market is complete is a rather restrictive assumption. The interest of complete markets is that it allows us to derive a simple theory of contingent claim pricing and hedging. The Cox-Ross-Rubinstein model, that we shall introduce in the next section, is a very simple example of complete market modelling. The following theorem gives a precise characterization of complete and viable financial markets.

Theorem 1.4.1 *Suppose that $\mathscr{F}_0 = \{\emptyset, \Omega\}$. Then a viable market is complete if and only if there exists a unique probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which discounted prices are martingales.*

Proof. (a) Let us assume that the market is viable and complete. By Theorem 1.3.1, there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which discounted prices are martingales. Moreover, any contingent claim h, which by definition is a non-negative $\mathscr{F}_N$-measurable random variable, is attainable. That is, there exists an admissible strategy ϕ such that $h = V_N(\phi)$. Since ϕ is self-financing, we know that

$$h/S_N^0 = \tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j;$$

see Proposition 1.2.1. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are two probability measures equivalent to $\mathbf{P}$ and the discounted prices $\{\tilde{V}_n(\phi) : 0 \le n \le N\}$ are martingales under both $\mathbf{P}_1$ and $\mathbf{P}_2$. Then for $i = 1$ or $i = 2$,

$$\mathbf{E}_i[\tilde{V}_N(\phi)] = \mathbf{E}_i[\tilde{V}_0(\phi)] = \mathbf{E}_i[V_0(\phi)] = V_0(\phi),$$

where the last equality coming from the fact that $\mathscr{F}_0 = \{\emptyset, \Omega\}$. It follows that

$$\mathbf{E}_1\big[h/S_N^0\big] = \mathbf{E}_2\big[h/S_N^0\big].$$

Since h is arbitrary, we have $\mathbf{P}_1 = \mathbf{P}_2$ on the whole σ-algebra $\mathscr{F} = \mathscr{F}_N$.

(b) Suppose the market is viable and incomplete. Then there exists some a random variable $h \ge 0$ which is not attainable. Let us call $\mathscr{V}$ the set of random variables of the form

$$U_0 + \sum_{n=1}^{N} \phi_n \cdot \Delta\tilde{S}_n,$$

where U_0 is $\mathscr{F}_0$-measurable and $\{(\phi_n^1, \cdots, \phi_n^d) : 0 \le n \le N\}$ is an $\mathbb{R}^d$-valued predictable process. It follows from Proposition 1.2.2 that h/S_N^0 does not belong to $\mathscr{V}$. Otherwise, there exists a self-financing strategy ϕ such that $h/S_N^0 = \tilde{V}_N(\phi)$ or $h = V_N(\phi)$, giving a contradiction. Hence, $\mathscr{V}$ is a strict subset of the set of all random variables on $(\Omega, \mathscr{F})$. Therefore, if $\mathbf{P}^*$ is a probability measure equivalent to $\mathbf{P}$ under which discounted prices are martingales and if we define the scalar product $(X, Y) = \mathbf{E}^*[XY]$ for random variables, there must exist a non-zero random variable X orthogonal to $\mathscr{V}$. We define

$$\mathbf{P}^{**}(\{\omega\}) = \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big)\mathbf{P}^*(\{\omega\}),$$

with $\|X\|_\infty = \sup_{\omega\in\Omega} |X(\omega)| > 0$. Clearly, $\mathbf{P}^{**} \ne \mathbf{P}^*$ and $\mathbf{P}^{**}(\{\omega\}) > 0$ for all $\omega \in \Omega$ so that $\mathbf{P}^{**}$ is equivalent to $\mathbf{P}$. For any predictable process $\{(\phi_n^1, \cdots, \phi_n^d) : 0 \le n \le N\}$ let

$$U = \sum_{n=1}^{N} \phi_n \cdot \Delta\tilde{S}_n.$$

Then $U \in \mathscr{V}$ and

$$\begin{aligned}
\mathbf{E}^{**}[U] &= \sum_{\omega\in\Omega} U(\omega)\mathbf{P}^{**}(\{\omega\}) \\
&= \sum_{\omega\in\Omega} U(\omega)\mathbf{P}^{*}(\{\omega\}) + \frac{1}{2\|X\|_\infty}\sum_{\omega\in\Omega} U(\omega)X(\omega)\mathbf{P}^{*}(\{\omega\}) \\
&= \mathbf{E}^*[U] + \frac{1}{2\|X\|_\infty}(X,U) = 0,
\end{aligned}$$

where the last equality comes from the fact that $\{\tilde{S}_n : 0 \le n \le N\}$ is a $\mathbf{P}^*$-martingale and X is orthogonal to U. It follows from Proposition 1.3.2 that $\{\tilde{S}_n : 0 \le n \le N\}$ is also a $\mathbf{P}^{**}$-martingale. $\square$

1.4.2 Pricing and hedging contingent claims

We assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and the market is viable and complete. By Theorem 1.4.1, there exists a unique probability measure $\mathbf{P}^*$ under which the discounted prices of financial assets are martingales. Let h be a non-negative $\mathscr{F}_N$-measurable random variable and ϕ an admissible strategy replicating the contingent claim, i.e., $V_N(\phi) = h$. Then the process $\{\tilde{V}_n : 0 \le n \le N\}$ is a $\mathbf{P}^*$-martingale. It follows that

$$\tilde{V}_n(\phi) = \mathbf{E}^*\left[\tilde{V}_N(\phi)|\mathscr{F}_n\right],$$

and hence

$$\frac{V_n(\phi)}{S_n^0} = \mathbf{E}^*\left[\frac{V_N(\phi)}{S_N^0}\middle|\mathscr{F}_n\right].$$

Therefore,

$$V_n(\phi) = S_n^0\mathbf{E}^*\left[\frac{h}{S_N^0}\middle|\mathscr{F}_n\right], \qquad n = 0, 1, \cdots, N. \tag{4.1}$$

This equation shows that, at any time, the value of an admissible strategy replicating h is completely determined by this contingent claim. Then it is natural to call $V_n(\phi)$ the *price* at time n of the option: that is the wealth needed at time n to replicate h at time N by following the strategy ϕ. If an investor sells the option at time zero for $\mathbf{E}^*[h/S_n^0]$, he can follow a replicating strategy ϕ to generate an amount h at time N. In other words, the investor is *perfectly hedged*.

It is important to notice that the computation of the option price only requires the knowledge of $\mathbf{P}^*$ and not $\mathbf{P}$. We could have just considered a *measurable* space $(\Omega, \mathscr{F})$ equipped with the filtration $(\mathscr{F})_{0\le n\le N}$. In other words, we would only define the set of all states and the evolution of the information over time. As soon as the probability space and the filtration are specified, we do not need to find the *true* probability of the possible events (say, by statistical means) in order to price the option. The analysis of the Cox-Ross-Rubinstein model will show how we can compute the option price and the hedging strategy in practice.

1.4.3 Introduction to American options

An *American option* can be exercised at any time between 0 and N. We define it as a non-negative adapted process $\{Z_n : 0 \leq n \leq N\}$, where Z_n is the immediate profit made by exercising the option at time n. In the case of an American call option on the stock $\{S_n^1 : 0 \leq n \leq N\}$ with strike price K we have $Z_n = (S_n^1 - K)_+$. In the case of a put, $Z_n = (K - S_n^1)_+$.

In order to define the price $\{U_n\}$ of the option associated with $\{Z_n\}$, we shall think in terms of a backward induction starting at time N. Indeed, we should obviously have $U_N = Z_N$. At what price we should sell the option at time $N-1$? If the holder exercises straight away he will earn Z_{N-1}, or he might exercises at time N in which case the writer must pay the amount Z_N. (The holder is the buyer, and the writer is the seller.) Therefore, at time $N-1$, the writer has to earn the maximum between Z_{N-1} and the amount necessary at time $N-1$ to generate Z_N at time N. In other words, the writer wants the maximum between Z_{N-1} and the value at time $N-1$ of an admissible strategy paying off Z_N at time N, i.e., $S_{N-1}^0 \mathbf{E}^*[\tilde{Z}_N | \mathscr{F}_{N-1}]$ with $\tilde{Z}_N = Z_N / S_N^0$. Therefore, it makes sense to price the option at time $N-1$ as

$$U_{N-1} = \max\big(Z_{N-1}, S_{N-1}^0 \mathbf{E}^*[\tilde{Z}_N | \mathscr{F}_{N-1}]\big).$$

By induction, we define the price of the American option as

$$U_{n-1} = \max\left(Z_{n-1}, S_{n-1}^0 \mathbf{E}^*\left[\frac{U_n}{S_n^0}\Big| \mathscr{F}_{n-1}\right]\right)$$

for $1 \leq n \leq N$. As before, let $\tilde{U}_n = U_n / S_n^0$ be the discounted price of the option. Then

$$\tilde{U}_{n-1} = \max\big(\tilde{Z}_{n-1}, \mathbf{E}^*[\tilde{U}_n | \mathscr{F}_{n-1}]\big). \tag{4.2}$$

In particular, if we assume the interest rate over one period is constant and equal to r, then $S_n^0 = (1+r)^n$ and hence

$$U_{n-1} = \max\Big(Z_{n-1}, \frac{1}{1+r} \mathbf{E}^*[U_n | \mathscr{F}_{n-1}]\Big). \tag{4.3}$$

We have noticed that the discounted price of the European option is a $\mathbf{P}^*$-martingale. However, the discounted price of the American option is generally a $\mathbf{P}^*$-supermartingale.

Proposition 1.4.1 *The sequence* $\{\tilde{U}_n\}$ *is the smallest* $\mathbf{P}^*$*-supermartingale that dominates the process* $\{\tilde{Z}_n\}$*.*

Proof. From the equality (4.2) we know that $\{\tilde{U}_n\}$ is a $\mathbf{P}^*$-supermartingale dominating $\{\tilde{Z}_n\}$. Let us consider another $\mathbf{P}^*$-supermartingale $\{\tilde{T}_n\}$ that also dominates $\{\tilde{Z}_n\}$. Then $\tilde{T}_N \geq \tilde{Z}_N = \tilde{U}_N$. Moreover, if $\tilde{T}_n \geq \tilde{U}_n$ we have

$$\tilde{T}_{n-1} \geq \mathbf{E}^*[\tilde{T}_n|\mathscr{F}_{n-1}] \geq \mathbf{E}^*[\tilde{U}_n|\mathscr{F}_{n-1}],$$

and hence

$$\tilde{T}_{n-1} \geq \max\left(\tilde{Z}_{n-1}, \mathbf{E}^*[\tilde{U}_n|\mathscr{F}_{n-1}]\right) = \tilde{U}_{n-1}.$$

A backward induction proves the assertion that $\{\tilde{T}_n\}$ dominates $\{\tilde{U}_n\}$. □

1.5 Cox-Ross-Rubinstein model

The Cox-Ross-Rubinstein model is a discrete-time version of the notable Black-Scholes model. It contains only one risky asset $\{S_n\}$ in addition to the riskless asset $\{S_n^0\}$. It is assumed that $S_n^0 = (1+r)^n$, where $r > -1$ is the return over one period of time. The risky asset is modelled as follows: between two consecutive periods the relative price change is ether a or b with $-1 < a < b$. It follows that

$$S_{n+1} = S_n(1+a) \quad \text{or} \quad S_n(1+b). \tag{5.1}$$

The initial stock price S_0 is given. Then the set of all possible events is $\Omega = \{1+a, 1+b\}^N$. Each N-tuple represents the successive values of the ratio $T_n := S_n/S_{n-1}$. We also assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}$ is the family of all subsets of Ω. For $1 \leq n \leq N$ let

$$\mathscr{F}_n = \sigma(\{T_1, \cdots, T_n\}) = \sigma(\{S_1, \cdots, S_n\}). \tag{5.2}$$

The assumption that each singleton in Ω has strictly positive probability implies that

$$\mathbf{P}(\{(x_1, \cdots, x_N)\}) = \mathbf{P}\{T_1 = x_1, \cdots, T_N = x_N\} > 0$$

for each $(x_1, \cdots, x_N) \in \Omega$. As a result, knowing $\mathbf{P}$ is equivalent to knowing the law of the N-tuple $(T_1, \cdots, T_N)$.

1.5.1 Viability and completeness

Proposition 1.5.1 *The discounted price* $\{\tilde{S}_n\}$ *is a martingale under* $\mathbf{P}$ *if and only if*

$$\mathbf{E}[T_{n+1}|\mathscr{F}_n] = 1 + r, \qquad 0 \leq n \leq N-1. \tag{5.3}$$

Proof. Since $\tilde{S}_n$ is $\mathscr{F}_n$-measurable, the equality $\mathbf{E}[\tilde{S}_{n+1}|\mathscr{F}_n] = \tilde{S}_n$ is equivalent to

$$\mathbf{E}[\tilde{S}_{n+1}/\tilde{S}_n|\mathscr{F}_n] = \mathbf{E}\left[\frac{T_{n+1}}{(1+r)}\middle|\mathscr{F}_n\right] = 1.$$

That is, $\mathbf{E}[T_{n+1}|\mathscr{F}_n] = 1 + r$. □

Proposition 1.5.2 *If the market is viable, then* $r \in (a, b)$.

Proof. Under the assumption, there exists a probability $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which $\{\tilde{S}_n\}$ is a martingale. According to Proposition 1.5.1, we have $\mathbf{E}^*[T_{n+1}|\mathscr{F}_n] = 1 + r$ and hance $\mathbf{E}^*[T_{n+1}] = 1 + r$. Since T_{n+1} is either equal to $1 + a$ or $1 + b$ with non-zero probability, we must have $r \in (a, b)$. □

If the condition of Proposition 1.5.2 does not hold, an arbitrage strategies can be constructed as follows. When $r \leq a$, we borrow an amount S_0 at time 0 and purchase one share of the risky asset. At time N, we pay the loan back and sell the risky asset. We realize a profit equal to $S_N - S_0(1+r)^N$, which is always non-negative as

$$S_N = S_0 \prod_{j=1}^{N} T_j \geq (1+a)^N \geq (1+r)^N.$$

Since T_{n+1} is either equal to $1 + a$ or $1 + b$ with non-zero probability and $b > a$, we have $S_N - S_0(1+r)^N > 0$ with non-zero probability. When $r \leq b$, we can short-sell one share of risky asset at price S_0 at time 0 and invest the amount S_0 to the riskless asset. At time N, we withdraw the amount $S_0(1+r)^N$, pay the buyer the amount S_N and realize a profit equal to $S_N - S_0(1+r)^N$. It can be seen that we have $S_0(1+r)^N - S_N > 0$ with non-zero probability.

From now on, we assume that $r \in (a, b)$ and write $p = (b - r)/(b - a)$. Then $r = ap + b(1 - p)$.

Proposition 1.5.3 *The process* $\{\tilde{S}_n\}$ *is a* $\mathbf{P}$*-martingale if and only if* $\{T_1, T_2, \cdots, T_N\}$ *are independent and identically distributed variables and*

$$\mathbf{P}\{T_1 = 1 + a\} = 1 - \mathbf{P}\{T_1 = 1 + b\} = p. \tag{5.4}$$

In this case, the market is viable and complete.

Proof. Under condition (5.4) we have

$$\mathbf{E}[T_{n+1}|\mathscr{F}_n] = \mathbf{E}[T_{n+1}] = p(1+a) + (1-p)(1+b) = 1 + r,$$

and hence $(\tilde{S}_n)$ is a $\mathbf{P}$-martingale by Proposition 1.5.1. Conversely, if $\mathbf{E}[T_{n+1}|\mathscr{F}_n] = 1 + r$ for $0 \le n \le N-1$, we can write

$$(1+a)\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+a\}}|\mathscr{F}_n] + (1+b)\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+b\}}|\mathscr{F}_n] = 1 + r.$$

On the other hand, since

$$\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+a\}}|\mathscr{F}_n] + \mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+b\}}|\mathscr{F}_n] = 1,$$

we conclude

$$a\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+a\}}|\mathscr{F}_n] + b\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+b\}}|\mathscr{F}_n] = r = ap + b(1-p).$$

Then it is easy to conclude that

$$\mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+a\}}|\mathscr{F}_n] = p \quad \text{and} \quad \mathbf{E}[\mathbf{1}_{\{T_{n+1}=1+b\}}|\mathscr{F}_n] = 1-p,$$

or equivalently,

$$\mathbf{P}\{T_{n+1} = 1 + a|\mathscr{F}_n\} = p \quad \text{and} \quad \mathbf{P}\{T_{n+1} = 1 + b|\mathscr{F}_n\} = 1 - p.$$

By induction, we prove that for any $x_i \in \{1+a, 1+b\}$,

$$\mathbf{P}\{T_1 = x_1, \cdots, T_n = x_n\} = \prod_{i=1}^{n} p_i,$$

where $p_i = p$ if $x_i = 1+a$ and $p_i = 1-p$ if $x_i = 1+b$. That shows that the variables T_i are i.i.d. under $\mathbf{P}$ and (5.4) holds.

What we have shown implies that there is a unique probability measure $\mathbf{P}$ on $(\Omega, \mathscr{F})$ such that $\{\tilde{S}_n\}$ is a $\mathbf{P}$-martingale. Thus the market is viable and complete according to Theorem 1.4.1. □

1.5.2 Pricing call and put options

Let C_n and P_n denote respectively the value at time n of a European call and put on a share of stock with strike price K and maturity N.

Proposition 1.5.4 *The put/call parity equation*

$$C_n - P_n = S_n - K(1+r)^{n-N}. \tag{5.5}$$

Proof. Since $\{\tilde{S}_n\}$ is a $\mathbf{P}^*$-martingale,

$$\mathbf{E}^*[S_N|\mathscr{F}_n] = \mathbf{E}^*\big[\tilde{S}_N(1+r)^N|\mathscr{F}_n\big] = \tilde{S}_n(1+r)^N = (1+r)^{N-n}S_n.$$

According to (4.1) we have

$$\begin{aligned} C_n - P_n &= (1+r)^{n-N}\mathbf{E}^*\big[(S_N-K)_+ - (K-S_N)_+|\mathscr{F}_n\big] \\ &= (1+r)^{n-N}\mathbf{E}^*[S_N - K|\mathscr{F}_n] \\ &= S_n - K(1+r)^{n-N}. \end{aligned}$$

□

Theorem 1.5.1 *We have $C_n = c(n, S_n)$, where the function $c(n,x)$ is equal to*

$$(1+r)^{N-n}\sum_{j=0}^{N-n}\frac{(N-n)!}{j!(N-n-j)!}p^j(1-p)^{N-n-j}\big[x(1+a)^j(1+b)^{N-n-j}-K\big]_+. \tag{5.6}$$

Proof. From $T_n = S_n/S_{n-1}$, we have

$$S_N = S_n\prod_{j=n+1}^{N}T_j.$$

By (4.1),

$$C_n = (1+r)^{n-N}\mathbf{E}^*\bigg[\Big(S_n\prod_{j=n+1}^{N}T_j - K\Big)_+\bigg|\mathscr{F}_n\bigg].$$

Note that $\prod_{j=n+1}^{N}T_j$ is independent of $\mathscr{F}_n$ under $\mathbf{P}^*$ and S_n is $\mathscr{F}_n$-measurable. It follows that $C_n = c(n, S_n)$ with

$$c(n,x) = (1+r)^{N-n}\mathbf{E}^*\bigg[\Big(x\prod_{i=n+1}^{N}T_i - K\Big)_+\bigg].$$

Then we obtain (5.6) by evaluating it according to the distribution of $(T_1,\cdots,T_N)$.
□

Theorem 1.5.2 *In the replicating strategy of a call the number of risky asset is $H_n = \Delta(n, S_{n-1})$ at time n, where Δ is defined by*

$$\Delta(n,x) = \frac{c\big(n,x(1+b)\big) - c\big(n,x(1+a)\big)}{x(b-a)}. \tag{5.7}$$

Proof. Let H_n^0 denote the number of riskless assets in the replicating portfolio. We have

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n).$$

Since H_n^0 and H_n are both $\mathscr{F}_{n-1}$-measurable, they are functions of $(S_1, \cdots, S_{n-1})$. But, since $c(n,x)$ is increasing in x and since S_n only takes the values $S_{n-1}(1+a)$ and $S_{n-1}(1+b)$, the above equality implies that

$$H_n^0(1+r)^n + H_n S_{n-1}(1+a) = c\big(n, S_{n-1}(1+a)\big),$$

and

$$H_n^0(1+r)^n + H_n S_{n-1}(1+b) = c\big(n, S_{n-1}(1+b)\big).$$

Subtracting the fist equality from the second one, we obtain:

$$H_n S_{n-1}(b-a) = c\big(n, S_{n-1}(1+b)\big) - c\big(n, S_{n-1}(1+a)\big).$$

This gives the desired result. □

1.5.3 Asymptotics of the prices

Lemma 1.5.1 *Let $Y_N = X_1^N + X_2^N + \cdots + X_N^N$, where $\{X_j^N : j \geq 1\}$ are i.i.d. random variables taking values from $\{-\sigma/\sqrt{N}, \sigma/\sqrt{N}\}$ with mean μ_N satisfying $\lim_{N\to\infty} N\mu_N = \mu$. Then the distribution of Y_N converges to the Gaussian distribution $N(\mu, \sigma^2)$.*

Proof. Since $\mathbf{E}[X_j^N] = \mu_N$ and $\mathbf{E}[(X_j^N)^2] = \sigma^2/N$, we get

$$\begin{aligned}
\mathbf{E}\big[\exp(iuY_N)\big] &= \Big(\mathbf{E}\big[\exp(iuX_1^N)\big]\Big)^N \\
&= \Big(\mathbf{E}\big[1 + iuX_1^N - (uX_1^N)^2 + o((X_1^N)^2)\big]\Big)^N \\
&= \big(1 + iu\mu_N - \sigma^2 u^2/2N + o(1/N)\big)^N.
\end{aligned}$$

It follows that

$$\lim_{N\to\infty} \mathbf{E}\big[\exp(iuY_N)\big] = \exp\{it\mu - \sigma^2 u^2/2\},$$

which proves the desired convergence in law. □

Now let us use the model to price a call or a put with maturity T on a single stock. Let $\sigma > 0$, $T > 0$ and $R \in \mathbb{R}$ be fixed constants. For the integer $N \geq 1$ we write $r = RT/N$. Then we define a and b respectively by

$$\log\Big(\frac{1+a}{1+r}\Big) = -\frac{\sigma}{\sqrt{N}} \quad \text{and} \quad \log\Big(\frac{1+b}{1+r}\Big) = \frac{\sigma}{\sqrt{N}}.$$

It follows that

$$p = \frac{b-r}{b-a} = \frac{e^{\sigma/\sqrt{N}} - 1}{e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}}.$$

The real number R is interpreted as the *instantaneous rate* at all times between 0 and T because

$$\lim_{N\to\infty} (1+r)^N = \lim_{N\to\infty} \Big(1 + \frac{RT}{N}\Big)^{\frac{N}{RT}\cdot RT} = e^{RT}.$$

On the other hand, σ^2 will be seen as the limit variance of the variable $\log(S_N/S_0)$ when N is large.

Theorem 1.5.3 *We have the following results on the asymptotic prices of the put and the call at time zero:*

$$\lim_{N\to\infty} P_0^{(N)} = Ke^{-RT}F(m) - S_0F(m-\sigma), \tag{5.8}$$

and

$$\lim_{N\to\infty} C_0^{(N)} = S_0F(\sigma - m) - Ke^{-RT}F(-m), \tag{5.9}$$

where $m = [\sigma^2/2 - \log(S_0/K) - RT]/\sigma$ *and*

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Proof. By (4.1),

$$\begin{aligned} P_0^{(N)} &= (1+r)^{-N}\mathbf{E}^*\Big[\Big(K - S_0 \prod_{n=1}^{N} T_n\Big)_+ |\mathscr{F}_0\Big] \\ &= (1 + RT/N)^{-N}\mathbf{E}^*\Big[\Big(K - S_0 \prod_{n=1}^{N} T_n\Big)_+\Big] \\ &= \mathbf{E}^*\Big[\big((1 + RT/N)^{-N}K - S_0e^{Y_N}\big)_+\Big] \end{aligned}$$

with $Y_N = \sum_{n=1}^N \log\big(T_n/(1+r)\big)$. By the assumption, the variables $X_j^N := \log(T_j/(1+r))$ are valued in $\{-\sigma/\sqrt{n}, \sigma/\sqrt{n}\}$ and are i.i.d. under $\mathbf{P}^*$. Moreover,

$$\mathbf{E}^*[X_j^N] = (1-2p)\frac{\sigma}{\sqrt{N}} = \frac{2 - e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}}{e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}}\frac{\sigma}{\sqrt{N}}.$$

Therefore, the sequence $\{Y_N\}$ satisfies the condition of Lemma 1.5.1 with

$$\mu = \lim_{N\to\infty} N\mathbf{E}^*[X_j^N] = -\sigma^2/2.$$

Then Y_N converges in distribution to the Gaussian variable $Y := \sigma\xi - \sigma^2/2$ where ξ has distribution $N(0,1)$. On the other hand, we may write $\psi(y) = (Ke^{-RT} - S_0 e^y)_+$ to see that

$$\begin{aligned}
&|P_0^{(N)} - \mathbf{E}^*[\psi(Y_N)]| \\
&= \left|\mathbf{E}^*\left[\left((1+RT/N)^{-N}K - S_0 e^{Y_N}\right)_+ - \left(Ke^{-RT} - S_0 e^{Y_N}\right)_+\right]\right| \\
&\leq \mathbf{E}^*\left|\left((1+RT/N)^{-N}K - S_0 e^{Y_N}\right)_+ - \left(Ke^{-RT} - S_0 e^{Y_N}\right)_+\right| \\
&\leq K|(1+RT/N)^{-N} - e^{-RT}|,
\end{aligned}$$

which goes to zero as $N \to \infty$. Since ψ is a bounded and continuous function, we conclude by dominated convergence that

$$\begin{aligned}
\lim_{N\to\infty} P_0^{(N)} &= \lim_{N\to\infty} \mathbf{E}^*[\psi(Y_N)] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (Ke^{-RT} - S_0 e^{-\sigma^2/2+\sigma y})_+ e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} (Ke^{-RT} - S_0 e^{-\sigma^2/2+\sigma y}) e^{-y^2/2} dy \\
&= Ke^{-RT}F(m) - S_0 \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi}} e^{-(y-\sigma)^2/2} dy \\
&= Ke^{-RT}F(m) - S_0 F(m-\sigma).
\end{aligned}$$

The price of the call follows easily from put/call parity

$$\begin{aligned}
\lim_{N\to\infty} C_0^{(N)} &= \lim_{N\to\infty} P_0^{(N)} + S_0 - \lim_{N\to\infty} K(1+r)^{-N} \\
&= Ke^{-RT}F(m) - S_0 F(m-\sigma) + S_0 - Ke^{-RT} \\
&= S_0 F(\sigma - m) - Ke^{-RT}F(-m).
\end{aligned}$$

□

Remark 1.5.1 Note that the only non-directly observable parameter is σ. Its interpretation as a limit variance suggests that it should be estimated by statistical methods. We shall return to this problem later.

Chapter 2

Optimal times and American options

2.1 Introduction

The purpose of this chapter is to address the pricing and hedging of American options. We will establish a link between these questions and the optimal stopping problem. To do so we will need to define the notion of optimal stopping time, which enables us to model exercise strategies for American options. We will also define the Snell envelope, which is the fundamental concept to solve the optimal stopping problem. The application of these concepts to American options will be described.

2.2 Stopping times

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a finite probability space. We assume that $\mathscr{F}$ be the class of all subsets of Ω and $\mathbf{P}(\{w\}) > 0$ for all $\omega \in \Omega$. This hypothesis is nonetheless not essential. If it does not hold, the results remain true almost surely. Let $(\mathscr{F}_n)_{0 \leq n \leq N}$ be a filtration defined on this probability space.

Definition 2.2.1 *A random variable τ taking values in $\{0, 1, \cdots, N\}$ is called a stopping time if $\{\tau = n\} \in \mathscr{F}_n$ for every $0 \leq n \leq N$.*

The reader can verify that τ is a stopping time if and only if $\{\tau \leq n\} \in \mathscr{F}_n$ for every $0 \leq n \leq N$. We can use this equivalent definition to generalize the concept of stopping time to the continuous-time setting.

For a stopping time τ, we denote by $\mathscr{F}_\tau$ the family of events A such that $A \cap \{\tau = n\} \in \mathscr{F}_n$ for every $0 \le n \le N$. It is easy to show that $\mathscr{F}_\tau \subseteq \mathscr{F}_N$ is a σ-algebra, which is often called the *σ-algebra determined prior to τ*. Clearly, the random variable τ is $\mathscr{F}_\tau$-measurable.

Given an adapted sequence $\{X_n\}$ and a stopping time τ, let $X_n^\tau(\omega) = X_{\tau(\omega)\wedge n}(\omega)$. We call $\{X_n^\tau\}$ the *sequence stopped at τ*.

Proposition 2.2.1 *Let $\{X_n\}$ be an adapted sequence and τ a stopping time. Then the stopped sequence $\{X_n^\tau\}$ is adapted. Moreover, if $\{X_n\}$ is a martingale (resp. supermartingale or submartingale), then $\{X_n^\tau\}$ is also a martingale (resp. supermartingale or submartingale).*

Proof. Let $\phi_j = 1_{\{j \le \tau\}}$. Since $\{j \le \tau\} = \{\tau < j\}^c = \{\tau \le j-1\}^c \in \mathscr{F}_{j-1}$, the process $\{\phi_n : 0 \le n \le N\}$ is predictable. Moreover, we see that

$$X_n^\tau = X_0 + \sum_{j=1}^{n} \phi_j (X_j - X_{j-1}).$$

It is then clear that $\{X_n^\tau\}$ is adapted to the filtration $(\mathscr{F}_n)$. Furthermore, if $\{X_n\}$ is a martingale, then $\{X_n^\tau\}$ is also a martingale with respect to $(\mathscr{F}_n)$ since it is the martingale transform of $\{X_n\}$. Similarly, we can show that if the sequence $\{X_n\}$ is a supermartingale (resp. submartingale), the stopped sequence $\{X_n^\tau\}$ is still a supermartingale (resp. a submartingale) using the predictability and the non-negativity of $\{\phi_j\}$. $\square$

For stopping times τ and σ satisfying $\tau \le \sigma$, it is easy to show that $\mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$. Moreover, we have the following

Proposition 2.2.2 *Suppose $\{M_n\}$ is a martingale and τ and σ are stopping times such that $\tau \le \sigma$, then we have*

$$M_\tau = \mathbf{E}[M_\sigma | \mathscr{F}_\tau]. \tag{2.1}$$

Proof. Since τ is a stopping time, for any event $A \in \mathscr{F}_\tau$ and any real random variable X we have

$$\begin{aligned} \mathbf{E}[1_A X] &= \sum_{j=0}^{N} \mathbf{E}[1_{A\cap\{\tau=j\}} X] = \sum_{j=0}^{N} \mathbf{E}\Big[\mathbf{E}(1_{A\cap\{\tau=j\}} X | \mathscr{F}_j)\Big] \\ &= \sum_{j=0}^{N} \mathbf{E}\Big[1_{A\cap\{\tau=j\}} \mathbf{E}(X|\mathscr{F}_j)\Big] = \mathbf{E}\Big[1_A \sum_{j=0}^{N} 1_{\{\tau=j\}} \mathbf{E}(X|\mathscr{F}_j)\Big]. \end{aligned}$$

It follows that

$$\mathbf{E}[X|\mathscr{F}_\tau] = \sum_{j=0}^{N} 1_{\{\tau=j\}}\mathbf{E}[X|\mathscr{F}_j].$$

Applying this to $X = M_N$ we obtain

$$\mathbf{E}[M_N|\mathscr{F}_\tau] = \sum_{j=0}^{N} 1_{\{\tau=j\}}\mathbf{E}[M_N|\mathscr{F}_j] = \sum_{j=0}^{N} 1_{\{\tau=j\}}M_j = M_\tau.$$

Similarly, we have $\mathbf{E}[M_N|\mathscr{F}_\sigma] = M_\sigma$. Therefore,

$$\mathbf{E}[M_\sigma|\mathscr{F}_\tau] = \mathbf{E}[\mathbf{E}(M_N|\mathscr{F}_\sigma)|\mathscr{F}_\tau] = \mathbf{E}[M_N|\mathscr{F}_\tau] = M_\tau,$$

proving the equality (2.1). □

2.3 Optimal stopping times

The theory of optimal stopping times play a very important role in the study of America options. In this section we give the characterizations of some optimal stopping times in terms of Snell envelopes. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a finite probability space. We assume that $\mathscr{F}$ be the class of all subsets of Ω and $\mathbf{P}(\{w\}) > 0$ for all $\omega \in \Omega$. Let $(\mathscr{F}_n)_{0\le n\le N}$ be a filtration defined on this probability space.

2.3.1 Snell envelopes and Optimal times

Let us consider a sequence $\{Z_n : 0 \le n \le N\}$ defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ which is adapted to the filtration $(\mathscr{F}_n)_{0\le n\le N}$. Let $\{U_n\}$ be defined inductively by

$$U_{n-1} = \begin{cases} Z_N & \text{for } n = N, \\ \max\big\{Z_{n-1}, \mathbf{E}[U_n|\mathscr{F}_{n-1}]\big\} & \text{for } 1 \le n \le N. \end{cases} \tag{3.1}$$

The sequence $\{U_n\}$ is called the *Snell envelope* of $\{Z_n\}$.

Proposition 2.3.1 *The Snell envelope $\{U_n\}$ defined by* (3.1) *is the smallest supermartingale that dominates $\{Z_n\}$.*

Proof. From (3.1) we know that $\{U_n\}$ is a supermartingale dominating $\{Z_n\}$. Let us consider another supermartingale $\{T_n\}$ that also dominates $\{Z_n\}$. Then $T_N \ge Z_N = U_N$. Moreover, if $T_n \ge U_n$ we have

$$T_{n-1} \ge \mathbf{E}[T_n|\mathscr{F}_{n-1}] \ge \mathbf{E}[U_n|\mathscr{F}_{n-1}],$$

and hence

$$T_{n-1} \geq \max\left\{Z_{n-1}, \mathbf{E}[U_n|\mathscr{F}_{n-1}]\right\} = U_{n-1}.$$

A backward induction proves that $\{T_n\}$ dominates $\{U_n\}$. □

By definition, we have $U_n \geq Z_n$ with equality for $n = N$. In the case of a strict inequality, we have $U_n = \mathbf{E}[U_{n+1}|\mathscr{F}_n]$. This suggests that by stopping adequately the sequence $\{U_n\}$ it is possible to obtain a martingale as the following proposition shows.

Proposition 2.3.2 *The random variable define by*

$$\tau_0 = \inf\{n \geq 0 : U_n = Z_n\} \tag{3.2}$$

is a stopping time and the stopped sequence $\{U_n^{\tau_0}\}$ is a martingale.

Proof. Since $U_N = Z_N$, we see that τ_0 is a well-defined element of $\{0, 1, \cdots, N\}$. Moreover, we have $\{\tau_0 = 0\} = \{U_0 = Z_0\} \in \mathscr{F}_0$ and

$$\{\tau_0 = k\} = \{U_0 > Z_0\} \cap \cdots \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \in \mathscr{F}_k$$

for $1 \leq k \leq N$. To demonstrate that $\{U_n^{\tau_0}\}$ is a martingale, we set $\phi_j = 1_{\{\tau_0 \geq j\}}$ and write

$$U_n^{\tau_0} = U_0 + \sum_{j=1}^{n} \phi_j \Delta U_j,$$

as in the proof of Proposition 2.2.1. It follows that

$$U_{n+1}^{\tau_0} - U_n^{\tau_0} = \phi_{n+1}(U_{n+1} - U_n) = 1_{\{n+1 \leq \tau_0\}}(U_{n+1} - U_n) \tag{3.3}$$

for $0 \leq n \leq N-1$. By the definition $U_n = \max\{Z_n, \mathbf{E}[U_{n+1}|\mathscr{F}_n]\}$ we find $U_n > Z_n$ on the set $\{n + 1 \leq \tau_0\}$. Consequently $U_n = \mathbf{E}[U_{n+1}|\mathscr{F}_n]$ and we deduce from (3.3) that

$$U_{n+1}^{\tau_0} - U_n^{\tau_0} = 1_{\{n+1 \leq \tau_0\}}(U_{n+1} - \mathbf{E}[U_{n+1}|\mathscr{F}_n]).$$

Since $\{n + 1 \leq \tau_0\} = \{\tau_0 < n + 1\}^c = \{\tau_0 \leq n\}^c \in \mathscr{F}_n$, we can take the conditional expectation on both sides of the equality to see that

$$\mathbf{E}[(U_{n+1}^{\tau_0} - U_n^{\tau_0})|\mathscr{F}_n] = 1_{\{n+1 \leq \tau_0\}}\mathbf{E}\{U_{n+1} - \mathbf{E}[U_{n+1}|\mathscr{F}_n]|\mathscr{F}_n\} = 0.$$

Then $\{U_n^{\tau_0}\}$ is a martingale. □

We denote by $\mathscr{T}_{n,N}$ the set of stopping times taking values in $\{n, n+1, \cdots, N\}$. Notice that $\mathscr{T}_{n,N}$ is a finite set since Ω is assumed to be finite.

Corollary 2.3.1 *The stopping time τ_0 defined by* (3.2) *satisfies*

$$U_0 = \mathbf{E}[Z_{\tau_0}|\mathscr{F}_0] = \sup_{\tau\in\mathscr{T}_{0,N}} \mathbf{E}[Z_\tau|\mathscr{F}_0]. \tag{3.4}$$

Proof. Since U^{τ_0} is a martingale, we have

$$U_0 = U_0^{\tau_0} = \mathbf{E}[U_N^{\tau_0}|\mathscr{F}_0] = \mathbf{E}[U_{\tau_0}|\mathscr{F}_0] = \mathbf{E}[Z_{\tau_0}|\mathscr{F}_0].$$

On the other hand, for $\tau \in \mathscr{T}_{0,N}$ the stopped sequence U^τ is a supermartingale. Then

$$U_0 \geq \mathbf{E}[U_N^\tau|\mathscr{F}_0] = \mathbf{E}[U_\tau|\mathscr{F}_0] \geq \mathbf{E}[Z_\tau|\mathscr{F}_0],$$

which yields the result. □

Definition 2.3.1 *A stopping time τ is said to be optimal for the sequence $\{Z_n\}$ if*

$$\mathbf{E}[Z_\tau|\mathscr{F}_0] = \sup_{\sigma\in\mathscr{T}_{0,N}} \mathbf{E}[Z_\sigma|\mathscr{F}_0]. \tag{3.5}$$

If we think of Z_n as the total winnings of a gamble after n games, we see from (3.5) that an optimal stopping time τ maximize the expected gain given $\mathscr{F}_0$. Corollary 2.3.1 shows that τ_0 is an optimal stopping time. By similar arguments we may generalize the result of Corollary 2.3.1 to the following equality:

$$U_n = \mathbf{E}[Z_{\tau_n}|\mathscr{F}_n] = \sup_{\tau\in\mathscr{T}_{n,N}} \mathbf{E}[Z_\tau|\mathscr{F}_n], \tag{3.6}$$

where $\tau_n = \inf\{j \geq n : U_j = Z_j\}$. This gives an alternate definition of the sequence $\{U_n\}$.

Proposition 2.3.3 *A stopping time τ is optimal if and only if*

$$\mathbf{E}[Z_\tau] = \sup_{\sigma\in\mathscr{T}_{0,N}} \mathbf{E}[Z_\sigma]. \tag{3.7}$$

Proof. If τ is an optimal stopping time, for any $\sigma \in \mathscr{T}_{0,N}$ we have $\mathbf{E}[Z_\tau|\mathscr{F}_0] \geq \mathbf{E}[Z_\sigma|\mathscr{F}_0]$ and hence

$$\mathbf{E}[Z_\tau] = \mathbf{E}[\mathbf{E}[Z_\tau|\mathscr{F}_0]] \geq \mathbf{E}[\mathbf{E}[Z_\sigma|\mathscr{F}_0]] = \mathbf{E}[Z_\sigma],$$

which clearly implies (3.7). Conversely, suppose that (3.7) holds. Corollary 2.3.1 implies that $\mathbf{E}[Z_\tau] = \mathbf{E}[U_0]$. But $\{U_n\}$ is a supermartingale dominating $\{Z_n\}$, so we conclude that

$$\mathbf{E}[Z_\tau] \leq \mathbf{E}[U_\tau] \leq \mathbf{E}[U_0] = \mathbf{E}[Z_\tau], \tag{3.8}$$

and hence $Z_\tau = U_\tau$. It follows that $\mathbf{E}[Z_\tau|\mathscr{F}_0] = \mathbf{E}[U_\tau|\mathscr{F}_0] \leq U_0$. In view of (3.8) we get $\mathbf{E}[Z_\tau|\mathscr{F}_0] = U_0$, which implies the optimality of τ. □

Theorem 2.3.1 *A stopping time τ is optimal if and only if $Z_\tau = U_\tau$ and $\{U_n^\tau\}$ is a martingale.*

Proof. If $Z_\tau = U_\tau$ and $\{U_n^\tau\}$ is a martingale, we have $U_0 = \mathbf{E}[U_N^\tau|\mathscr{F}_0] = \mathbf{E}[U_\tau|\mathscr{F}_0] = \mathbf{E}[Z_\tau|\mathscr{F}_0]$. Then the optimality of τ is ensured by Corollary 2.3.1. Conversely, if τ is optimal, we have

$$U_0 = \mathbf{E}[Z_\tau|\mathscr{F}_0] \le \mathbf{E}[U_\tau|\mathscr{F}_0]$$

since $U_\tau \ge Z_\tau$. But, U_τ is a supermartingale, so it is always true that $\mathbf{E}[U_\tau|\mathscr{F}_0] \le U_0$. It follows that

$$U_0 = \mathbf{E}[Z_\tau|\mathscr{F}_0] = \mathbf{E}[U_\tau|\mathscr{F}_0].$$

Since $U_\tau \ge Z_\tau$, we must have $U_\tau = Z_\tau$. From $\mathbf{E}[U_\tau|\mathscr{F}_0] = U_0$ and the inequalities

$$U_0 \ge \mathbf{E}[U_{\tau\wedge n}|\mathscr{F}_0] \ge \mathbf{E}[U_\tau|\mathscr{F}_0] = \mathbf{E}\{\mathbf{E}[U_\tau|\mathscr{F}_n]|\mathscr{F}_0\}$$

based on the supermartingale property of $\{U_n^\tau\}$ we get

$$U_0 = \mathbf{E}[U_{\tau\wedge n}|\mathscr{F}_0] = \mathbf{E}[U_\tau|\mathscr{F}_0] = \mathbf{E}\{\mathbf{E}[U_\tau|\mathscr{F}_n]|\mathscr{F}_0\}.$$

But we have $U_{\tau\wedge n} \ge \mathbf{E}[U_\tau|\mathscr{F}_n]$, so $U_{\tau\wedge n} = \mathbf{E}[U_\tau|\mathscr{F}_n]$. That proves $\{U_n^\tau\}$ is a martingale. □

By Theorem 2.3.1 the optimal stopping time τ_0 defined by (3.2) is the smallest optimal stopping time. In the next section, we shall give a characterization of the largest optimal stopping time in terms of Doob's decomposition.

2.3.2 The Largest optimal time

Suppose then that $\{U_n\}$ is the Snell envelope of an adapted sequence $\{Z_n\}$. Then $\{U_n\}$ is a supermartingale. By Doob's decomposition, there is a unique martingale $\{M_n\}$ and a unique non-decreasing process $\{A_n\}$ such that

$$U_n = M_n - A_n, \qquad 0 \le n \le N-1. \tag{3.9}$$

We can give a characterization of the largest optimal stopping time for $\{Z_n\}$ using the non-decreasing process $\{A_n\}$:

Proposition 2.3.4 *The largest optimal stopping time for $\{Z_n\}$ is given by*

$$\tau_{\max} = \begin{cases} N & \text{if } A_N = 0, \\ \inf\{n : \ A_{n+1} \ne 0\} & \text{if } A_N \ne 0. \end{cases} \tag{3.10}$$

Proof. Since $\{A_n\}$ is predictable, it is straightforward to see that $\tau_{\max}$ is a stopping time. From $U_n = M_n - A_n$ and because $A_j = 0$ for $j \le \tau_{\max}$, we deduce that $U^{\tau_{\max}} = M^{\tau_{\max}}$ and then conclude that $U^{\tau_{\max}}$ is a martingale. To show the optimality of $\tau_{\max}$ it is sufficient to prove $U_{\tau_{\max}} = Z_{\tau_{\max}}$. Note that

$$\begin{aligned} U_{\tau_{\max}} &= \sum_{j=0}^{N-1} 1_{\{\tau_{\max}=j\}} U_j + 1_{\{\tau_{\max}=N\}} U_N \\ &= \sum_{j=0}^{N-1} 1_{\{\tau_{\max}=j\}} \max\{Z_j, \mathbf{E}[U_{j+1}|\mathscr{F}_j]\} + 1_{\{\tau_{\max}=N\}} Z_N. \end{aligned}$$

We have $\mathbf{E}[U_{j+1}|\mathscr{F}_j] = \mathbf{E}[M_{j+1} - A_{j+1}|\mathscr{F}_j] = M_j - A_{j+1}$. On the set $\{\tau_{\max} = j\}$ we have $A_j = 0$ and $A_{j+1} > 0$, so $U_j = M_j$ and $\mathbf{E}[U_{j+1}|\mathscr{F}_j] = M_j - A_{j+1} < U_j$. It follows that $U_j = \max\{Z_j, \mathbf{E}[U_{j+1}|\mathscr{F}_j]\} = Z_j$. Finally, we get $U_{\tau_{\max}} = Z_{\tau_{\max}}$. It remains to show that it is the largest optimal stopping time. Suppose that τ is an optimal stopping time such that $\tau \ge \tau_{\max}$ and $\mathbf{P}\{\tau > \tau_{\max}\} > 0$. We have $\mathbf{E}[M_\tau] = \mathbf{E}[M_0] = \mathbf{E}[U_0]$ and hence

$$\mathbf{E}[U_\tau] = \mathbf{E}[M_\tau] - \mathbf{E}[A_\tau] = \mathbf{E}[U_0] - \mathbf{E}[A_\tau] < \mathbf{E}[U_0].$$

Then U^τ cannot be a martingale, which is in contradiction to the assumption that τ is an optimal stopping time. That establishes the claim. □

2.3.3 Snell envelopes of Markov chains

The aim of this paragraph is to compute Snell envelopes in a Markovian setting. A sequence $\{X_n : 0 \le n \le N\}$ of random variables taking their values in a finite set E is called a *Markov chain* if we have

$$\mathbf{P}\{X_{n+1} = y|X_0 = x_0, \cdots, X_{n-1} = x_{n-1}, X_n = x\} = \mathbf{P}\{X_{n+1} = y|X_n = x\} \quad (3.11)$$

for any $0 \le n \le N-1$ and any element $x_0, \cdots, x_{n-1}, x, y$ of E. The chain is said to be homogeneous if the value $P(x,y) = \mathbf{P}\{X_{n+1} = y|X_n = x\}$ does not depend on n. The matrix $P = (P(x,y))_{(x,y)\in E\times E}$ indexed by $E \times E$, is then called the *transition matrix* of the chain. Clearly, the matrix P has non-negative entries and satisfies $\sum_{y\in E} P(x,y) = 1$ for all $x \in E$. We can also define the notion of a Markov chain with respect to the filtration $(\mathscr{F})_{0\le n\le N}$.

Definition 2.3.2 *A sequence $\{X_n : 0 \le n \le N\}$ of random variables taking values in E is a homogeneous Markov chain with respect to the filtration $(\mathscr{F}_n)_{0\le n\le N}$ with transition matrix P if $\{X_n\}$ is adapted and if we have*

$$\mathbf{E}[f(X_{n+1})|\mathscr{F}_n] = Pf(X_n), \quad (3.12)$$

for every $0 \leq n \leq N-1$ and every real-valued function f on E, where Pf represents the function such that

$$Pf(x) = \sum_{y \in E} P(x,y)f(y).$$

If one interprets real-valued functions on E as matrices with a single column indexed by E, then Pf is indeed the product of the two matrices P and f. It can also be easily seen that a Markov chain, as defined at the beginning of the section, is a Markov chain with respect to its natural filtration defined by $\mathscr{F}_n = \sigma(X_0, \cdots, X_n)$. The following proposition is an immediate consequence of the latter definition and the definition of a Snell envelope.

Proposition 2.3.5 *Let $\{Z_n : 0 \leq n \leq N\}$ be an adapted sequence defined by $Z_n = \psi(n, X_n)$, where $\{X_n : 0 \leq n \leq N\}$ is a homogeneous Markov chain with transition matrix P and ψ is a function from $\{0, 1, \cdots, N\} \times E$ to $\mathbb{R}$. Then the Snell envelope $\{U_n\}$ of the sequence $\{Z_n\}$ is given by $U_n = u(n, X_n)$, where the function $u(\cdot,\cdot)$ is defined by $u(N,\cdot) = \psi(N,\cdot)$ and*

$$u(n-1,\cdot) = \max\{\psi(n-1,\cdot), Pu(n,\cdot)\}, \qquad 1 \leq n \leq N. \tag{3.13}$$

2.4 Applications to American options

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a finite probability space where $\mathscr{F}$ is the class of all subsets of Ω and $\mathbf{P}(\{w\}) > 0$ for all $\omega \in \Omega$. Suppose that $(\mathscr{F}_n)_{0 \leq n \leq N}$ is a filtration satisfying $\mathscr{F}_0 = \{\phi, \Omega\}$ and $\mathscr{F}_N = \mathscr{F}$. We consider a viable and complete market and denote by $\mathbf{P}^*$ the unique probability under which the discounted asset prices are martingales.

2.4.1 Pricing American options

An *American option* can be exercised at any time between 0 and N. We define it as a non-negative adapted process $\{Z_n : 0 \leq n \leq N\}$, where Z_n is the immediate profit made by exercising the option at time n. In the case of an American call option on the stock $\{S_n^1 : 0 \leq n \leq N\}$ with strike price K we have $Z_n = (S_n^1 - K)_+$. In the case of a put, $Z_n = (K - S_n^1)_+$.

In order to define the price $\{U_n\}$ of the option associated with the sequence of payoff $\{Z_n\}$, we shall think in terms of a backward induction starting at time N. Indeed, we should obviously have $U_N = Z_N$. Let us consider at what price we should sell the option at time $N-1$. If the holder exercises straight away the writer will have to pay Z_{N-1}. The holder might also choose to exercise the option at time N in which case the writer must pay the amount Z_N. (The holder is the buyer, and the writer is the seller.) Therefore, at time $N-1$ the writer has to earn the maximum between Z_{N-1} and the amount necessary at time $N-1$ to generate Z_N at time N. In other words, the writer wants the maximum between Z_{N-1} and the value at time $N-1$ of an admissible strategy paying off Z_N at time N, that is, $S^0_{N-1}\mathbf{E}^*[\tilde{Z}_N|\mathscr{F}_{N-1}]$ with $\tilde{Z}_N = Z_N/S^0_N$. Therefore, it makes sense to price the option at time $N-1$ as

$$U_{N-1} = \max\left\{Z_{N-1}, S^0_{N-1}\mathbf{E}^*[\tilde{Z}_N|\mathscr{F}_{N-1}]\right\}.$$

By induction, we define the *price of the American option* as

$$U_{n-1} = \max\left\{Z_{n-1}, S^0_{n-1}\mathbf{E}^*\left[U_n/S^0_n\middle|\mathscr{F}_{n-1}\right]\right\}, \qquad 1 \le n \le N. \tag{4.1}$$

As before, let $\tilde{U}_n = U_n/S^0_n$ be the discounted price of the option. Then we have

$$\tilde{U}_{n-1} = \max\left\{\tilde{Z}_{n-1}, \mathbf{E}^*[\tilde{U}_n|\mathscr{F}_{n-1}]\right\}, \qquad 1 \le n \le N. \tag{4.2}$$

In other words, the discounted price $\{\tilde{U}_n\}$ is the Snell envelope under $\mathbf{P}^*$ of the discounted payoff $\{\tilde{Z}_n\}$. We have noticed that the discounted price of the European option is a $\mathbf{P}^*$-martingale. From (4.2) we see that the discounted price of the American option is generally a $\mathbf{P}^*$-supermartingale.

2.4.2 Hedging American options

We have defined the value process $\{U_n\}$ of an American option described by the adapted non-negative sequence $\{Z_n\}$. We know the discounted price of the option $\{\tilde{U}_n\}$ is the Snell envelope of $\{\tilde{Z}_n\}$ under $\mathbf{P}^*$. By (3.6) we have

$$\tilde{U}_n = \sup_{\tau\in\mathscr{T}_{n,N}} \mathbf{E}^*[\tilde{Z}_\tau|\mathscr{F}_n] \tag{4.3}$$

and consequently

$$U_n = S^0_n \sup_{\tau\in\mathscr{T}_{n,N}} \mathbf{E}^*[Z_\tau/S^0_\tau|\mathscr{F}_n]. \tag{4.4}$$

By Doob's decomposition, we can write $\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$, where $\{\tilde{M}_n\}$ is a $\mathbf{P}^*$-martingale and $\{\tilde{A}_n\}$ is an increasing predictable process, null at time zero. Since the market is complete, there is a self-financing strategy ϕ such that $V_N(\phi) = S^0_N\tilde{M}_N$ and so $\tilde{V}_N(\phi) = \tilde{M}_N$. Since $\{\tilde{V}_N(\phi)\}$ is a $\mathbf{P}^*$-martingale, we have

$$\tilde{V}_n(\phi) = \mathbf{E}^*[\tilde{V}_N(\phi)|\mathscr{F}_n] = \mathbf{E}^*[\tilde{M}_N|\mathscr{F}_n] = \tilde{M}_n, \tag{4.5}$$

and consequently $\tilde{U}_n = \tilde{V}_n(\phi) - \tilde{A}_n$. Then we have $U_n = V_n(\phi) - A_n$ with $A_n = S^0_n\tilde{A}_n$. From this equality, it is obvious that the writer of the option can hedge himself perfectly. Indeed, once the writer receives the premium $U_0 = V_0(\phi)$, he can generate a wealth equal to $V_n(\phi)$ at time n which is bigger than U_n and a fortiori Z_n.

Now what is the optimal date to exercise the option? The date of exercise is to be chosen among all the stopping times. For the buyer of the option, there is no point in exercising at time n when $U_n > Z_n$, because he would trade an asset worth U_n (the option) for an amount Z_n (by exercising the option). Thus, as the first condition, an optimal date τ of exercise should satisfy $U_\tau = Z_\tau$. On the other hand, there is no point in exercising after the time

$$\tau_{\max} = \inf\{j : A_{j+1} \neq 0\} = \inf\{j : \tilde{A}_{j+1} \neq 0\} \tag{4.6}$$

because selling the option at that time provides the holder with a wealth $U_{\tau_{\max}} = V_{\tau_{\max}}(\phi)$ and, following the strategy ϕ from that time, he creates a portfolio whose value is strictly bigger than the option's at times $\tau_{\max}+1, \tau_{\max}+2, \cdots, N$. Therefore we set $\tau \leq \tau_{\max}$ as the second condition for an optimal date τ of exercise, which implies that $\{\tilde{U}_n^\tau\}$ is a martingale. As a result of those two conditions, optimal dates of exercise are optimal stopping times for the sequence $\{\tilde{Z}_n\}$ under probability $\mathbf{P}^*$.

Let us consider the writer's point of view. If he hedges himself using the strategy ϕ as defined above, he gets the wealth $V_n(\phi)$ at time n. If the buyer exercises at a stopping time τ which is not optimal, then $\mathbf{P}^*\{U_\tau > Z_\tau\} > 0$ or $\mathbf{P}^*\{A_\tau > 0\} > 0$. In both cases, the writer makes a profit $V_\tau(\phi) - Z_\tau = U_\tau + A_\tau - Z_\tau$ which is non-negative and is strictly positive with non-zero probability.

2.4.3 Relations of two options

We know that the discounted price of an European option is a $\mathbf{P}^*$-martingale and that of an American option is generally a $\mathbf{P}^*$-supermartingale. Let us give some more comparison of the two type of options.

Proposition 2.4.1 *Let C_n be the value at time n of the American option described by a non-negative adapted sequence $\{Z_n\}$ and let c_n be the value at time n of the European option defined by the $\mathscr{F}_N$-measurable random variable $h = Z_N$. Then we have $C_n \geq c_n$ for every $0 \leq n \leq N$. Moreover, if $c_n \geq Z_n$ for every $0 \leq n \leq N$, then $c_n = C_n$ for every $0 \leq n \leq N$.*

Proof. Under $\mathbf{P}^*$ the discounted value $\{\tilde{c}_n\}$ is a martingale and $\{\tilde{C}_n\}$ is a supermartingale. Since $C_N = c_N = Z_N$ clearly, we have

$$\tilde{C}_n \geq \mathbf{E}^*[\tilde{C}_N|\mathscr{F}_n] = \mathbf{E}^*[\tilde{c}_N|\mathscr{F}_n] = \tilde{c}_n.$$

If $c_n \geq Z_n$ for every $0 \leq n \leq N$, the sequence $\{\tilde{c}_n\}$ is a martingale that dominates $\{\tilde{Z}_n\}$. Since $\{\tilde{C}_n\}$ is the smallest supermartingale that dominates $\{\tilde{Z}_n\}$, we have

$\tilde{C}_n \leq \tilde{c}_n$ and so $C_n \leq c_n$ for every $0 \leq n \leq N$. Finally, we get $C_n = c_n$ for every $0 \leq n \leq N$. □

The inequality $C_n \geq c_n$ makes sense since the American option entitles the holder to more rights than its European counterpart. One checks readily that if the relationships of Proposition 2.6.1 did not hold, there would be some arbitrage opportunities by trading the options.

Proposition 2.4.2 *Consider a market with a single risky asset with price $\{S_n\}$ and a riskless asset $\{S_n^0\}$ with constant interest rate $r \geq 0$ on each period. With the notation of Proposition 2.4.1, if $Z_n = (S_n - K)_+$ is a call option with strike price K on one unit of the risky asset, then $c_n = C_n$ for every $0 \leq n \leq N$.*

Proof. By using the martingale properties of $\{\tilde{S}_n\}$ and $\{\tilde{c}_n\}$ we have

$$\tilde{c}_n = (1+r)^{-N}\mathbf{E}^*[(S_N - K)_+|\mathscr{F}_n] \geq \mathbf{E}^*[\tilde{S}_N - K(1+r)^{-N}|\mathscr{F}_n] = \tilde{S}_n - K(1+r)^{-N}.$$

It follows that

$$c_n \geq S_n - K(1+r)^{-(N-n)} \geq S_n - K.$$

Since $c_n \geq 0$, we have $c_n \geq (S_n - K)_+ = Z_n$ and then Proposition 2.4.1 implies $C_n = c_n$. □

The above proposition asserts an equality between the price of the European call and the price of the corresponding American call. This property does not hold for the put, nor in the case of calls on currencies or dividend paying stocks. For further discussions on the Snell envelope and optimal stopping, one may consult Neveu (1972, Chapter VI) and Dacunha-Castelle and Duflo (1986, Section 5.1). For the theory of optimal stopping in the continuous case, see El Karoui (1981) and Shiryayev (1978).

2.4.4 American options of Markovian assets

Let us consider a market which consists of a riskless asset and a risky asset. Suppose that the interest rate over one period is constant and equal to r. The price of the riskless asset is then $S_n^0 = (1+r)^n$. Under this assumption, (4.1) becomes

$$U_{n-1} = \max\left\{Z_{n-1}, (1+r)^{-1}\mathbf{E}^*[U_n|\mathscr{F}_{n-1}]\right\}, \qquad 1 \leq n \leq N. \tag{4.7}$$

Suppose that the price of the risky asset $\{S_n : 0 \leq n \leq N\}$ is a homogeneous Markov chain under the probability $\mathbf{P}^*$ with transition matrix P. Let us consider the American option given by $\{Z_n : 0 \leq n \leq N\}$ with $Z_n = \psi(n, X_n)$ for a function

ψ from $\{0, 1, \cdots, N\} \times E$ to $\mathbb{R}_+$. Then the price sequence $\{U_n\}$ is given by $U_n = u(n, X_n)$, where the function $u(\cdot, \cdot)$ is defined by $u(N, \cdot) = \psi(N, \cdot)$ and

$$u(n-1, \cdot) = \max\big\{\psi(n-1, \cdot), (1+r)^{-1}Pu(n, \cdot)\big\}, \qquad 1 \le n \le N. \tag{4.8}$$

From (4.4) we get the pricing formula:

$$U_n = (1+r)^n \sup_{\tau \in \mathscr{T}_{n,N}} \mathbf{E}^*\big[(1+r)^{-\tau}\psi(\tau, S_\tau)\big|\mathscr{F}_n\big], \qquad 0 \le n \le N. \tag{4.9}$$

Let us consider a CRR model. Suppose that $b > a > -1$ are given constants and let $p = (b-r)/(b-a)$. In the CRR model, we have $S_n = S_0 \prod_{j=1}^n T_j$, where $\{T_j\}$ are i.i.d. random variables under the probability $\mathbf{P}^*$ with $\mathbf{P}^*\{T_j = 1+a\} = p$ and $\mathbf{P}^*\{T_j = 1+b\} = 1-p$. In this case, the price at time n of the American call with strike price $K \ge 0$ can be written as $P_n = P(n, S_n)$ where the function $P(\cdot, \cdot)$ is defined by $P(N, x) = (K-x)_+$ and

$$P(n-1, x) = \max\big\{(K-x)_+, (1+r)^{-1}f(n, x)\big\}, \qquad 1 \le n \le N, \tag{4.10}$$

with

$$f(n, x) = pP(n, x(1+a)) + (1-p)P(n, x(1+b)). \tag{4.11}$$

Alternatively, we can also define $P(\cdot, \cdot)$ by

$$P(n, x) = \sup_{\tau \in \mathscr{T}_{n,N}} \mathbf{E}^*\Big[(1+r)^{-\tau}\Big(K - x\prod_{j=1}^{\tau} T_j\Big)_+\Big]. \tag{4.12}$$

Proposition 2.4.3 *The function $x \mapsto P(0, x)$ is convex and non-increasing.*

Proof. From (4.12) it is clearly seen that $x \mapsto P(0, x)$ is non-increasing. Suppose that $x, y \ge 0$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Setting $\eta_\tau = \prod_{j=1}^{\tau} T_j$, we have by the convexity of $x \mapsto (K-x)_+$ that

$$\begin{aligned}
\alpha P(0, x) + \beta P(0, y) &= \alpha \sup_{\tau \in \mathscr{T}_{0,N}} \mathbf{E}^*\Big[(1+r)^{-\tau}\big(K - x\eta_\tau\big)_+\Big] \\
&\quad + \beta \sup_{\tau \in \mathscr{T}_{0,N}} \mathbf{E}^*\Big[(1+r)^{-\tau}\big(K - y\eta_\tau\big)_+\Big] \\
&\ge \sup_{\tau \in \mathscr{T}_{0,N}} \mathbf{E}^*\Big[(1+r)^{-\tau}\alpha\big(K - x\eta_\tau\big)_+ + \beta\big(K - y\eta_\tau\big)_+\Big] \\
&\ge \sup_{\tau \in \mathscr{T}_{0,N}} \mathbf{E}^*\Big[(1+r)^{-\tau}\big(K - \alpha x\eta_\tau - \beta y\eta_\tau\big)_+\Big] \\
&= P(0, \alpha x + \beta y).
\end{aligned}$$

Then $x \mapsto P(0, x)$ is a convex function. $\square$

Proposition 2.4.4 *Suppose that $-1 < a < 0$. Then there is a constant $x^* \in [0, K)$ such that $P(0, x) > (K - x)_+$ if $x^* < x < K/(1 + a)^N$ and $P(0, x) = (K - x)_+$ otherwise.*

Proof. By Proposition 2.3.4, the function $x \mapsto P(0, x)$ is convex so it is continuous. It is to see that $P(0, 0) = K$. Let $x^* = \inf\{x \geq 0 : P(0, x) \neq (K - x)_+\}$. Then $P(0, x) = (K - x)_+$ for $0 \leq x \leq x^*$. If $x \geq K/(1 + a)^N$, we have $x \prod_{j=1}^{\tau} T_j \geq x(1 + a)^N \geq K$ for any $\tau \leq N$. Then $P(0, x) = 0$ for $x \geq K/(1 + a)^N$. From (4.10) and (4.11) we have

$$P(n-1, x) \geq \frac{1}{1+r} f(n, x) \geq \frac{p}{1+r} P(n, x(1 + a)).$$

Applying this relation inductively we get

$$P(0, x) \geq \frac{p^N}{(1+r)^N} P(N, x(1 + a)^N) = \frac{p^N}{(1+r)^N} [K - x(1 + a)^N].$$

That shows $P(0, x) > 0$ for $K < x < K/(1+a)^N$. Now the desired result will follows by the monotonicity and convexity of $x \mapsto P(0, x)$. □

By the result of Proposition 2.4.4, the holder of the American option would keep his option if $x^* < S_0 < K/(1 + a)^N$. Otherwise he should exercise the option immediately.

2.5 Strategies with consumption

2.5.1 Definition and basic properties

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a finite probability space where $\mathscr{F}$ is the class of all subsets of Ω and $\mathbf{P}(\{w\}) > 0$ for all $\omega \in \Omega$. Suppose that $(\mathscr{F}_n)_{0 \leq n \leq N}$ is a filtration satisfying $\mathscr{F}_0 = \{\phi, \Omega\}$ and $\mathscr{F}_N = \mathscr{F}$. A strategy with consumption can be introduced in the following way: At time n, once the new price $S_n^0, \cdots, S_n^d$ are quoted, the investor readjusts his positions from ϕ_n to ϕ_{n+1} and selects the wealth γ_{n+1} to consume at time $n + 1$. Any endowment being excluded and the new positions being decided given prices at time n, we deduce

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n + \gamma_{n+1}. \tag{5.1}$$

Then a *strategy with consumption* is defined as a pair (ϕ, γ), where the *strategy* $\phi = \{(\phi_n^0, \phi_n^1, \cdots, \phi_n^d) : 0 \leq n \leq N\}$ is a predictable process taking values in $\mathbb{R}^{d+1}$, representing the numbers of assets held in the portfolio, and the *consumption process*

$\gamma = \{\gamma_n : 0 \le n \le N\}$ is a predictable process taking values in $\mathbb{R}^+$, representing the wealth consumed at each step. The above equation gives the relationship between the processes ϕ and γ, which replaces the self-financing condition introduced before.

Recall that the *value* and *discounted value* of the strategy $\phi = \{(\phi_n^0, \phi_n^1, \cdots, \phi_n^d)\}$ at time n are defined respectively as $V_n(\phi) = \phi_n \cdot S_n$ and $\tilde{V}_n(\phi) = \phi_n \cdot \tilde{S}_n$. We have the following

Proposition 2.5.1 *The following conditions are equivalent:*

(i) The pair (ϕ, γ) defines a trading strategy with consumption.

(ii) For every $0 \le n \le N$ we have

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j - \sum_{j=1}^{n} \gamma_j. \tag{5.2}$$

(iii) For every $0 \le n \le N$ we have

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j - \sum_{j=1}^{n} \gamma_j / S_{j-1}^0. \tag{5.3}$$

Proof. Obviously, (5.1) holds if and only if

$$\phi_{n+1} \cdot S_{n+1} - \phi_n \cdot S_n = \phi_{n+1} \cdot S_{n+1} - \phi_{n+1} \cdot S_n - \gamma_{n+1},$$

that is,

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n) - \gamma_{n+1}. \tag{5.4}$$

The last equation is clear equivalent to (ii). On the other hand, (5.1) is equivalent to

$$\phi_n \cdot \tilde{S}_n = \phi_{n+1} \cdot \tilde{S}_n + \gamma_{n+1}/S_n^0,$$

which holds if and only if

$$\phi_{n+1} \cdot \tilde{S}_{n+1} - \phi_n \cdot \tilde{S}_n = \phi_{n+1} \cdot \tilde{S}_{n+1} - \phi_{n+1} \cdot \tilde{S}_n - \gamma_{n+1}/S_n^0,$$

that is,

$$\tilde{V}_{n+1}(\phi) - \tilde{V}_n(\phi) = \phi_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n) - \gamma_{n+1}/S_n^0. \tag{5.5}$$

The last equality holds for all $0 \le n \le N$ if and only (iii) is true. $\square$

2.5.2 Hedging with consumption strategies

Suppose that the market is viable and complete. Then there is a unique probability $\mathbf{P}^*$ under which the discounted asset prices are martingales. By Proposition 2.5.1, the discounted value process $\{\tilde{V}_n(\phi)\}$ of a consumption strategy is a supermartingale under $\mathbf{P}^*$. The following result shows the converse is also true.

Proposition 2.5.2 *Let $\{\tilde{U}_n\}$ is a supermartingale under $\mathbf{P}^*$. Then there is a trading strategy with consumption (ϕ, γ) such that $\tilde{V}_n(\phi) = \tilde{U}_n$ for every $0 \leq n \leq N$.*

Proof. By Doob's decomposition, we can write $\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$, where $\{\tilde{M}_n\}$ is a $\mathbf{P}^*$-martingale and $\{\tilde{A}_n\}$ is an increasing predictable process, null at time zero. Since the market is complete, there is a self-financing strategy $\psi = \{\psi_n : 0 \leq n \leq N\}$ that hedges the contingent claim $S_N^0 \tilde{M}_N$. Then $\tilde{V}_N(\psi) = \tilde{M}_N$. Since $\{\tilde{V}_n(\psi)\}$ is a $\mathbf{P}^*$-martingale, we have

$$\tilde{V}_n(\psi) = \mathbf{E}^*[\tilde{V}_N(\psi)|\mathscr{F}_n] = \mathbf{E}^*[\tilde{M}_N|\mathscr{F}_n] = \tilde{M}_n.$$

Now we define the strategy ϕ by setting $\phi_n^0 = \psi_n^0 - \tilde{A}_n$ and $\phi_n^i = \psi_n^i$ for $0 \leq n \leq N$ and $1 \leq i \leq d$. It follows that

$$\tilde{V}_n(\phi) = \tilde{V}_n(\psi) - \tilde{A}_n = \tilde{M}_n - \tilde{A}_n = \tilde{U}_n. \tag{5.6}$$

By Proposition 2.5.1 we have

$$\tilde{V}_n(\psi) = \tilde{V}_0(\psi) + \sum_{j=1}^{n} \psi_n \cdot \Delta\tilde{S}_j = \tilde{V}_0(\phi) + \sum_{j=1}^{n} \phi_n \cdot \Delta\tilde{S}_j \tag{5.7}$$

since $\Delta\tilde{S}_j^0 = 0$. Let $\gamma_n = 0$ and $\gamma_n = S_{n-1}^0(\tilde{A}_n - \tilde{A}_{n-1})$ for $1 \leq n \leq N$. We see immediately that $\{\gamma_n\}$ is a non-negative predictable process. From (5.6) and (5.7) it follows that

$$\tilde{V}_n(\phi) = \tilde{V}_n(\psi) - \tilde{A}_n = \tilde{V}_0(\phi) + \sum_{j=1}^{n} \phi_n \cdot \Delta\tilde{S}_j - \sum_{j=1}^{n} \gamma_j / S_{j-1}^0.$$

Then (ϕ, γ) is a trading strategy with consumption. □

We say that a trading strategy with consumption (ϕ, γ) *hedges the American option* defined by the non-negative adapted sequence $\{Z_n\}$ if $V_n(\phi) \geq Z_n$ for every $0 \leq n \leq N$.

Proposition 2.5.3 *Let $\{Z_n\}$ be a non-negative adapted sequence. Then there is at least one trading strategy with consumption that hedges $\{Z_n\}$, whose value is precisely the value $\{U_n\}$ of the American option. Moreover, any trading strategy with consumption (ϕ, γ) hedging $\{Z_n\}$ satisfies $V_n(\phi) \geq U_n$ for every $0 \leq n \leq N$.*

Proof. Let $\tilde{Z}_n = Z_n/S_n^0$. We know that the discounted value $\{\tilde{U}_n\}$ is a supermartingale. By Proposition 2.5.2, there is a trading strategy with consumption (ϕ, γ) such that $\tilde{V}_n(\phi) = \tilde{U}_n$ and so $V_n(\phi) = U_n$ for all $0 \leq n \leq N$. Since the discounted value $\{\tilde{U}_n\}$ is the smallest supermartingale that dominates $\{\tilde{Z}_n\}$, any trading strategy with consumption (ϕ, γ) hedging $\{Z_n\}$ satisfies $V_n(\phi) \geq U_n$ for all $0 \leq n \leq N$. □

2.5.3 Budget-feasible consumptions

Let us consider a viable and complete market and denote by $\mathbf{P}^*$ the unique probability under which the discounted asset prices are martingales. The *endowment* of an investor at time zero can be represented by a number $x \geq 0$. A consumption process $\{\gamma_n\}$ is said to be *budget-feasible* from the endowment $x \geq 0$ if there is a strategy ϕ such that the pair (ϕ, γ) defines a trading strategy with consumption satisfying $V_0(\phi) = x$ and $V_n(\phi) \geq 0$ for every $0 \leq n \leq N$.

Proposition 2.5.4 *A consumption process $\{\gamma_n\}$ is budget-feasible from the endowment $x \geq 0$ if and only if*

$$\mathbf{E}^* \left[\sum_{j=1}^{N} \gamma_j / S_{j-1}^0 \right] \leq x. \tag{5.8}$$

Proof. Suppose that $\{\gamma_n\}$ is budget-feasible from the endowment $x \geq 0$. Then there is a strategy ϕ such that the pair (ϕ, γ) defines a trading strategy with consumption satisfying $V_0(\phi) = x$ and $V_n(\phi) \geq 0$ for every $0 \leq n \leq N$. By Proposition 2.5.1,

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j - \sum_{j=1}^{n} \gamma_j / S_{j-1}^0,$$

where the second term on the right hand side is a martingale under $\mathbf{P}^*$. Taking the expectation we find

$$0 \leq \mathbf{E}^*[\tilde{V}_n(\phi)] = x - \mathbf{E}^* \left[\sum_{j=1}^{N} \gamma_j / S_{j-1}^0 \right],$$

which yields (5.8). Conversely, suppose that (5.8) holds. We define a $\mathbf{P}^*$-martingale by

$$M_n = x + \mathbf{E}^* \left[\sum_{j=1}^{N} \gamma_j / S_{j-1}^0 \middle| \mathscr{F}_n \right] - \mathbf{E}^* \left[\sum_{j=1}^{N} \gamma_j / S_{j-1}^0 \right]. \tag{5.9}$$

By Proposition 2.5.1 and 2.5.2, there is a trading strategy with consumption (ϕ, γ) such that

$$\tilde{V}_n(\phi) = M_n - \sum_{j=1}^{n} \gamma_j / S_{j-1}^0. \tag{5.10}$$

From (5.9) and (5.10) it follows that

$$\tilde{V}_n(\phi) = x + \mathbf{E}^* \left[\sum_{j=n+1}^{N} \gamma_j / S_{j-1}^0 \middle| \mathscr{F}_n \right] - \mathbf{E}^* \left[\sum_{j=1}^{N} \gamma_j / S_{j-1}^0 \right] \geq 0.$$

In particular, we have $\tilde{V}_0(\phi) = x$. $\square$

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