CONTINUOUS-STATE BRANCHING PROCESSES

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Preface

These notes were used in a short graduate course on branching processes the author gave in Beijing Normal University. The following main topics are covered: scaling limits of Galton–Watson processes, continuous-state branching processes, extinction probabilities, conditional limit theorems, decompositions of sample paths, martingale problems, stochastic equations, Lamperti's transformations, independent and dependent immigration processes. Some of the results are simplified versions of those in the author's book "Measure-valued branching Markov processes" (Springer, 2011). We hope these simplified results will set out the main ideas in an easy way and lead the reader to a quick access of the subject.

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Chapter 1

Preliminaries

In this chapter, we discuss the basic properties of Laplace transforms of finite measures on the positive half line. In particular, we give some characterizations of the weak convergence of those measures in terms of their Laplace transforms. Based on these results, a general representation for infinitely divisible distributions on the positive half line is established. We also give some characterizations of continuous functions on the positive half line with Lévy–Khintchine type representations.

1.1 Laplace transforms of measures

In this section, we discuss the basic properties of Laplace transforms of finite measures on the positive half line $\mathbb{R}_+ := [0, \infty)$. Let $B(\mathbb{R}_+) = \mathrm{b}\mathscr{B}(\mathbb{R}_+)$ be the set of bounded Borel functions on $\mathbb{R}_+$. Given a finite measure μ on $\mathbb{R}_+$, we define the *Laplace transform* L_μ of μ by

$$L_\mu(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda x} \mu(\mathrm{d}x), \qquad \lambda \geq 0. \tag{1.1.1}$$

Theorem 1.1.1 *A finite measure on $\mathbb{R}_+$ is uniquely determined by its Laplace transform.*

Proof. Suppose that μ_1 and μ_2 are finite measures on $\mathbb{R}_+$ and $L_{\mu_1}(\lambda) = L_{\mu_2}(\lambda)$ for all $\lambda \geq 0$. Let $\mathscr{K} = \{x \mapsto \mathrm{e}^{-\lambda x} : \lambda \geq 0\}$ and let $\mathscr{L}$ be the class of functions $F \in B(\mathbb{R}_+)$ so that

$$\int_0^\infty F(x)\mu_1(\mathrm{d}x) = \int_0^\infty F(x)\mu_2(\mathrm{d}x).$$

Then $\mathscr{K}$ is closed under multiplication and $\mathscr{L}$ is a monotone vector space containing $\mathscr{K}$. It is easy to see $\sigma(\mathscr{K}) = \mathscr{B}(\mathbb{R}_+)$. Then the monotone class theorem implies $\mathscr{L} \supset \mathrm{b}\sigma(\mathscr{K}) = B(\mathbb{R}_+)$. That proves the desired result. $\square$

Theorem 1.1.2 *Let $\{\mu_n\}$ be finite measures on $\mathbb{R}_+$ and let $\lambda \mapsto L(\lambda)$ be a continuous function on $[0,\infty)$. If there is a dense subset D of $(0,\infty)$ to that $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L(\lambda)$ for every $\lambda \in D$, then there is a finite measure μ on $\mathbb{R}_+$ such that $L_\mu = L$ and $\lim_{n\to\infty} \mu_n = \mu$ by weak convergence.*

Proof. We can regard each μ_n as a finite measure on $\bar{\mathbb{R}}_+ := [0,\infty]$, the one-point compactification of $[0,\infty)$. Let F_n denote the distribution function of μ_n. By applying Helly's theorem one can see that any subsequence of $\{F_n\}$ contains a weakly convergent subsequence $\{F_{n_k}\}$. Then the corresponding subsequence $\{\mu_{n_k}\}$ converges weakly on $\bar{\mathbb{R}}_+$ to a finite measure μ. It follows that

$$\mu(\bar{\mathbb{R}}_+) = \lim_{k\to\infty} \mu_{n_k}(\mathbb{R}_+) = \lim_{k\to\infty} L_{\mu_{n_k}}(0) = L(0).$$

Moreover, for $\lambda \in D$ we have

$$\int_{\bar{\mathbb{R}}_+} \mathrm{e}^{-\lambda x}\mu(\mathrm{d}x) = \lim_{k\to\infty} \int_0^\infty \mathrm{e}^{-\lambda x}\mu_{n_k}(\mathrm{d}x) = L(\lambda), \tag{1.1.2}$$

where $\mathrm{e}^{-\lambda\cdot\infty} = 0$ by convention. By letting $\lambda \to 0+$ along D in (1.1.2) and using the continuity of L at $\lambda = 0$ we find $\mu(\mathbb{R}_+) = L(0)$, so μ is supported by $\mathbb{R}_+$. Then $\lim_{n\to\infty} \mu_{n_k} = \mu$ weakly on $\mathbb{R}_+$. It is easy to see that (1.1.2) in fact holds for all $\lambda \geq 0$, so we have $L_\mu = L$. By a standard argument one sees $\lim_{n\to\infty} \mu_n = \mu$ weakly on $\mathbb{R}_+$. $\square$

Theorem 1.1.3 *Let $\mu_1, \mu_2, \ldots$ and μ be finite measures on $\mathbb{R}_+$. Then $\mu_n \to \mu$ weakly if and only if $L_{\mu_n}(\lambda) \to L_\mu(\lambda)$ for every $\lambda \geq 0$.*

Proof. If $\mu_n \to \mu$ weakly, we have $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L_\mu(\lambda)$ for every $\lambda \geq 0$ by dominated convergence. The converse assertion is a consequence of Theorem 1.1.2. $\square$

We next give a necessary and sufficient condition for a continuous real function to be the Laplace transform of a finite measure on $\mathbb{R}_+$. For a constant $c \geq 0$ and a function f on an interval $T \subset \mathbb{R}$ we write

$$\Delta_c f(\lambda) = f(\lambda + c) - f(\lambda), \qquad \lambda, \lambda + c \in T.$$

Let Δ_c^0 be the identity and define $\Delta_c^n = \Delta_c^{n-1}\Delta_c$ for $n \geq 1$ inductively. Then we have

$$\Delta_c^m f(\lambda) = (-1)^m \sum_{i=0}^m \binom{m}{i} (-1)^i f(\lambda + ic).$$

The *Bernstein polynomials* of a function f on $[0,1]$ are given by

$$B_{f,m}(s) = \sum_{i=0}^m \binom{m}{i} \Delta_{1/m}^i f(0) s^i, \qquad 0 \leq s \leq 1, m = 1, 2, \ldots. \tag{1.1.3}$$

It is well-known that $B_{f,m}(s) \to f(s)$ uniformly as $m \to \infty$; see, e.g., Feller (1971, p.222). A real function θ on $[0, \infty)$ is said to be *completely monotone* if it satisfies

$$(-1)^i \Delta_c^i \theta(\lambda) \geq 0, \qquad \lambda \geq 0, c \geq 0, i = 0, 1, 2, \ldots. \tag{1.1.4}$$

Theorem 1.1.4 *A continuous real function θ on $[0, \infty)$ is the Laplace transform of a finite measure μ on $\mathbb{R}_+$ if and only if it is completely monotone.*

Proof. If θ is the Laplace transform of a finite measure on $\mathbb{R}_+$ it is clearly a completely monotone function. Conversely, suppose that (1.1.4) holds. For fixed $a > 0$, we let $\gamma_a(s) = \theta(a - as)$ for $0 \leq s \leq 1$. The complete monotonicity of θ implies

$$\Delta_{1/m}^i \gamma_a(0) \geq 0, \qquad i = 0, 1, \ldots, m.$$

Then the Bernstein polynomial $B_{\gamma_a,m}(s)$ has positive coefficients, so $B_{\gamma_a,m}(\mathrm{e}^{-\lambda/a})$ is the Laplace transform of a finite measure on $\mathbb{R}_+$. By Theorem 1.1.2,

$$\theta(\lambda) = \lim_{a\to\infty} \lim_{m\to\infty} B_{\gamma_a,m}(\mathrm{e}^{-\lambda/a}), \qquad \lambda \geq 0,$$

is the Laplace transform of a finite measure on $\mathbb{R}_+$. $\square$

We often use a variation of the Laplace transform in dealing with σ-finite measures on $(0, \infty)$. A typical case is considered in the following:

Theorem 1.1.5 *Let μ_1 and μ_2 be two σ-finite measures on $(0, \infty)$. If for every $\lambda \geq 0$,*

$$\int_0^\infty (1 - \mathrm{e}^{-\lambda x}) \mu_1(\mathrm{d}x) = \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) \mu_2(\mathrm{d}x) \tag{1.1.5}$$

and the value is finite, then we have $\mu_1 = \mu_2$.

Proof. By setting $\mu_1(\{0\}) = \mu_2(\{0\}) = 0$ we extend μ_1 and μ_2 to σ-finite measures on $[0, \infty)$. Taking the difference of (1.1.5) for λ and $\lambda + 1$ we obtain

$$\int_0^\infty \mathrm{e}^{-\lambda x}(1 - \mathrm{e}^{-x}) \mu_1(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-\lambda x}(1 - \mathrm{e}^{-x}) \mu_2(\mathrm{d}x).$$

Then the result of Theorem 1.1.1 implies that

$$(1 - \mathrm{e}^{-x}) \mu_1(\mathrm{d}x) = (1 - \mathrm{e}^{-x}) \mu_2(\mathrm{d}x)$$

as finite measures on $[0, \infty)$. Since $1 - \mathrm{e}^{-x}$ is strictly positive on $(0, \infty)$, it follows that $\mu_1 = \mu_2$ as σ-finite measures on $(0, \infty)$. $\square$

Now let us consider a complete separable metric space E with the Borel σ-algebra denoted by $\mathscr{B}(E)$. Suppose that h is a strictly positive bounded Borel function on E. Let $B_h(E)$ be the set of Borel functions f on E such that $|f| \leq \mathrm{const} \cdot h$. Let M_h be the set of Borel measures μ on E such that $\int_E h \mathrm{d}\mu < \infty$. Let $\mathscr{M}_h$ be the σ-algebra on M_h generated by the mappings

$$\mu \mapsto \mu(f) := \int_E f(x)\mu(\mathrm{d}x), \qquad f \in B_h(E).$$

Given a finite measure Q on $(M_h, \mathscr{M}_h)$, we define the *Laplace functional* L_Q of Q by

$$L_Q(f) = \int_{M_h} \mathrm{e}^{-\nu(f)} Q(\mathrm{d}\nu), \qquad f \in B_h(E)^+. \tag{1.1.6}$$

A random element X taking values on $(M_h, \mathscr{M}_h)$ is called a *random measure* on E. The *Laplace functional* of a random measure means the Laplace functional of its distribution on $(M_h, \mathscr{M}_h)$. The reader may refer to Kallenberg (1975) or Li (2011) for the basic theory of random measure. In particular, the proofs of the following results can be found in the two references:

Theorem 1.1.6 *A finite measure on $(M_h, \mathscr{M}_h)$ is uniquely determined by its Laplace functional.*

Suppose that λ is a σ-finite measure on $(E, \mathscr{B}(E))$. A random measure X on E is called a *Poisson random measure* with *intensity* λ provided:

(1) for each $B \in \mathscr{B}(E)$ with $\lambda(B) < \infty$, the random variable $X(B)$ has the Poisson distribution with parameter $\lambda(B)$, that is,

$$\mathbf{P}\{X(B) = n\} = \frac{\lambda(B)^n}{n!}\mathrm{e}^{-\lambda(B)}, \qquad n = 0, 1, 2, \ldots;$$

(2) if $B_1, \ldots, B_n \in \mathscr{B}(E)$ are disjoint and $\lambda(B_i) < \infty$ for each $i = 1, \ldots, n$, then $X(B_1), \ldots, X(B_n)$ are mutually independent random variables.

Theorem 1.1.7 *A random measure X on E is Poissonian with intensity $\lambda \in M_h(E)$ if and only if its Laplace functional is given by*

$$\mathbf{E}\exp\{-X(f)\} = \exp\Big\{ - \int_E (1 - \mathrm{e}^{-f(x)})\lambda(\mathrm{d}x)\Big\}, \quad f \in B_h(E)^+. \tag{1.1.7}$$

Proof. Suppose that X is a Poisson random measure on E with intensity λ. Let $B_1, \dots, B_n \in \mathscr{B}(E)$ be disjoint sets satisfying $\lambda(B_i) < \infty$ for each $i = 1, \dots, n$. For any constants $\alpha_1, \dots, \alpha_n \geq 0$ we can use the above two properties to see

$$\mathbf{E} \exp\Big\{ -\sum_{i=1}^{n} \alpha_i X(B_i) \Big\} = \exp\Big\{ -\sum_{i=1}^{n} (1 - \mathrm{e}^{-\alpha_i}) \lambda(B_i) \Big\}. \tag{1.1.8}$$

Then we get (1.1.7) by approximating $f \in B_h(E)^+$ by simple functions and using dominated convergence. Conversely, if the Laplace functional of X is given by (1.1.7), we may apply the equality to the simple function $f = \sum_{i=1}^{n} \alpha_i 1_{B_i}$ to get (1.1.8). Then X satisfies the above two properties in the definition of a Poisson random measure on E with intensity λ.

1.2 Infinitely divisible distributions

For probability measures μ_1 and μ_2 on $\mathbb{R}_+$, the product $\mu_1 \times \mu_2$ is a probability measure on $\mathbb{R}_+^2$. The image of $\mu_1 \times \mu_2$ under the mapping $(x_1, x_2) \mapsto x_1 + x_2$ is called the *convolution* of μ_1 and μ_2 and is denoted by $\mu_1 * \mu_2$, which is a probability measure on $\mathbb{R}_+$. According to the definition, for any $F \in B(\mathbb{R}_+)$ we have

$$\int_0^\infty F(x)(\mu_1 * \mu_2)(\mathrm{d}x) = \int_0^\infty \mu_1(\mathrm{d}x_1) \int_0^\infty F(x_1 + x_2) \mu_2(\mathrm{d}x_2). \tag{1.2.1}$$

Clearly, if ξ_1 and ξ_2 are independent random variables with distributions μ_1 and μ_2 on $\mathbb{R}_+$, respectively, then the random variable $\xi_1 + \xi_2$ has distribution $\mu_1 * \mu_2$. It is easy to show that

$$L_{\mu_1 * \mu_2}(\lambda) = L_{\mu_1}(\lambda) L_{\mu_2}(\lambda), \qquad \lambda \geq 0. \tag{1.2.2}$$

Let $\mu^{*0} = \delta_0$ and define $\mu^{*n} = \mu^{*(n-1)} * \mu$ inductively for integers $n \geq 1$. We say a probability distribution μ on $\mathbb{R}_+$ is *infinitely divisible* if for each integer $n \geq 1$, there is a probability μ_n such that $\mu = \mu_n^{*n}$. In this case, we call μ_n the *n-th root* of μ. A positive random variable ξ is said to be *infinitely divisible* if it has infinitely divisible distribution on $\mathbb{R}_+$.

We next give a characterization for the class of infinitely divisible probability measures on $\mathbb{R}_+$. Write $\psi \in \mathscr{I}$ if $\lambda \mapsto \psi(\lambda)$ is a positive function on $[0, \infty)$ with the representation

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda u}) l(\mathrm{d}u), \tag{1.2.3}$$

where $h \geq 0$ and $(1 \wedge u) l(\mathrm{d}u)$ is a finite measure on $(0, \infty)$.

Proposition 1.2.1 *The pair (h, l) in (1.2.3) is uniquely determined by the function $\psi \in \mathscr{I}$.*

Proof. Suppose that ψ can also be represented by (1.2.3) with (h, l) replaced by (h', l'). For $\lambda > 0$ and $\theta \geq 0$, we can evaluate $\psi(\lambda + \theta) - \psi(\theta)$ with the two representations and get

$$h\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-\lambda u}\big)\mathrm{e}^{-\theta u} l(\mathrm{d}u) = h'\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-\lambda u}\big)\mathrm{e}^{-\theta u} l'(\mathrm{d}u).$$

By letting $\theta \to \infty$ we get $h = h'$, and so $l(\mathrm{d}u) = l'(\mathrm{d}u)$ by Theorem 1.1.5. □

Theorem 1.2.2 *Suppose that ψ is a continuous function on $[0, \infty)$. If there is a sequence $\{\psi_n\} \subset \mathscr{I}$ such that $\psi(\lambda) = \lim_{n\to\infty} \psi_n(\lambda)$ for all $\lambda \geq 0$, then $\psi \in \mathscr{I}$.*

Proof. Suppose that $\psi_n \in \mathscr{I}$ is given by (1.2.3) with (h, l) replaced by (h_n, l_n). We can define a finite measure F_n on $\bar{\mathbb{R}}_+$ by setting $F_n(\{0\}) = h_n$, $F_n(\{\infty\}) = 0$ and $F_n(\mathrm{d}u) = (1 - \mathrm{e}^{-u})l_n(\mathrm{d}u)$ for $0 < u < \infty$. For $\lambda > 0$ let

$$\xi(u, \lambda) = \begin{cases} (1 - \mathrm{e}^{-u})^{-1}(1 - \mathrm{e}^{-u\lambda}) & \text{if } 0 < u < \infty, \\ \lambda & \text{if } u = 0, \\ 1 & \text{if } u = \infty. \end{cases} \tag{1.2.4}$$

Then we have

$$\psi_n(\lambda) = \int_{\bar{\mathbb{R}}_+} \xi(u, \lambda) F_n(\mathrm{d}u), \qquad \lambda > 0.$$

It is evident that $\{F_n(\bar{\mathbb{R}}_+)\}$ is a bounded sequence. Take any subsequence $\{F_{n_k}\} \subset \{F_n\}$ such that $\lim_{k\to\infty} F_{n_k} = F$ weakly for a finite measure F on $\bar{\mathbb{R}}_+$. Since $u \mapsto \xi(u, \lambda)$ is continuous on $\bar{\mathbb{R}}_+$, we have

$$\psi(\lambda) = \int_{\bar{\mathbb{R}}_+} \xi(u, \lambda) F(\mathrm{d}u), \qquad \lambda > 0.$$

Observe also that $\lim_{n\to\infty} \psi(1/n) = \psi(0) = 0$ implies $F(\{\infty\}) = 0$. Then the desired conclusion follows by a change of the integration variable. □

Theorem 1.2.3 *The relation $\psi = -\log L_\mu$ establishes a one-to-one correspondence between the functions $\psi \in \mathscr{I}$ and infinitely divisible probability measures μ on $\mathbb{R}_+$.*

Proof. Suppose that $\psi \in \mathscr{I}$ is given by (1.2.3). Let N be a Poisson random measure on $(0, \infty)$ with intensity $l(\mathrm{d}u)$ and let

$$\xi = h + \int_0^\infty xN(\mathrm{d}x).$$

By Theorem 1.1.7 for any $\lambda \geq 0$ we have

$$\mathbf{E}\,\mathrm{e}^{-\lambda\xi} = \exp\Big\{ - h\lambda - \int_0^\infty \big(1 - \mathrm{e}^{-\lambda u}\big) l(\mathrm{d}u)\Big\}.$$

Then $\psi = -\log L_\mu$ for a probability measure μ on $\mathbb{R}_+$. Similarly, for each integer $n \geq 1$ there is a probability measure μ_n on $\mathbb{R}_+$ so that $\psi/n = -\log L_{\mu_n}$. It is easy to see that $\mu_n^{*n} = \mu$. That gives the infinite divisibility of μ. Conversely, suppose that $\psi = -\log L_\mu$ for an infinitely divisible probability measure μ on $\mathbb{R}_+$. For $n \geq 1$ let μ_n be the n-th root of μ. Then

$$\psi(\lambda) = \lim_{n\to\infty} n[1 - \mathrm{e}^{-n^{-1}\psi(\lambda)}] = \lim_{n\to\infty} \int_0^\infty \big(1 - \mathrm{e}^{-\lambda x}\big) n\mu_n(\mathrm{d}x).$$

By Theorem 1.2.2 we have $\psi \in \mathscr{I}$. □

The above theorem gives a complete characterization of infinitely divisible probability measures on $\mathbb{R}_+$. We write $\mu = I(h, l)$ if μ is an infinitely divisible probability measure on $\mathbb{R}_+$ with $\psi := -\log L_\mu$ given by (1.2.3).

Theorem 1.2.4 *If $\psi_1, \psi_2 \in \mathscr{I}$, then $\psi_1 \circ \psi_2 \in \mathscr{I}$.*

Proof. For every $x \geq 0$ we clearly have $x\psi_2 \in \mathscr{I}$, so there is an infinitely divisible probability measure ν_x on $\mathbb{R}_+$ satisfying $-\log L_{\nu_x} = x\psi_2$. By a monotone class argument one can see $\nu_x(\mathrm{d}y)$ is a probability kernel on $\mathbb{R}_+$. Let μ be the infinitely divisible probability measure on $\mathbb{R}_+$ with $-\log L_\mu = \psi_1$ and define

$$\eta(\mathrm{d}y) = \int_0^\infty \mu(\mathrm{d}x)\nu_x(\mathrm{d}y), \qquad y \geq 0.$$

It is not hard to show that $-\log L_\eta = \psi_1 \circ \psi_2$. By the same reasoning, for each integer $n \geq 1$ there is a probability measure η_n such that $-\log L_{\eta_n} = n^{-1}\psi_1 \circ \psi_2$. Then $\eta = \eta_n^{*n}$ and hence η is infinitely divisible. By Theorem 1.2.3 we conclude that $\psi_1 \circ \psi_2 \in \mathscr{I}$. □

Example 1.2.1 Let $b > 0$ and $\alpha > 0$. The *Gamma distribution* γ on $\mathbb{R}_+$ with parameters (b, α) is defined by

$$\gamma(B) = \frac{\alpha^b}{\Gamma(b)} \int_B x^{b-1}\mathrm{e}^{-\alpha x}\mathrm{d}x, \qquad B \in \mathscr{B}(\mathbb{R}_+),$$

which reduces to the *exponential distribution* when $b = 1$. The Laplace transform of γ is

$$L_\gamma(\lambda) = \Big(\frac{\alpha}{\alpha+\lambda}\Big)^b, \qquad \lambda \geq 0.$$

It is easily seen that γ is infinitely divisible and its n-th root is the Gamma distribution with parameters $(b/n, \alpha)$.

Example 1.2.2 For $c > 0$ and $0 < \alpha < 1$ the function $\lambda \mapsto c\lambda^\alpha$ admits the representation (1.2.3). Indeed, by integration by parts we have

$$\begin{aligned}\int_0^\infty (1-\mathrm{e}^{-\lambda u})\frac{\mathrm{d}u}{u^{1+\alpha}} &= \lambda^\alpha \int_0^\infty (1-\mathrm{e}^{-v})\frac{\mathrm{d}v}{v^{1+\alpha}} \\ &= \lambda^\alpha \Big[-(1-\mathrm{e}^{-v})\frac{1}{\alpha v^\alpha}\Big|_0^\infty + \int_0^\infty \mathrm{e}^{-v}\frac{\mathrm{d}v}{\alpha v^\alpha}\Big] \\ &= \frac{1}{\alpha}\Gamma(1-\alpha)\lambda^\alpha.\end{aligned}$$

It follows that

$$\lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-\mathrm{e}^{-\lambda u})\frac{\mathrm{d}u}{u^{1+\alpha}}, \qquad \lambda \geq 0. \tag{1.2.5}$$

The infinitely divisible probability measure ν on $\mathbb{R}_+$ satisfying $-\log L_\nu(\lambda) = c\lambda^\alpha$ is known as the *one-sided stable distribution* with index $0 < \alpha < 1$. This distribution does not charge zero and is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$ with continuous density. For $\alpha = 1/2$ it has density

$$q(x) := \frac{c}{2\sqrt{\pi}}x^{-3/2}\mathrm{e}^{-c^2/4x}, \qquad x > 0.$$

For a general index the density can be given using an infinite series; see, e.g., Sato (1999, p.88).

1.3 Lévy–Khintchine type representations

In this section, we give some criteria for continuous functions on $[0, \infty)$ to have Lévy–Khintchine type representations. The results are useful in the study of high-density limits of discrete branching processes. For $u \geq 0$ and $\lambda \geq 0$ let

$$\xi_n(u, \lambda) = \mathrm{e}^{-\lambda u} - 1 - (1+u^n)^{-1}\sum_{i=1}^{n-1}\frac{(-\lambda u)^i}{i!}, \qquad n = 1, 2, \ldots.$$

We are interested in functions ϕ on $[0,\infty)$ with the representation

$$\phi(\lambda) = \sum_{i=0}^{n-1} a_i \lambda^i + \int_0^\infty \xi_n(u,\lambda)(1-\mathrm{e}^{-u})^{-n} G(\mathrm{d}u), \quad \lambda \geq 0, \tag{1.3.1}$$

where $n \geq 1$ is an integer, $\{a_0, \ldots, a_{n-1}\}$ is a set of constants and $G(\mathrm{d}u)$ is a finite measure on $\mathbb{R}_+$. The value at $u=0$ of the integrand in (1.3.1) is defined by continuity as $(-\lambda)^n/n!$. The following theorem was proved in Li (1991, 2011):

Theorem 1.3.1 *A continuous real function* ϕ *on* $[0,\infty)$ *has the representation* (1.3.1) *if and only if for every* $c \geq 0$ *the function*

$$\theta_c(\lambda) := (-1)^n \Delta_c^n \phi(\lambda), \qquad \lambda \geq 0 \tag{1.3.2}$$

is the Laplace transform of a finite measure on $\mathbb{R}_+$.

Based on the above theorem we can give canonical representations for the limit functions of some sequences involving probability generating functions. Let $\{\alpha_k\}$ be a sequence of positive numbers and let $\{g_k\}$ be a sequence of probability generating functions, that is,

$$g_k(z) = \sum_{i=0}^\infty p_{ki} z^i, \qquad |z| \leq 1,$$

where $p_{ki} \geq 0$ and $\sum_{i=0}^\infty p_{ki} = 1$. We first consider the sequence of functions $\{\psi_k\}$ defined by

$$\psi_k(\lambda) = \alpha_k[1 - g_k(1-\lambda/k)], \qquad 0 \leq \lambda \leq k. \tag{1.3.3}$$

Theorem 1.3.2 *If the sequence* $\{\psi_k\}$ *defined by* (1.3.3) *converges to a continuous real function* ψ *on* $[0,\infty)$*, then the limit function belongs to the class* $\mathscr{I}$ *defined by* (1.2.3).

Proof. For any $c, \lambda \geq 0$ and sufficiently large $k \geq 1$ we have

$$\Delta_c \psi_k(\lambda) = -\alpha_k \Delta_c g_k(1 - \cdot/k)(\lambda).$$

Since for each integer $i \geq 1$ the i-th derivative $g_k^{(i)}$ is a power series with positive coefficients, we have

$$(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} \Delta_c \psi_k(\lambda) = -k^{-i} \alpha_k \Delta_c g_k^{(i)}(1 - \cdot/k)(\lambda) \geq 0.$$

By the mean-value theorem, one sees inductively $(-1)^i \Delta_h^i \Delta_c \psi_k(\lambda) \geq 0$. Letting $k \to \infty$ we obtain $(-1)^i \Delta_h^i \Delta_c \psi(\lambda) \geq 0$. Then $\Delta_c \psi(\lambda)$ is a completely monotone function of

$\lambda \geq 0$, so by Theorem 1.1.4 it is the Laplace transform of a finite measure on $\mathbb{R}_+$. Since $\psi(0) = \lim_{k\to\infty} \psi_k(0) = 0$, by Theorem 1.3.1 there is a finite measure F on $\mathbb{R}_+$ so that

$$\psi(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda u})(1 - \mathrm{e}^{-u})^{-1} F(\mathrm{d}u),$$

where the value of the integrand at $u = 0$ is defined as λ by continuity. Then (1.2.3) follows with $\beta = F(\{0\})$ and $n(\mathrm{d}u) = (1 - \mathrm{e}^{-u})^{-1} F(\mathrm{d}u)$ for $u > 0$. $\square$

Example 1.3.1 Suppose that g is a probability generating function so that $\beta := g'(1-) < \infty$. Let $\alpha_k = k$ and $g_k(z) = g(z)$. Then the sequence $\psi_k(\lambda)$ defined by (1.3.3) converges to $\beta\lambda$ as $k \to \infty$.

Example 1.3.2 For any $0 < \alpha \leq 1$ the function $\psi(\lambda) = \lambda^\alpha$ has the representation (1.2.3). For $\alpha = 1$ that is trivial, and for $0 < \alpha < 1$ that follows from (1.2.5). Let $\psi_k(\lambda)$ be defined by (1.3.3) with $\alpha_k = k^\alpha$ and $g_k(z) = 1 - (1 - z)^\alpha$. Then $\psi_k(\lambda) = \lambda^\alpha$ for $0 \leq \lambda \leq k$.

In the study of limit theorems of branching models, we shall also need to consider the limit of another function sequence defined as follows. Let $\{\alpha_k\}$ and $\{g_k\}$ be given as above and let

$$\phi_k(\lambda) = \alpha_k[g_k(1 - \lambda/k) - (1 - \lambda/k)], \qquad 0 \leq \lambda \leq k. \tag{1.3.4}$$

Theorem 1.3.3 *If the sequence* $\{\phi_k\}$ *defined by* (1.3.4) *converges to a continuous real function* ϕ *on* $[0, \infty)$*, then the limit function has the representation*

$$\phi(\lambda) = a\lambda + c\lambda^2 + \int_0^\infty \Big(\mathrm{e}^{-\lambda u} - 1 + \frac{\lambda u}{1 + u^2}\Big) m(\mathrm{d}u), \tag{1.3.5}$$

where $c \geq 0$ *and* a *are constants, and* $m(\mathrm{d}u)$ *is a* σ*-finite measure on* $(0, \infty)$ *satisfying*

$$\int_0^\infty (1 \wedge u^2) m(\mathrm{d}u) < \infty. \tag{1.3.6}$$

Proof. Since $\phi(0) = \lim_{k\to\infty} \phi_k(0) = 0$, arguing as in the proof of Theorem 1.3.2 we see that ϕ has the representation (1.3.1) with $n = 2$ and $a_0 = 0$, which can be rewritten into the equivalent form (1.3.5). $\square$

As observed in the above proof, the representation (1.3.5) is essentially a special form of (1.3.1). For computational convenience we may rewrite (1.3.5) as

$$\phi(\lambda) = b_1\lambda + c\lambda^2 + \int_0^\infty \big(\mathrm{e}^{-\lambda u} - 1 + \lambda u 1_{\{u\leq 1\}}\big) m(\mathrm{d}u), \tag{1.3.7}$$

where

$$b_1 := a + \int_0^\infty \Big(\frac{u}{1+u^2} - u1_{\{u\le 1\}}\Big)m(\mathrm{d}u).$$

If the measure $m(\mathrm{d}u)$ satisfies the integrability condition

$$\int_0^\infty (u \wedge u^2)\, m(\mathrm{d}u) < \infty, \tag{1.3.8}$$

we have

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty \big(\mathrm{e}^{-\lambda u} - 1 + \lambda u\big)m(\mathrm{d}u), \tag{1.3.9}$$

where

$$b = a - \int_0^\infty \frac{u^3}{1+u^2} m(\mathrm{d}u).$$

Proposition 1.3.4 *A function ϕ with the representation* (1.3.5) *is locally Lipschitz if and only if* (1.3.8) *holds.*

Proof. By applying dominated convergence to (1.3.7), for each $\lambda > 0$ we have

$$\phi'(\lambda) = b_1 + 2c\lambda + \int_{(0,1]} u\big(1 - \mathrm{e}^{-\lambda u}\big)m(\mathrm{d}u) - \int_{(1,\infty)} u\mathrm{e}^{-\lambda u} m(\mathrm{d}u).$$

Then we use monotone convergence to the two integrals to get

$$\phi'(0+) = b_1 - \int_{(1,\infty)} um(\mathrm{d}u).$$

If ϕ is locally Lipschitz, we have $\phi'(0+) > -\infty$ and the integral on the right-hand side is finite. This together with (1.3.6) implies (1.3.8). Conversely, if (1.3.8) holds, then ϕ' is bounded on each bounded interval and so ϕ is locally Lipschitz. $\square$

Corollary 1.3.5 *If the sequence $\{\phi_k\}$ defined by* (1.3.4) *is uniformly Lipschitz on each bounded interval and $\phi_k(\lambda) \to \phi(\lambda)$ for all $\lambda \ge 0$ as $k \to \infty$, then the limit function has the representation* (1.3.9).

Example 1.3.3 Suppose that g is a probability generating function so that $g'(1-) = 1$ and $c := g''(1-)/2 < \infty$. Let $\alpha_k = k^2$ and $g_k(z) = g(z)$. By Taylor's expansion it is easy to show that the sequence $\phi_k(\lambda)$ defined by (1.3.4) converges to $c\lambda^2$ as $k \to \infty$.

Example 1.3.4 For $0 < \alpha < 1$ the function $\phi(\lambda) = -\lambda^\alpha$ has the representation (1.3.5). That follows from (1.2.5) as we notice

$$\int_0^\infty \Big(\frac{u}{1+u^2}\Big)\frac{\mathrm{d}u}{u^{1+\alpha}} = \int_0^\infty \Big(\frac{1}{1+u^2}\Big)\frac{\mathrm{d}u}{u^\alpha} < \infty.$$

The function is the limit of the sequence $\phi_k(\lambda)$ defined by (1.3.4) with $\alpha_k = k^\alpha$ and $g_k(z) = 1 - (1-z)^\alpha$.

Example 1.3.5 The function $\phi(\lambda) = \lambda\log\lambda$ can be represented in the form of (1.3.7). In fact, we have

$$\begin{aligned}\int_0^\infty (\mathrm{e}^{-\lambda u} - 1 + \lambda u 1_{\{u\le 1\}})\frac{\mathrm{d}u}{u^2} &= \lambda\int_0^\infty (\mathrm{e}^{-v} - 1 + v1_{\{v\le\lambda\}})\frac{\mathrm{d}v}{v^2} \\ &= h\lambda + \lambda\int_1^\lambda \frac{\mathrm{d}v}{v} = h\lambda + \lambda\log\lambda,\end{aligned}$$

where

$$h = \int_0^\infty (\mathrm{e}^{-v} - 1 + v1_{\{v\le 1\}})\frac{\mathrm{d}v}{v^2}.$$

It follows that

$$\lambda\log\lambda = -h\lambda + \int_0^\infty (\mathrm{e}^{-\lambda u} - 1 + \lambda u1_{\{u\le 1\}})\frac{\mathrm{d}u}{u^2}, \qquad \lambda \ge 0.$$

For $k \ge 1$ sufficiently large, let $\phi_k(\lambda)$ be defined by (1.3.4) with $\alpha_k = k(\log k - 1)$ and

$$g_k(z) = z + k\alpha_k^{-1}(1-z)\log[k(1-z)].$$

Then we have $\phi_k(\lambda) = \lambda\log\lambda$ for $0 \le \lambda \le k$.

Example 1.3.6 For any $1 \le \alpha \le 2$ the function $\phi(\lambda) = \lambda^\alpha$ can be represented in the form of (1.3.9). In particular, for $1 < \alpha < 2$ we can use integration by parts to see

$$\begin{aligned}\int_0^\infty (\mathrm{e}^{-\lambda u} - 1 + \lambda u)\frac{\mathrm{d}u}{u^{1+\alpha}} &= \lambda^\alpha\int_0^\infty (\mathrm{e}^{-v} - 1 + v)\frac{\mathrm{d}v}{v^{1+\alpha}} \\ &= \lambda^\alpha\Big[-(\mathrm{e}^{-v} - 1 + v)\frac{1}{\alpha v^\alpha}\Big|_0^\infty + \int_0^\infty (1-\mathrm{e}^{-v})\frac{\mathrm{d}v}{\alpha v^\alpha}\Big] \\ &= \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\lambda^\alpha,\end{aligned}$$

where the last equality follows by the calculations in Example 1.2.2. Thus we have

$$\lambda^\alpha = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty (\mathrm{e}^{-\lambda u} - 1 + \lambda u)\frac{\mathrm{d}u}{u^{1+\alpha}}, \qquad \lambda \ge 0.$$

Let $\phi_k(\lambda)$ be defined by (1.3.4) with $\alpha_k = \alpha k^\alpha$ and $g_k(z) = z + \alpha^{-1}(1-z)^\alpha$. Then $\phi_k(\lambda) = \lambda^\alpha$ for $0 \le \lambda \le k$.

Chapter 2

Continuous-state branching processes

In this chapter, we first give a construction of CB-processes as the scaling limits of discrete Galton–Watson branching processes. This approach also gives the interpretations of the CB-processes. We shall study some basic properties of the CB-processes. In particular, some conditional limit theorems will be given. We also give a reconstruction of the sample paths of the CB-processes in terms of excursions.

2.1 Construction by scaling limits

Suppose that $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ is a family of positive integer-valued i.i.d. random variables with distribution given by the probability generating function g. Given the positive integer $x(0) = m$, we define inductively

$$x(n) = \sum_{i=1}^{x(n-1)} \xi_{n,i}, \qquad n = 1, 2, \ldots. \tag{2.1.1}$$

It is easy to show that $\{x(n) : n = 0, 1, 2, \ldots\}$ is a discrete-time positive integer-valued Markov chain with transition matrix $P(i, j)$ defined by

$$\sum_{j=0}^{\infty} P(i, j) z^j = g(z)^i, \qquad i = 0, 1, 2, \ldots, \ |z| \le 1. \tag{2.1.2}$$

The random variable $x(n)$ can be thought of as the number of individuals in generation $n \ge 0$ of an evolving particle system. After one unit time, each of the $x(n)$ particles splits independently of others into a random number of offspring according to the distribution given by g; see, e.g., Athreya and Ney (1972). For $n \ge 0$ the n-step transition matrix

$P^n(i,j)$ is determined by

$$\sum_{j=0}^{\infty} P^n(i,j)z^j = g^n(z)^i, \qquad i = 0,1,2,\ldots,\ |z| \leq 1, \tag{2.1.3}$$

where $g^n(z)$ is defined by $g^n(z) = g(g^{n-1}(z))$ successively with $g^0(z) = z$. We call any positive integer-valued Markov chain with transition matrix given by (2.1.2) or (2.1.3) a *Galton–Watson branching process* (GW-process). If $g'(1-) < \infty$, the first moment of the discrete distribution $\{P^n(i,j); j = 0,1,2,\ldots\}$ is given by

$$\sum_{j=1}^{\infty} jP^n(i,j) = ig'(1-)^n, \tag{2.1.4}$$

which can be obtained by differentiating both sides of (2.1.3).

Now suppose we have a sequence of GW-processes $\{x_k(n) : n \geq 0\}$ with offspring distribution given by the sequence of probability generating functions $\{g_k\}$. Let $z_k(n) = x_k(n)/k$ for $n \geq 0$. Then $\{z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k := \{0, 1/k, 2/k, \ldots\}$ and n-step transition probability $P_k^n(x, dy)$ determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y} P_k^n(x, \mathrm{d}y) = g_k^n(\mathrm{e}^{-\lambda/k})^{kx}, \qquad \lambda \geq 0. \tag{2.1.5}$$

Suppose that $\{\gamma_k\}$ is a positive sequence so that $\gamma_k \to \infty$ as $k \to \infty$. Let $[\gamma_k t]$ denote the integer part of $\gamma_k t \geq 0$. We are interested in the asymptotic behavior of the sequence of continuous time processes $\{z_k([\gamma_k t]) : t \geq 0\}$. By (2.1.5) we have

$$\int_{E_k} \mathrm{e}^{-\lambda y} P_k^{[\gamma_k t]}(x, \mathrm{d}y) = \exp\{-x v_k(t, \lambda)\}, \tag{2.1.6}$$

where

$$v_k(t,\lambda) = -k \log g_k^{[\gamma_k t]}(\mathrm{e}^{-\lambda/k}), \qquad \lambda \geq 0. \tag{2.1.7}$$

Clearly, if $z_k(0) = x \in E_k$, then the probability $P_k^{[\gamma_k t]}(x, \cdot)$ gives the distribution of $z_k([\gamma_k t])$ on $\mathbb{R}_+$. Let us consider the function sequences

$$G_k(z) = k\gamma_k[g_k(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}], \qquad z \geq 0, \tag{2.1.8}$$

and

$$\phi_k(z) = k\gamma_k[g_k(1 - z/k) - (1 - z/k)], \quad 0 \leq z \leq k. \tag{2.1.9}$$

Proposition 2.1.1 *The sequence $\{G_k\}$ is uniformly Lipschitz on each bounded interval if and only if so is $\{\phi_k\}$. In this case, we have* $\lim_{k\to\infty} |\phi_k(z) - G_k(z)| = 0$ *uniformly on each bounded interval.*

Proof. From (2.1.8) and (2.1.9) it is simple to see that

$$G'_k(z) = \gamma_k \mathrm{e}^{-z/k}[1 - g'_k(\mathrm{e}^{-z/k})], \qquad z \geq 0, \tag{2.1.10}$$

and

$$\phi'_k(z) = \gamma_k[1 - g'_k(1 - z/k)], \qquad 0 \leq z \leq k. \tag{2.1.11}$$

Clearly, the sequence $\{G'_k\}$ is uniformly bounded on each bounded interval if and only if so is $\{\phi'_k\}$. Then the first assertion is immediate. We next assume $\{G_k\}$ is uniformly Lipschitz on each bounded interval. Let $a \geq 0$. By the mean-value theorem, for $k \geq a$ and $0 \leq z \leq a$ we have

$$\begin{aligned} G_k(z) - \phi_k(z) &= k\gamma_k\big[g_k(\mathrm{e}^{-z/k}) - g_k(1 - z/k) - \mathrm{e}^{-z/k} + (1 - z/k)\big] \\ &= k\gamma_k[g'_k(\eta_k) - 1](\mathrm{e}^{-z/k} - 1 + z/k), \end{aligned} \tag{2.1.12}$$

where

$$1 - a/k \leq 1 - z/k \leq \eta_k \leq \mathrm{e}^{-z/k} \leq 1.$$

Choose $k_0 \geq a$ so that $\mathrm{e}^{-2a/k_0} \leq 1 - a/k_0$. Then $\mathrm{e}^{-2a/k} \leq 1 - a/k$ for $k \geq k_0$ and hence

$$\gamma_k|g'_k(\eta_k) - 1| \leq \sup_{0 \leq z \leq 2a} \gamma_k|g'_k(\mathrm{e}^{-z/k}) - 1|, \qquad k \geq k_0.$$

Since $\{G_k\}$ is uniformly Lipschitz on each bounded interval, the sequence (2.1.10) is uniformly bounded on $[0, 2a]$. Then $\{\gamma_k|g'_k(\eta_k) - 1| : k \geq k_0\}$ is a bounded sequence. Now the desired result follows from (2.1.12). □

By the above proposition, if either $\{G_k\}$ or $\{\phi_k\}$ is uniformly Lipschitz on each bounded interval, then they converge or diverge simultaneously and in the convergent case they have the same limit. For the convenience of statement of the results, we formulate the following conditions:

Condition 2.1.2 *The sequence $\{G_k\}$ is uniformly Lipschitz on $[0, a]$ for every $a \geq 0$ and there is a function ϕ on $[0, \infty)$ so that $G_k(z) \to \phi(z)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.*

Proposition 2.1.3 *Suppose that Condition 2.1.2 is satisfied. Then the function ϕ has representation*

$$\phi(z) = bz + cz^2 + \int_0^\infty \big(\mathrm{e}^{-zu} - 1 + zu\big)m(\mathrm{d}u), \quad z \geq 0, \tag{2.1.13}$$

where $c \geq 0$ and b are constants and $(u \wedge u^2)m(\mathrm{d}u)$ is a finite measure on $(0, \infty)$.

Proof. By Proposition 2.1.1, the sequence $\{\phi_k\}$ is uniformly Lipschitz on $[0, a]$ and $\phi_k(z) \to \phi(z)$ uniformly on $[0, a]$ for every $a \geq 0$. Then the result follows by Corollary 1.3.5. $\square$

Proposition 2.1.4 *For any function ϕ with representation* (2.1.13) *there is a sequence $\{G_k\}$ in the form of* (2.1.8) *satisfying Condition 2.1.2.*

Proof. By Proposition 2.1.1 it suffices to construct a sequence $\{\phi_k\}$ in the form of (2.1.9) uniformly Lipschitz on $[0, a]$ and $\phi_k(z) \to \phi(z)$ uniformly on $[0, a]$ for every $a \geq 0$. To simplify the formulations we decompose the function ϕ into two parts. Let $\phi_0(z) = \phi(z) - bz$. We first define

$$\gamma_{0,k} = (1 + 2c)k + \int_0^\infty u(1 - \mathrm{e}^{-ku})m(\mathrm{d}u)$$

and

$$g_{0,k}(z) = z + k^{-1}\gamma_{0,k}^{-1}\phi_0(k(1 - z)), \qquad |z| \leq 1.$$

It is easy to see that $z \mapsto g_{0,k}(z)$ is an analytic function in $(-1, 1)$ satisfying $g_{0,k}(1) = 1$ and

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} g_{0,k}(0) \geq 0, \qquad n \geq 0.$$

Therefore $g_{0,k}(\cdot)$ is a probability generating function. Let $\phi_{0,k}$ be defined by (2.1.9) with (γ_k, g_k) replaced by $(\gamma_{0,k}, g_{0,k})$. Then $\phi_{0,k}(z) = \phi_0(z)$ for $0 \leq z \leq k$. That completes the proof if $b = 0$. In the case $b \neq 0$, we set

$$g_{1,k}(z) = \frac{1}{2}\Big(1 + \frac{b}{|b|}\Big) + \frac{1}{2}\Big(1 - \frac{b}{|b|}\Big)z^2.$$

Let $\gamma_{1,k} = |b|$ and let $\phi_{1,k}(z)$ be defined by (2.1.9) with (γ_k, g_k) replaced by $(\gamma_{1,k}, g_{1,k})$. Thus we have

$$\phi_{1,k}(z) = bz + \frac{1}{2k}(|b| - b)z^2.$$

Finally, let $\gamma_k = \gamma_{0,k} + \gamma_{1,k}$ and $g_k = \gamma_k^{-1}(\gamma_{0,k}g_{0,k} + \gamma_{1,k}g_{1,k})$. Then the sequence $\phi_k(z)$ defined by (2.1.9) is equal to $\phi_{0,k}(z) + \phi_{1,k}(z)$ which satisfies the required condition. $\square$

Lemma 2.1.5 *Suppose that the sequence $\{G_k\}$ defined by* (2.1.8) *is uniformly Lipschitz on $[0, 1]$. Then there are constants $B, N \geq 0$ such that $v_k(t, \lambda) \leq \lambda \mathrm{e}^{Bt}$ for every $t, \lambda \geq 0$ and $k \geq N$.*

Proof. Let $b_k := G_k'(0+)$ for $k \geq 1$. Since $\{G_k\}$ is uniformly Lipschitz on $[0,1]$, the sequence $\{b_k\}$ is bounded. From (2.1.8) we have $b_k = \gamma_k[1 - g_k'(1-)]$. By (2.1.4) it is not hard to obtain

$$\int_{E_k} yP_k^{[\gamma_k t]}(x, \mathrm{d}y) = xg_k'(1-)^{[\gamma_k t]} = x\Big(1 - \frac{b_k}{\gamma_k}\Big)^{[\gamma_k t]}.$$

Let $B \geq 0$ be a constant such that $2|b_k| \leq B$ for all $k \geq 1$. Since $\gamma_k \to \infty$ as $k \to \infty$, there is $N \geq 1$ so that

$$0 \leq \Big(1 - \frac{b_k}{\gamma_k}\Big)^{\frac{\gamma_k}{B}} \leq \Big(1 + \frac{B}{2\gamma_k}\Big)^{\frac{\gamma_k}{B}} \leq \mathrm{e}, \qquad k \geq N.$$

It follows that, for $t \geq 0$ and $k \geq N$,

$$\int_{E_k} yP_k^{[\gamma_k t]}(x, \mathrm{d}y) \leq x \exp\Big\{\frac{B}{\gamma_k}[\gamma_k t]\Big\} \leq x\mathrm{e}^{Bt}. \tag{2.1.14}$$

Then the desired estimate follows from (2.1.6), (2.1.14) and Jensen's inequality. □

Theorem 2.1.6 *Suppose that Condition 2.1.2 holds. Then for every $a \geq 0$ we have $v_k(t, \lambda) \to$ some $v_t(\lambda)$ uniformly on $[0, a]^2$ as $k \to \infty$ and the limit function solves the integral equation*

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))\mathrm{d}s, \qquad \lambda, t \geq 0. \tag{2.1.15}$$

Proof. The following argument is a modification of that of Aliev and Shchurenkov (1982) and Aliev (1985). For any $n \geq 0$ we may write

$$\begin{aligned}\log g_k^{n+1}(\mathrm{e}^{-\lambda/k}) &= \log\big[g_k(g_k^n(\mathrm{e}^{-\lambda/k}))g_k^n(\mathrm{e}^{-\lambda/k})^{-1}\big] + \log g_k^n(\mathrm{e}^{-\lambda/k}) \\ &= (k\gamma_k)^{-1}\bar{G}_k\big(-k\log g_k^n(\mathrm{e}^{-\lambda/k})\big) + \log g_k^n(\mathrm{e}^{-\lambda/k}),\end{aligned}$$

where

$$\bar{G}_k(z) = k\gamma_k \log\big[g_k(\mathrm{e}^{-z/k})\mathrm{e}^{z/k}\big].$$

From this and (2.1.7) it follows that

$$v_k(t + \gamma_k^{-1}, \lambda) = v_k(t, \lambda) - \gamma_k^{-1}\bar{G}_k(v_k(t, \lambda)).$$

By applying the above equation to $t = 0, 1/\gamma_k, \ldots, ([\gamma_k t]-1)/\gamma_k$ and adding the resulting equations we obtain

$$v_k(t, \lambda) = \lambda - \sum_{i=1}^{[\gamma_k t]} \gamma_k^{-1}\bar{G}_k(v_k(\gamma_k^{-1}(i-1), \lambda)).$$

Then we can write

$$v_k(t,\lambda) = \lambda + \varepsilon_k(t,\lambda) - \int_0^t \bar{G}_k(v_k(s,\lambda))ds, \tag{2.1.16}$$

where

$$\varepsilon_k(t,\lambda) = \big(t - \gamma_k^{-1}[\gamma_k t]\big)\bar{G}_k\big(v_k(\gamma_k^{-1}[\gamma_k t],\lambda)\big).$$

It is not hard to see

$$\bar{G}_k(z) = k\gamma_k \log\big[1 + (k\gamma_k)^{-1}G_k(z)\mathrm{e}^{z/k}\big].$$

By Condition 2.1.2, for any $0 < \varepsilon \leq 1$ we can enlarge $N \geq 1$ so that

$$|\bar{G}_k(z) - \phi(z)| \leq \varepsilon, \qquad 0 \leq z \leq a\mathrm{e}^{Ba}, k \geq N. \tag{2.1.17}$$

It then follows that

$$|\varepsilon_k(t,\lambda)| \leq \gamma_k^{-1}M, \qquad 0 \leq t,\lambda \leq a, \tag{2.1.18}$$

where

$$M = 1 + \sup_{0\leq z\leq a\mathrm{e}^{Ba}} |\phi(z)|.$$

For $n \geq k \geq N$ let

$$K_{k,n}(t,\lambda) = \sup_{0\leq s\leq t} |v_n(s,\lambda) - v_k(s,\lambda)|.$$

By (2.1.16), (2.1.17) and (2.1.18) we obtain

$$K_{k,n}(t,\lambda) \leq 2(\gamma_k^{-1}M + \varepsilon a) + L\int_0^t K_{k,n}(s,\lambda)ds, \qquad 0 \leq t,\lambda \leq a,$$

where $L = \sup_{0\leq s\leq a\mathrm{e}^{Ba}} |\phi'(z)|$. By Gronwall's inequality,

$$K_{k,n}(t,\lambda) \leq 2(\gamma_k^{-1}M + \varepsilon a)\exp\{Lt\}, \qquad 0 \leq t,\lambda \leq a.$$

Then $v_k(t,\lambda) \to$ some $v_t(\lambda)$ uniformly on $[0,a]^2$ as $k \to \infty$ for every $a \geq 0$. From (2.1.16) we get (2.1.15). □

Theorem 2.1.7 *Suppose that ϕ is given by* (2.1.13). *Then for any $\lambda \geq 0$ there is a unique locally bounded positive solution $t \mapsto v_t(\lambda)$ to* (2.1.15). *Moreover, the solution satisfies the semigroup property*

$$v_{r+t}(\lambda) = v_r \circ v_t(\lambda) = v_r(v_t(\lambda)), \qquad r,t,\lambda \geq 0. \tag{2.1.19}$$

Proof. By Propositions 2.1.4 and 2.1.6 there is a locally bounded positive solution to (2.1.15). The proof of the uniqueness of the solution is a standard application of Gronwall's inequality. The relation (2.1.19) follows from the uniqueness of the solution to (2.1.15). □

Theorem 2.1.8 *Suppose that ϕ is given by* (2.1.13). *Then there is a Feller transition semigroup $(Q_t)_{t\geq 0}$ on $\mathbb{R}_+$ defined by*

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-xv_t(\lambda)}, \qquad \lambda \geq 0, x \geq 0. \tag{2.1.20}$$

Moreover, if $E_k \ni x_k \to x \geq 0$, we have $P_k^{[\gamma_k t]}(x_k, \cdot) \to Q_t(x, \cdot)$ weakly.

Proof. By Proposition 2.1.4 and Theorems 1.1.2 and 2.1.6 there is a probability kernel $Q_t(x, \mathrm{d}y)$ on $\mathbb{R}_+$ defined by (2.1.20). Moreover, we have $P_k^{[\gamma_k t]}(x_k, \cdot) \to Q_t(x, \cdot)$ weakly if $x_k \to x$. The semigroup property of the family of kernels $(Q_t)_{t\geq 0}$ follows from (2.1.19) and (2.1.20). For $\lambda > 0$ and $x \geq 0$ set $e_\lambda(x) = \mathrm{e}^{-\lambda x}$. We denote by D_1 the linear span of $\{e_\lambda : \lambda > 0\}$. Clearly, the operator Q_t preserves D_1 for every $t \geq 0$. By the continuity of $t \mapsto v_t(\lambda)$ it is easy to show that $t \mapsto Q_t e_\lambda(x)$ is continuous for $\lambda > 0$ and $x \geq 0$. Then $t \mapsto Q_t f(x)$ is continuous for every $f \in D_1$ and $x \geq 0$. Let $C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ vanishing at infinity. By the Stone–Weierstrass theorem, the set D_1 is uniformly dense in $C_0(\mathbb{R}_+)$; see, e.g., Hewitt and Stromberg (1965, pp.98-99). Then each operator Q_t preserves $C_0(\mathbb{R}_+)$ and $t \mapsto Q_t f(x)$ is continuous for every $f \in C_0(\mathbb{R}_+)$ and $x \geq 0$. That gives the Feller property of the semigroup $(Q_t)_{t\geq 0}$. □

A Markov process is called a *continuous-state branching process* (CB-process) with *branching mechanism* ϕ if it has transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.1.20). It is simple to see that

$$Q_t(x_1 + x_2, \cdot) = Q_t(x_1, \cdot) * Q_t(x_2, \cdot), \qquad t, x_1, x_2 \geq 0, \tag{2.1.21}$$

which is called the *branching property* of $(Q_t)_{t\geq 0}$. The family of functions $(v_t)_{t\geq 0}$ is called the *cumulant semigroup* of the CB-process. By Theorem 2.1.8 the process has a càdlàg realization. Let $\Omega = D([0, \infty), \mathbb{R}_+)$ denote the space of càdlàg paths from $[0, \infty)$ to $\mathbb{R}_+$ furnished with the Skorokhod topology. The following theorem gives an interpretation of the CB-process as the approximation of the GW-process.

Theorem 2.1.9 *Suppose that Condition 2.1.2 holds. Let $\{x(t) : t \geq 0\}$ be a CB-process with transition semigroup $(Q_t)_{t\geq 0}$ defined by* (2.1.20). *If $z_k(0)$ converges to $x(0)$ in distribution, then $\{z_k([\gamma_k t]) : t \geq 0\}$ converges to $\{x(t) : t \geq 0\}$ in distribution on $D([0, \infty), \mathbb{R}_+)$.*

Proof. For $\lambda > 0$ and $x \geq 0$ set $e_\lambda(x) = \mathrm{e}^{-\lambda x}$. Let $C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ vanishing at infinity. By Theorem 2.1.6 it is easy to show

$$\lim_{k\to\infty} \sup_{x\in E_k} \left| P_k^{[\gamma_k t]} e_\lambda(x) - Q_t e_\lambda(x) \right| = 0, \qquad \lambda > 0.$$

Then the Stone–Weierstrass theorem implies

$$\lim_{k\to\infty} \sup_{x\in E_k} \left| P_k^{[\gamma_k t]} f(x) - Q_t f(x) \right| = 0, \qquad f \in C_0(\mathbb{R}_+).$$

By Ethier and Kurtz (1986, p.226 and pp.233-234) we conclude that $\{z_k([\gamma_k t]) : t \geq 0\}$ converges to the CB-process $\{x(t) : t \geq 0\}$ in distribution on $D([0,\infty), \mathbb{R}_+)$. □

The convergence of rescaled Galton–Watson branching processes to diffusion processes was first studied by Feller (1951). Jiřina (1958) introduced CB-processes in both discrete and continuous times. Lamperti (1967a) showed that the continuous-time processes are weak limits of rescaled Galton–Watson branching processes. We have followed Aliev and Shchurenkov (1982) and Li (2006) in some of the above calculations; see also Li (2011).

2.2 Simple properties of CB-processes

In this section we prove some basic properties of CB-processes. Most of the results presented here can be found in Grey (1974) and Li (2000). We shall follow the treatments in Li (2011). Suppose that ϕ is a branching mechanism defined by (2.1.13). Then a CB-process has transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.1.15) and (2.1.20). It is easy to see for each $x \geq 0$, the probability measure $Q_t(x,\cdot)$ is infinitely divisible. Then $(v_t)_{t\geq 0}$ can be expressed canonically as

$$v_t(\lambda) = h_t\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda u}) l_t(\mathrm{d}u), \qquad t \geq 0, \lambda \geq 0, \tag{2.2.1}$$

where $h_t \geq 0$ and $ul_t(\mathrm{d}u)$ is a finite measure on $(0,\infty)$. From (2.1.15) we see that $t \mapsto v_t(\lambda)$ is first continuous and then continuously differentiable. Moreover, we have the backward differential equation:

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{2.2.2}$$

By (2.2.2) and the semigroup property $v_r \circ v_t = v_{r+t}$ for $r, t \geq 0$ we also have the forward differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(\lambda) \frac{\partial}{\partial \lambda} v_t(\lambda), \quad v_0(\lambda) = \lambda. \tag{2.2.3}$$

By differentiating both sides of (2.1.15) it is easy to find

$$\frac{\partial}{\partial\lambda} v_t(0+) = \mathrm{e}^{-bt}, \qquad t \geq 0, \tag{2.2.4}$$

which together with (2.1.20) yields

$$\int_0^\infty y Q_t(x, \mathrm{d}y) = x\mathrm{e}^{-bt}, \qquad t \geq 0, x \geq 0. \tag{2.2.5}$$

We say the CB-process is *critical*, *subcritical* or *supercritical* according as $b = 0$, ≥ 0 or ≤ 0.

Proposition 2.2.1 *For every $t \geq 0$ the function $\lambda \mapsto v_t(\lambda)$ is strictly increasing on $[0, \infty)$.*

Proof. By the continuity of $t \mapsto v_t(\lambda)$, for any $\lambda_0 > 0$ there is $t_0 > 0$ so that $v_t(\lambda_0) > 0$ for $0 \leq t \leq t_0$. Then (2.1.20) implies $Q_t(x, \{0\}) < 1$ for $x > 0$ and $0 \leq t \leq t_0$, and so $\lambda \mapsto v_t(\lambda)$ is strictly increasing for $0 \leq t \leq t_0$. By the semigroup property of $(v_t)_{t\geq 0}$ we infer $\lambda \mapsto v_t(\lambda)$ is strictly increasing for all $t \geq 0$. □

Corollary 2.2.2 *The transition semigroup $(Q_t)_{t\geq 0}$ defined by* (2.1.20) *is a Feller semigroup.*

Proof. By Proposition 2.2.1 for $t \geq 0$ and $\lambda > 0$ we have $v_t(\lambda) > 0$. From (2.1.20) we see the operator Q_t maps $\{x \mapsto \mathrm{e}^{-\lambda x} : \lambda > 0\}$ to itself. By the Stone–Weierstrass theorem, the linear span of $\{x \mapsto \mathrm{e}^{-\lambda x} : \lambda > 0\}$ is dense in $C_0(\mathbb{R}_+)$ in the supremum norm. Then Q_t maps $C_0(\mathbb{R}_+)$ to itself. The Feller property of $(Q_t)_{t\geq 0}$ follows by the continuity of $t \mapsto v_t(\lambda)$. □

Proposition 2.2.3 *Suppose that $\lambda > 0$ and $\phi(\lambda) \neq 0$. Then the equation $\phi(z) = 0$ has no root between λ and $v_t(\lambda)$. Moreover, we have*

$$\int_{v_t(\lambda)}^{\lambda} \phi(z)^{-1} \mathrm{d}z = t, \qquad t \geq 0. \tag{2.2.6}$$

Proof. By (2.1.13) we see $\phi(0) = 0$ and $z \mapsto \phi(z)$ is a convex function. Since $\phi(\lambda) \neq 0$ for some $\lambda > 0$ according to the assumption, the equation $\phi(z) = 0$ has at most one root in $(0, \infty)$. Suppose that $\lambda_0 \geq 0$ is a root of $\phi(z) = 0$. Then (2.2.3) implies $v_t(\lambda_0) = \lambda_0$ for all $t \geq 0$. By Proposition 2.2.1 we have $v_t(\lambda) > \lambda_0$ for $\lambda > \lambda_0$ and $0 < v_t(\lambda) < \lambda_0$ for $0 < \lambda < \lambda_0$. Then $\lambda > 0$ and $\phi(\lambda) \neq 0$ imply there is no root of $\phi(z) = 0$ between λ and $v_t(\lambda)$. From (2.2.2) we get (2.2.6). □

Proposition 2.2.4 *For any $t \geq 0$ and $\lambda \geq 0$ let $v_t'(\lambda) = (\partial/\partial\lambda)v_t(\lambda)$. Then we have*

$$v_t'(\lambda) = \exp\Big\{ - \int_0^t \phi'(v_s(\lambda))\mathrm{d}s\Big\}, \tag{2.2.7}$$

where

$$\phi'(z) = b + 2cz + \int_0^\infty u\big(1 - \mathrm{e}^{-zu}\big)m(\mathrm{d}u). \tag{2.2.8}$$

Proof. Based on (2.1.15) and (2.2.2) it is elementary to see that

$$\frac{\partial}{\partial t}v_t'(\lambda) = -\phi'(v_t(\lambda))v_t'(\lambda) = \frac{\partial}{\partial\lambda}\frac{\partial}{\partial t}v_t(\lambda).$$

It follows that

$$\frac{\partial}{\partial t}\big[\log v_t'(\lambda)\big] = v_t'(\lambda)^{-1}\frac{\partial}{\partial t}v_t'(\lambda) = -\phi'(v_t(\lambda)).$$

Then we have (2.2.7) since $v_0'(\lambda) = 1$. □

Since $(Q_t)_{t\geq 0}$ is a Feller semigroup by Corollary 2.2.2, the CB-process has a Hunt realization $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$. Let $\tau_0 := \inf\{s \geq 0 : x(s) = 0\}$ denote the *extinction time* of the CB-process.

Theorem 2.2.5 *For every $t \geq 0$ the limit $\bar{v}_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0, \infty]$. Moreover, the mapping $t \mapsto \bar{v}_t$ is decreasing and for any $t \geq 0$ and $x > 0$ we have*

$$\mathbf{Q}_x\{\tau_0 \leq t\} = \mathbf{Q}_x\{x(t) = 0\} = \exp\{-x\bar{v}_t\}. \tag{2.2.9}$$

Proof. By Proposition 2.2.1 the limit $\bar{v}_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0, \infty]$ for every $t \geq 0$. For $t \geq r \geq 0$ we have

$$\bar{v}_t = \uparrow\lim_{\lambda\to\infty} v_r(v_{t-r}(\lambda)) = v_r(\bar{v}_{t-r}) \leq \bar{v}_r. \tag{2.2.10}$$

Since zero is a trap for the CB-process, we get (2.2.9) by letting $\lambda \to \infty$ in (2.1.20). □

For the convenience of statement of the results in the sequel, we formulate the following condition on the branching mechanism:

Condition 2.2.6 *There is some constant $\theta > 0$ so that*

$$\phi(z) > 0 \textit{ for } z \geq \theta \textit{ and } \int_\theta^\infty \phi(z)^{-1}\mathrm{d}z < \infty.$$

Theorem 2.2.7 *We have $\bar{v}_t < \infty$ for some and hence all $t > 0$ if and only if Condition 2.2.6 holds.*

Proof. By (2.2.10) it is simple to see that $\bar{v}_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda) < \infty$ for all $t > 0$ if and only if this holds for some $t > 0$. If Condition 2.2.6 holds, we can let $\lambda \to \infty$ in (2.2.6) to obtain

$$\int_{\bar{v}_t}^{\infty} \phi(z)^{-1}\mathrm{d}z = t \tag{2.2.11}$$

and hence $\bar{v}_t < \infty$ for $t > 0$. For the converse, suppose that $\bar{v}_t < \infty$ for some $t > 0$. By (2.2.2) there exists some $\theta > 0$ so that $\phi(\theta) > 0$, for otherwise we would have $v_t(\lambda) \geq \lambda$, yielding a contradiction. Then $\phi(z) > 0$ for all $z \geq \theta$ by the convexity of the branching mechanism. As in the above we see that (2.2.11) still holds, so Condition 2.2.6 is satisfied. $\square$

Theorem 2.2.8 *Let $\bar{v} = \downarrow\lim_{t\to\infty} \bar{v}_t \in [0,\infty]$. Then for any $x > 0$ we have*

$$\mathbf{Q}_x\{\tau_0 < \infty\} = \exp\{-x\bar{v}\}. \tag{2.2.12}$$

Moreover, we have $\bar{v} < \infty$ if and only if Condition 2.2.6 holds, and in this case $\bar{v}$ is the largest root of $\phi(z) = 0$.

Proof. The first assertion follows immediately from Theorem 2.2.5. By Theorem 2.2.7 we have $\bar{v}_t < \infty$ for some and hence all $t > 0$ if and only if Condition 2.2.6 holds. This is clearly equivalent to $\bar{v} < \infty$. From (2.2.11) it is easy to see that $\bar{v}$ is the largest root of $\phi(z) = 0$. $\square$

Corollary 2.2.9 *Suppose that Condition 2.2.6 holds. Then for any $x > 0$ we have $\mathbf{Q}_x\{\tau_0 < \infty\} = 1$ if and only if $b \geq 0$.*

Let $(Q_t^\circ)_{t\geq 0}$ be the restriction to $(0,\infty)$ of the semigroup $(Q_t)_{t\geq 0}$. A family of σ-finite measures $(H_t)_{t>0}$ on $(0,\infty)$ is called an *entrance law* for $(Q_t^\circ)_{t\geq 0}$ if $H_r Q_t^\circ = H_{r+t}$ for all $r,t > 0$. The special case of the canonical representation (2.2.1) with $h_t = 0$ for all $t > 0$ is particularly interesting. In this case, we have

$$v_t(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda u}) l_t(\mathrm{d}u), \qquad t > 0, \lambda \geq 0. \tag{2.2.13}$$

Theorem 2.2.10 *The cumulant semigroup admits representation* (2.2.13) *if and only if*

$$\phi'(\infty) := b + 2c\cdot\infty + \int_0^\infty u\, m(\mathrm{d}u) = \infty \tag{2.2.14}$$

with $0\cdot\infty = 0$ by convention. If condition (2.2.14) *is satisfied, then $(l_t)_{t>0}$ is an entrance law for the restricted semigroup $(Q_t^\circ)_{t\geq 0}$.*

Proof. From (2.2.8) it is clear that the limit $\phi'(\infty) = \lim_{z\to\infty}\phi'(z)$ always exists in $(-\infty,\infty]$. By (2.2.1) we have

$$v_t'(\lambda) = h_t + \int_0^\infty u\mathrm{e}^{-\lambda u} l_t(\mathrm{d}u), \qquad t\geq 0, \lambda\geq 0. \tag{2.2.15}$$

From (2.2.7) and (2.2.15) it follows that

$$h_t = v_t'(\infty) = \exp\Big\{-\int_0^t \phi'(\bar{v}_s)\mathrm{d}s\Big\}. \tag{2.2.16}$$

Then $h_t = 0$ for any $t>0$ implies $\phi'(\infty)=\infty$. For the converse, assume that $\phi'(\infty)=\infty$. If Condition 2.2.6 holds, by Theorem 2.2.7 for every $t>0$ we have $\bar{v}_t<\infty$, so $h_t=0$ by (2.2.1). If Condition 2.2.6 does not hold, then $\bar{v}_t=\infty$ for $t>0$ by Theorem 2.2.7. Then (2.2.16) implies $h_t=0$ for $t>0$. That proves the first assertion of the theorem. If $(v_t)_{t>0}$ admits the representation (2.2.13), we can use the semigroup property of $(v_t)_{t\geq 0}$ to see

$$\begin{aligned}\int_0^\infty (1-\mathrm{e}^{-\lambda u}) l_{r+t}(\mathrm{d}u) &= \int_0^\infty (1-\mathrm{e}^{-uv_t(\lambda)}) l_r(\mathrm{d}u)\\ &= \int_0^\infty l_r(\mathrm{d}x)\int_0^\infty (1-\mathrm{e}^{-\lambda u}) Q_t^\circ(x,\mathrm{d}u)\end{aligned}$$

for $r,t>0$ and $\lambda\geq 0$. Then $(l_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t\geq 0}$. □

Corollary 2.2.11 *If Condition 2.2.6 holds, the cumulant semigroup admits the representation* (2.2.13) *and $t\mapsto \bar{v}_t = l_t(0,\infty)$ is the minimal solution of the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_t = -\phi(\bar{v}_t), \qquad t>0 \tag{2.2.17}$$

with singular initial condition $\bar{v}_{0+}=\infty$.

Proof. Under Condition 2.2.6, for every $t>0$ we have $\bar{v}_t<\infty$ by Theorem 2.2.7. Moreover, the condition and the convexity of $z\mapsto\phi(z)$ imply $\phi'(\infty)=\infty$. Then we have the representation (2.2.13) by Theorem 2.2.10. The semigroup property of $(v_t)_{t\geq 0}$ implies $\bar{v}_{t+s} = v_t(\bar{v}_s)$ for $t>0$ and $s>0$. Then $t\mapsto\bar{v}_t$ satisfies (2.2.17). From (2.2.11) it is easy to see $\bar{v}_{0+}=\infty$. Using the relation $\bar{v}_t = \lim_{\lambda\to\infty} v_t(\lambda)$ it is easy to show that any solution $t\mapsto u_t$ of (2.2.17) with $u_{0+}=\infty$ satisfies $u_t\geq\bar{v}_t$ for $t>0$. □

Corollary 2.2.12 *Suppose that Condition 2.2.6 holds. Then for any $t>0$ the function $\lambda\mapsto v_t(\lambda)$ is strictly increasing and concave on $[0,\infty)$, and $\bar{v}$ is the largest solution of the equation $v_t(\lambda)=\lambda$. Moreover, we have $\bar{v} =\uparrow\lim_{t\to\infty} v_t(\lambda)$ for $0<\lambda<\bar{v}$ and $\bar{v} =\downarrow\lim_{t\to\infty} v_t(\lambda)$ for $\lambda>\bar{v}$.*

Proof. By Corollary 2.2.11 we have the canonical representation (2.2.13) for every $t > 0$. Since $\lambda \mapsto v_t(\lambda)$ is strictly increasing by Proposition 2.2.1, the measure $l_t(\mathrm{d}u)$ is non-trivial, so $\lambda \mapsto v_t(\lambda)$ is strictly concave. The equality $\bar{v} = v_t(\bar{v})$ follows by letting $s \to \infty$ in $\bar{v}_{t+s} = v_t(\bar{v}_s)$, where $\bar{v}_{t+s} \leq \bar{v}_s$. Then $\bar{v}$ is clearly the largest solution to $v_t(\lambda) = \lambda$. When $b \geq 0$, we have $\bar{v} = 0$ by Theorem 2.2.8 and Corollary 2.2.9. Furthermore, since $\phi(z) \geq 0$, from (2.2.2) we see $t \mapsto v_t(\lambda)$ is decreasing, and hence $\downarrow \lim_{t\to\infty} v_t(\lambda) =\downarrow \lim_{t\to\infty} \bar{v}_t = 0$. If $b < 0$ and $0 < \lambda < \bar{v}$, we have $\lambda \leq v_t(\lambda) < v_t(\bar{v}) = \bar{v}$ for all $t \geq 0$. Then the limit $v_\infty(\lambda) =\uparrow \lim_{t\uparrow\infty} v_t(\lambda)$ exists. From the relation $v_t(v_s(\lambda)) = v_{t+s}(\lambda)$ we have $v_t(v_\infty(\lambda)) = v_\infty(\lambda)$, and hence $v_\infty(\lambda) = \bar{v}$ since $\bar{v}$ is the unique solution to $v_t(\lambda) = \lambda$ in $(0, \infty)$. The assertion for $b < 0$ and $\lambda > \bar{v}$ can be proved similarly. $\square$

Let us consider the entrance law $(l_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$ defined by (2.2.13). In view of (2.1.20), for any $t > 0$ we have

$$\int_0^\infty (1 - \mathrm{e}^{-y\lambda}) l_t(\mathrm{d}y) = \lim_{x\to 0} x^{-1} \int_{\mathbb{R}_+} (1 - \mathrm{e}^{-y\lambda}) Q_t^\circ(x, \mathrm{d}y), \quad \lambda \geq 0.$$

Then we formally have

$$l_t = \lim_{x\to 0} x^{-1} Q_t(x, \cdot) \quad t > 0. \tag{2.2.18}$$

Under Condition 2.2.6, the above relation holds rigorously by the convergence of finite measures on $(0, \infty)$. In Theorem 2.2.10 one usually cannot extend $(l_t)_{t>0}$ to a σ-finite entrance law for the semigroup $(Q_t)_{t\geq 0}$ on $\mathbb{R}_+$. For example, let us assume Condition 2.2.6 holds and $(\bar{l}_t)_{t>0}$ is such an extension. For any $0 < r < \varepsilon < t$ we have

$$\begin{aligned}\bar{l}_t(\{0\}) &\geq \int_0^\infty Q_{t-r}^\circ(x, \{0\}) l_r(\mathrm{d}x) \geq \int_0^\infty \mathrm{e}^{-x\bar{v}_{t-\varepsilon}} l_r(\mathrm{d}x) \\ &= \bar{v}_r - \int_0^\infty (1 - \mathrm{e}^{-u\bar{v}_{t-\varepsilon}}) l_r(\mathrm{d}u) = \bar{v}_r - v_r(\bar{v}_{t-\varepsilon}).\end{aligned}$$

The right-hand side tends to infinity as $r \to 0$. Then $\bar{l}_t(\mathrm{d}x)$ cannot be a σ-finite measure on $\mathbb{R}_+$.

Example 2.2.1 Suppose that there are constants $c > 0$, $0 < \alpha \leq 1$ and b so that $\phi(z) = cz^{1+\alpha} + bz$. Then Condition 2.2.6 is satisfied. Let $q_\alpha^0(t) = \alpha t$ and

$$q_\alpha^b(t) = b^{-1}(1 - \mathrm{e}^{-\alpha b t}), \qquad b \neq 0.$$

By solving the equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -c v_t(\lambda)^{1+\alpha} - b v_t(\lambda), \qquad v_0(\lambda) = \lambda$$

we get

$$v_t(\lambda) = \frac{\mathrm{e}^{-bt}\lambda}{\big[1 + cq_\alpha^b(t)\lambda^\alpha\big]^{1/\alpha}}, \qquad t \geq 0, \lambda \geq 0. \tag{2.2.19}$$

Thus $\bar{v}_t = c^{-1/\alpha}\mathrm{e}^{-bt}q_\alpha^b(t)^{-1/\alpha}$ for $t > 0$. In particular, if $\alpha = 1$, then (2.2.13) holds with

$$l_t(\mathrm{d}u) = \frac{\mathrm{e}^{-bt}}{c^2 q_1^b(t)^2}\exp\Big\{-\frac{u}{cq_1^b(t)}\Big\}\mathrm{d}u, \qquad t > 0, u > 0.$$

2.3 Conditional limit theorems

Let $(Q_t)_{t\geq 0}$ denote the transition semigroup of the CB-process with branching mechanism ϕ given by (2.1.13). Let $(Q_t^\circ)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ to $(0,\infty)$. It is easy to check that $Q_t^b(x,\mathrm{d}y) := \mathrm{e}^{bt}x^{-1}yQ_t^\circ(x,\mathrm{d}y)$ defines a Markov semigroup on $(0,\infty)$. Let $q_t(\lambda) = \mathrm{e}^{bt}v_t(\lambda)$ and let $q_t'(\lambda) = (\partial/\partial\lambda)q_t(\lambda)$. Recall that $z \mapsto \phi'(z)$ is defined by (2.2.8). From (2.2.7) we have

$$q_t'(\lambda) = \exp\Big\{-\int_0^t \phi_0'(v_s(\lambda))\mathrm{d}s\Big\}, \tag{2.3.1}$$

where $\phi_0'(z) = \phi'(z) - b$. By differentiating both sides of (2.1.20) we see

$$\int_0^\infty \mathrm{e}^{-\lambda y}Q_t^b(x,\mathrm{d}y) = \exp\{-xv_t(\lambda)\}q_t'(\lambda), \quad \lambda \geq 0. \tag{2.3.2}$$

It follows that

$$\int_0^\infty \mathrm{e}^{-\lambda y}Q_t^b(x,\mathrm{d}y) = \exp\Big\{-xv_t(\lambda) - \int_0^t \phi_0'(v_s(\lambda))\mathrm{d}s\Big\}. \tag{2.3.3}$$

Using (2.3.3) it is easy to extend $(Q_t^b)_{t\geq 0}$ to a Feller semigroup on $\mathbb{R}_+$. Recall that $(v_t)_{t\geq 0}$ has the representation (2.2.1).

Theorem 2.3.1 *For any $t \geq 0$ we have $Q_t^b(0,\mathrm{d}u) = \mathrm{e}^{bt}h_t\delta_0(\mathrm{d}u) + \mathrm{e}^{bt}ul_t(\mathrm{d}u)$.*

Proof. By (2.2.13) and the definition of $q_t(\lambda)$ we have

$$q_t(\lambda) = \mathrm{e}^{bt}h_t\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-\lambda u}\big)\mathrm{e}^{bt}l_t(\mathrm{d}u), \tag{2.3.4}$$

and hence

$$q_t'(\lambda) = \mathrm{e}^{bt}h_t + \int_0^\infty u\mathrm{e}^{-\lambda u}\mathrm{e}^{bt}l_t(\mathrm{d}u). \tag{2.3.5}$$

Then the result follows from (2.3.2) and (2.3.5). □

Corollary 2.3.2 *Suppose that* $\phi'(z) \to \infty$ *as* $z \to \infty$. *Let* $(l_t)_{t>0}$ *be defined by* (2.2.13). *Then for* $t > 0$ *the probability measure* $Q_t^b(0, \cdot)$ *is supported by* $(0, \infty)$ *and* $Q_t^b(0, \mathrm{d}u) = u\mathrm{e}^{bt} l_t(\mathrm{d}u)$.

Now let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$ be a Hunt realization of the CB-process with the augmented natural σ-algebras. Let $\tau_0 := \inf\{s \geq 0 : x(s) = 0\}$ denote the extinction time of X.

Theorem 2.3.3 *Suppose that* $b \geq 0$ *and Condition 2.2.6 holds. Let* $t \geq 0$ *and* $x > 0$. *Then the distribution of* $x(t)$ *under* $\mathbf{Q}_x\{\cdot | r + t < \tau_0\}$ *converges as* $r \to \infty$ *to* $Q_t^b(x, \cdot)$.

Proof. Since zero is a trap for the CB-process, for any $r > 0$ we can use the Markov property of $\{x(t) : t \geq 0\}$ to see

$$\begin{aligned}\mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)} | r + t < \tau_0\big] &= \frac{\mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)} 1_{\{r+t<\tau_0\}}\big]}{\mathbf{Q}_x[1_{\{r+t<\tau_0\}}]} \\ &= \lim_{\theta\to\infty} \frac{\mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)}(1 - \mathrm{e}^{-\theta x(r+t)})\big]}{\mathbf{Q}_x\big[(1 - \mathrm{e}^{-\theta x(r+t)})\big]} \\ &= \frac{\mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)}(1 - \mathrm{e}^{-x(t)\bar{v}_r})\big]}{1 - \mathrm{e}^{-x\bar{v}_{r+t}}}. \end{aligned} \tag{2.3.6}$$

Recall that $\bar{v}_{r+t} = v_t(\bar{v}_r)$ and $v_t'(0) = \mathrm{e}^{-bt}$. By Theorem 2.2.8 and Corollary 2.2.9 we have $\lim_{r\to\infty} \bar{v}_r = 0$. Then

$$\begin{aligned}\lim_{r\to\infty} \mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)} | r + t < \tau_0\big] &= \lim_{r\to\infty} \frac{\mathbf{Q}_x\big[\mathrm{e}^{-\lambda x(t)} \bar{v}_r^{-1}(1 - \mathrm{e}^{-x(t)\bar{v}_r})\big]}{\bar{v}_r^{-1}(1 - \mathrm{e}^{-x v_t(\bar{v}_r)})} \\ &= \frac{1}{x}\mathrm{e}^{bt}\mathbf{Q}_x[x(t)\mathrm{e}^{-\lambda x(t)}].\end{aligned}$$

That gives the desired convergence result. □

It is easy to see that $t \mapsto Z(t) := \mathrm{e}^{bt}x(t)$ is a positive $(\mathscr{F}_t)$-martingale. By the theory of Markov processes, for each $x > 0$ there is a unique probability measure $\mathbf{Q}_x^b$ on $(\Omega, \mathscr{F})$ so that

$$\mathbf{Q}_x^b(F) = \mathbf{Q}_x[Z(t)F] \tag{2.3.7}$$

for any $\mathscr{F}_t$-measurable bounded random variable F. Moreover, under this new probability measure $\{x(t) : t \geq 0\}$ is a Markov process in $(0, \infty)$ with transition semigroup $(Q_t^b)_{t\geq 0}$. By a modification of the proof of Theorem 2.3.3 we get the following:

Theorem 2.3.4 *Suppose that* $b \geq 0$ *and Condition 2.2.6 holds. Let* $x > 0$ *and* $t \geq 0$. *Then for any* $\mathscr{F}_t$*-measurable bounded random variable* F *we have*

$$\mathbf{Q}_x^b[F] = \lim_{r\to\infty} \mathbf{Q}_x[F | r + t < \tau_0]. \tag{2.3.8}$$

By the above theorem, in the critical and subcritical cases, $\mathbf{Q}_x^b$ is intuitively the law of $\{x(t) : t \geq 0\}$ conditioned on large extinction times. See Lambert (2007), Li (2000, 2011) and Pakes (1999) for more conditional limit theorems.

2.4 A reconstruction from excursions

In this section, we give a reconstruction of the sample paths of the CB-process from excursions. Let $(Q_t)_{t\geq 0}$ denote the transition semigroup of the process with branching mechanism ϕ given by (2.1.13). Recall that $(Q_t^\circ)_{t\geq 0}$ is the restriction of $(Q_t)_{t\geq 0}$ to $(0,\infty)$. Let $D(0,\infty)$ be the space of càdlàg paths $t \mapsto w_t$ from $(0,\infty)$ to $\mathbb{R}_+$ having zero as a trap. Let $(\mathscr{A}^0, \mathscr{A}_t^0)$ be the natural σ-algebras on $D(0,\infty)$ generated by the coordinate process. For any entrance law $(H_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$ there is a unique σ-finite measure $\mathbf{Q}_H(\mathrm{d}w)$ on $\mathscr{A}^0$ such that $\mathbf{Q}_H(\{0\}) = 0$ and

$$\begin{aligned} &\mathbf{Q}_H(w_{t_1} \in \mathrm{d}x_1, w_{t_2} \in \mathrm{d}x_2, \ldots, w_{t_n} \in \mathrm{d}x_n) \\ &\quad = H_{t_1}(\mathrm{d}x_1) Q^\circ_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \end{aligned} \tag{2.4.1}$$

for every $\{t_1 < \cdots < t_n\} \subset (0,\infty)$ and $\{x_1, \ldots, x_n\} \subset (0,\infty)$. See, e.g., Getoor and Glover (1987) for the proof of the existence of $\mathbf{Q}_H$ in the setting of Borel right processes. Roughly speaking, the above formula means that $\{w_t : t > 0\}$ under $\mathbf{Q}_H$ is a Markov process in $(0,\infty)$ with transition semigroup $(Q_t^\circ)_{t\geq 0}$ and one-dimensional distributions $(H_t)_{t>0}$.

Theorem 2.4.1 *Suppose that the condition* (2.2.14) *is satisfied. Let $\mathbf{Q}_{(0)}$ be the σ-finite measure on $D(0,\infty)$ determined by the entrance law $(l_t)_{t>0}$ given by* (2.2.13). *Then for $\mathbf{Q}_{(0)}$-a.e. $w \in D(0,\infty)$ we have $w_t \to 0$ as $t \to 0$.*

Proof. Let $(Q_t^b)_{t\geq 0}$ be the transition semigroup on $\mathbb{R}_+$ defined by (2.3.3). Then we have $Q_t^b(x,\mathrm{d}y) = x^{-1}\mathrm{e}^{bt} y Q_t^\circ(x,\mathrm{d}y)$ for $x, y > 0$. Observe that

$$\int_{D(0,\infty)} \mathrm{e}^{bt} w_t \mathbf{Q}_{(0)}(\mathrm{d}w) = \int_0^\infty \mathrm{e}^{bt} y l_t(\mathrm{d}y) = \mathrm{e}^{bt} \frac{\partial}{\partial\lambda} v_t(0+) = 1.$$

Then for fixed $u > 0$ we can define a probability measure $\mathbf{Q}^u_{(0)}(\mathrm{d}w) := \mathrm{e}^{bu} w_u \mathbf{Q}_{(0)}(\mathrm{d}w)$ on $D(0,\infty)$. Under this measure, the coordinate process $\{w_t : 0 < t \leq u\}$ is an immigration process with transition semigroup $(Q_t^b)_{t\geq 0}$ and one-dimensional distributions

$$Q_t^b(0,\mathrm{d}y) = \mathrm{e}^{bt} y l_t(\mathrm{d}y), \qquad 0 < t \leq u.$$

By the uniqueness of the transition law of the immigration process we have $w_t \to 0$ as $t \to 0$ for $\mathbf{Q}^u_{(0)}$-a.e. $w \in D(0,\infty)$. Note that $\mathbf{Q}^u_{(0)}(\mathrm{d}w)$ and $\mathbf{Q}_{(0)}(\mathrm{d}w)$ are absolutely continuous with respect to each other on $D_u(0,\infty) := \{w \in D(0,\infty) : w_u > 0\}$. Since $D(0,\infty) = \{0\} \cup (\cup_{n=1}^\infty D_{1/n}(0,\infty))$ and $\mathbf{Q}_{(0)}(\{0\}) = 0$, we have $w_t \to 0$ as $t \to 0$ for $\mathbf{Q}_{(0)}$-a.e. $w \in D(0,\infty)$. $\square$

Let $D_0[0,\infty)$ be the set of paths $w \in D(0,\infty)$ satisfying $w(0) = w(t) = 0$ for $t \geq \tau_0(w) := \inf\{s > 0 : w(s) = 0\}$. Those paths are called *excursions*. By Theorem 2.4.1 we can regard $\mathbf{Q}_{(0)}$ as a σ-finite measure on $D_0[0,\infty)$. We call $\mathbf{Q}_{(0)}$ an *excursion law* for the CB-process. In view of (2.2.18), we can formally write

$$\mathbf{Q}_{(0)} = \lim_{x\to 0} x^{-1}\mathbf{Q}_x, \tag{2.4.2}$$

which explains why $w_t \to 0$ as $t \to 0$ for $\mathbf{Q}_{(0)}$-a.e. $w \in D(0,\infty)$.

We can give a reconstruction of the CB-process using a Poisson random measure based on the excursion law specified above. Let $x \geq 0$ and let

$$N(\mathrm{d}w) = \sum_{i=1}^{|N|} \delta_{w_i}$$

be a Poisson random measure on $D_0[0,\infty)$ with intensity $x\mathbf{Q}_{(0)}(\mathrm{d}w)$, where $|N| = N(D_0[0,\infty))$. We define the process $\{X_t : t \geq 0\}$ by $X_0 = x$ and

$$X_t = \int_{D_0[0,\infty)} w(t)N(\mathrm{d}w) = \sum_{i=1}^{|N|} w_i(t), \qquad t > 0. \tag{2.4.3}$$

The following theorem shows that (2.4.3) gives a reconstruction of the sample paths of the CB-process.

Theorem 2.4.2 *For $t \geq 0$ let $\mathscr{G}_t$ be the σ-algebra generated by the collection of random variables $\{N(A) : A \in \mathscr{A}_t^0\}$. Then $\{(X_t, \mathscr{G}_t) : t \geq 0\}$ is a realization of the CB-process.*

Proof. We first remark that the random variable X_t has distribution $Q_t(x,\cdot)$ on $\mathbb{R}_+$. In fact, for any $t > 0$ and $\lambda \geq 0$ we have

$$\begin{aligned}\mathbf{P}\big[\exp\{-\lambda X_t\}\big] &= \exp\Big\{-x\int_{D_0[0,\infty)} (1-\mathrm{e}^{-\lambda w(t)})\mathbf{Q}_{(0)}(\mathrm{d}w)\Big\} \\ &= \exp\Big\{-x\int_0^\infty (1-\mathrm{e}^{-\lambda z})l_t(\mathrm{d}z)\Big\} = \exp\{-xv_t(\lambda)\}.\end{aligned}$$

Let $t > r > 0$ and let h be a bounded positive function on $D_0[0,\infty)$ measurable relative to $\mathscr{A}_r^0$. For any $\lambda \geq 0$ we have

$$\mathbf{P}\Big[\exp\Big\{-\int_{D_0[0,\infty)} h(w)N(\mathrm{d}w) - \lambda X_t\Big\}\Big]$$

$$
\begin{aligned}
&= \exp\Big\{ -x \int_{D_0[0,\infty)} \big(1 - \mathrm{e}^{-h(w)-\lambda w(t)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w) \Big\} \\
&= \exp\Big\{ -x \int_{D_0[0,\infty)} \big(1 - \mathrm{e}^{-h(w)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w) \Big\} \\
&\quad \cdot \exp\Big\{ -x \int_{D_0[0,\infty)} \mathrm{e}^{-h(w)} \big(1 - \mathrm{e}^{-\lambda w(t)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w) \Big\},
\end{aligned}
$$

where we made the convention $\mathrm{e}^{-\infty} = 0$. By the Markov property of $\mathbf{Q}_{(0)}$ we have

$$
\begin{aligned}
&\int_{D_0[0,\infty)} \mathrm{e}^{-h(w)} \big(1 - \mathrm{e}^{-\lambda w(t)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w) \\
&\qquad = \int_{D_0[0,\infty)} \mathrm{e}^{-h(w)} \mathbf{Q}_{(0)}(\mathrm{d}w) \int_0^\infty (1 - \mathrm{e}^{-\lambda z}) Q^\circ_{t-r}(w(r), \mathrm{d}z) \\
&\qquad = \int_{D_0[0,\infty)} \mathrm{e}^{-h(w)} \big(1 - \mathrm{e}^{-w(r) v_{t-r}(\lambda)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathbf{P}\Big[\exp\Big\{ - \int_{D_0[0,\infty)} h(w) N(\mathrm{d}w) - \lambda X_t \Big\} \Big] \\
&\qquad = \exp\Big\{ - \int_{D_0[0,\infty)} \big(1 - \mathrm{e}^{-h(w)} \mathrm{e}^{-w(r) v_{t-r}(\lambda)}\big) \mathbf{Q}_{(0)}(\mathrm{d}w) \Big\} \\
&\qquad = \mathbf{P}\Big[\exp\Big\{ - \int_{D_0[0,\infty)} h(w) N(\mathrm{d}w) - X_r v_{t-r}(\lambda) \Big\} \Big].
\end{aligned}
$$

Then $\{(X_t, \mathscr{G}_t) : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. $\square$

The reconstruction (2.4.3) of the CB-process means that the population at time $t > 0$ consists of the descendants of at most countably many individuals at time zero, which evolve as the excursions $\{w_i : i = 1, \cdots, |N|\}$ selected randomly by the Poisson random measure $N(\mathrm{d}w)$.

Chapter 3

Structures of independent immigration

In this chapter we study independent immigration structures associated with CB-processes. We first give a formulation of the structures using skew convolution semigroups. Those semigroups are in one-to-one correspondence with infinitely divisible distributions on $\mathbb{R}_+$. We show the corresponding immigration process arise as scaling limits of Galton–Watson processes with immigration. We discuss briefly limit theorems and stationary distributions of the immigration superprocesses. The trajectories of the immigration processes are constructed using stochastic integrals with respect to Poisson random measures determined by entrance laws.

3.1 Formulation of immigration processes

In this section, we introduce a generalization of the CB-process. Let $(Q_t)_{t\geq 0}$ be the transition semigroup defined by (2.1.15) and (2.1.20). Let $(\gamma_t)_{t\geq 0}$ be a family of probability measures on $\mathbb{R}_+$. We call $(\gamma_t)_{t\geq 0}$ a *skew convolution semigroup* (SC-semigroup) associated with $(Q_t)_{t\geq 0}$ provided

$$\gamma_{r+t} = (\gamma_r Q_t) * \gamma_t, \qquad r, t \geq 0. \tag{3.1.1}$$

It is easy to show that (3.1.1) holds if and only if

$$u_{r+t}(\lambda) = u_t(\lambda) + u_r(v_t(\lambda)), \qquad r, t, \lambda \geq 0, \tag{3.1.2}$$

where

$$u_t(\lambda) = -\log \int_0^\infty \mathrm{e}^{-y\lambda} \gamma_t(\mathrm{d}y). \tag{3.1.3}$$

The concept of SC-semigroup is of interest because of the following:

Theorem 3.1.1 *The family of probability measures* $(\gamma_t)_{t\geq 0}$ *on* $\mathbb{R}_+$ *is an SC-semigroup if and only if*

$$Q_t^\gamma(x,\cdot) := Q_t(x,\cdot) * \gamma_t, \qquad t,x \geq 0 \tag{3.1.4}$$

defines a Markov semigroup $(Q_t^\gamma)_{t\geq 0}$ *on* $\mathbb{R}_+$.

Proof. Let $(\gamma_t)_{t\geq 0}$ probability measures on $\mathbb{R}_+$ and let $Q_t^\gamma(x,\cdot)$ be the probability kernel defined by (3.1.4). Then we have

$$\int_0^\infty \mathrm{e}^{-y\lambda} Q_t^\gamma(x,\mathrm{d}y) = \exp\big\{-xv_t(\lambda) - u_t(\lambda)\big\}, \qquad t,x,\lambda \geq 0. \tag{3.1.5}$$

Using this relation it is easy to show that $(Q_t^\gamma)_{t\geq 0}$ satisfies the Chapman-Kolmogorov equation if and only if (3.1.2) is satisfied. That proves the result. $\square$

If $\{y(t) : t \geq 0\}$ is a positive Markov process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ given by (3.1.4), we call it an *immigration process* or a *CBI-process* associated with $(Q_t)_{t\geq 0}$. The intuitive meaning of the model is clear in view of (3.1.1) and (3.1.4). From (3.1.4) we see that the population at any time $t \geq 0$ is made up of two parts; the native part generated by the mass $x \geq 0$ has distribution $Q_t(x,\cdot)$ and the immigration in the time interval $(0,t]$ gives the distribution γ_t. In a similar way, the equation (3.1.1) decomposes the mass immigrating to the population during the time interval $(0,r+t]$ into two parts; the immigration in the interval $(r,r+t]$ gives the distribution γ_t while the immigration in the interval $(0,r]$ generates the distribution γ_r at time r and gives the distribution $\gamma_r Q_t$ at time $r+t$. It is not hard to understand that (3.1.4) gives a general formulation of the immigration independent of the state of the population.

Theorem 3.1.2 *The family of probability measures* $(\gamma_t)_{t\geq 0}$ *on* $\mathbb{R}_+$ *is an SC-semigroup if and only if there exists* $\psi \in \mathscr{I}$ *so that*

$$\int_0^\infty \mathrm{e}^{-\lambda y}\gamma_t(\mathrm{d}y) = \exp\Big\{-\int_0^t \psi(v_s(\lambda))\mathrm{d}s\Big\}, \qquad t,\lambda \geq 0. \tag{3.1.6}$$

Proof. It is easy to check that for any $\psi \in \mathscr{I}$ the family $(\gamma_t)_{t\geq 0}$ defined by (3.1.6) is an SC-semigroup. Conversely, suppose that $(\gamma_t)_{t\geq 0}$ is an SC-semigroup. For $t,\lambda \geq 0$ let $u_t(\lambda)$ be defined by (3.1.3). Then $t \mapsto u_t(\lambda)$ is increasing. By Lebesgue's theorem, the limit

$$\psi_t(\lambda) := \lim_{s\to 0^+} s^{-1}[u_{t+s}(\lambda) - u_t(\lambda)] = \lim_{s\to 0^+} s^{-1}u_s(v_t(\lambda))$$

exists for almost all $t \geq 0$; see, e.g., Hewitt and Stromberg (1965, p.264). By the continuity of $t \mapsto v_t(\lambda)$, there is a dense subset D of $(0,\infty)$ so that the following limits exist:

$$\psi_0(\lambda) := \lim_{s\to 0^+} s^{-1}u_s(\lambda) = \lim_{s\to 0^+} s^{-1}[1-\mathrm{e}^{-u_s(\lambda)}], \qquad \lambda \in D. \tag{3.1.7}$$

For $u \in [0,\infty]$ and $\lambda \in (0,\infty)$ let $\xi(u,\lambda)$ be defined by (1.2.4). Then $\xi(u,\lambda)$ is jointly continuous in (u,λ). By (3.1.7) we have

$$\psi_0(\lambda) = \lim_{s\to 0^+} s^{-1}u_s(\lambda) = \lim_{s\to 0^+} s^{-1}\int_{[0,\infty]} \xi(u,\lambda)G_s(\mathrm{d}u) \tag{3.1.8}$$

for $\lambda \in D$, where $G_s(\mathrm{d}u) = (1-\mathrm{e}^{-u})\gamma_s(\mathrm{d}u)$. Then for some $\delta > 0$ we have

$$\sup_{0<s<\delta} s^{-1}G_s([0,\infty]) < \infty,$$

so the family of finite measures $\{s^{-1}G_s : 0 < s < \delta\}$ on $[0,\infty]$ is relatively compact. Suppose that $s_n \to 0$ and $s_n^{-1}G_{s_n} \to$ some G weakly as $n\to\infty$. Then

$$\lim_{n\to\infty} s_n^{-1}u_{s_n}(\lambda) = \lim_{n\to\infty} s_n^{-1}\int_{[0,\infty]} \xi(u,\lambda)G_{s_n}(\mathrm{d}u) = \int_{[0,\infty]} \xi(u,\lambda)G(du), \quad \lambda > 0.$$

Thus we can extend ψ_0 to a continuous function on $(0,\infty)$ given by

$$\psi_0(\lambda) = \int_{[0,\infty]} \xi(u,\lambda)G(du) = G(\{\infty\}) + h\lambda + \int_{(0,\infty)} (1-\mathrm{e}^{-u\lambda})l(du),$$

where $h = G(\{0\})$ and $l(\mathrm{d}u) = (1-\mathrm{e}^{-u})^{-1}G_s(\mathrm{d}u)$. By a standard argument one sees that (3.1.8) holds actually for all $\lambda > 0$. From (3.1.2) and (3.1.8) it follows that

$$D^+u_s(\lambda)\big|_{s=t} = D^+u_s(v_t(\lambda))\big|_{s=0} = \psi_0(v_t(\lambda)),$$

where D^+ denotes the right derivative relative to $s \geq 0$. Here the right-hand side is continuous in $t \geq 0$. Thus $t \mapsto u_t(\lambda)$ is continuously differentiable and (3.1.6) holds with $\psi(\lambda) = \psi_0(\lambda)$ for $\lambda > 0$. By letting $\lambda \to 0+$ in (3.1.6) one sees $G(\{\infty\}) = 0$. Then $\psi = \psi_0 \in \mathscr{I}$. $\Box$

By Theorem 3.1.2 there is a 1-1 correspondence between SC-semigroups and infinitely divisible distributions on $\mathbb{R}_+$. Then the theorem generalizes the 1-1 correspondence between classical convolution semigroups and infinitely divisible distributions. In fact, from (3.1.1) it is easy to see that $(\gamma_t)_{t\geq 0}$ reduces to a classical convolution semigroup if Q_t is the identity operator for all $t \geq 0$. As a consequence of Theorems 1.2.4 and 3.1.2, an SC-semigroup $(\gamma_t)_{t\geq 0}$ always consists of infinitely divisible distributions.

Now let us consider a transition semigroup $(Q_t^\gamma)_{t\geq 0}$ defined by (3.1.4) with the SC-semigroup $(\gamma_t)_{t\geq 0}$ given by (3.1.6). If an immigration process has transition semigroup $(Q_t^\gamma)_{t\geq 0}$, we say it has *branching mechanism* ϕ and *immigration mechanism* ψ. It is easy to see that

$$\int_0^\infty \mathrm{e}^{-\lambda y}Q_t^\gamma(x,\mathrm{d}y) = \exp\Big\{-xv_t(\lambda) - \int_0^t \psi(v_s(\lambda))\mathrm{d}s\Big\}. \tag{3.1.9}$$

The following results are immediate consequences of (3.1.6) and (3.1.9).

Theorem 3.1.3 *Suppose that $(\gamma'_t)_{t\geq 0}$ and $(\gamma''_t)_{t\geq 0}$ are two SC-semigroups associated with $(Q_t)_{t\geq 0}$. Let $\gamma_t = \gamma'_t * \gamma''_t$ for $t \geq 0$. Then $(\gamma_t)_{t\geq 0}$ is also an SC-semigroup associated with $(Q_t)_{t\geq 0}$.*

Theorem 3.1.4 *Suppose that $\{(y'(t), \mathscr{G}'_t) : t \geq 0\}$ and $\{(y''(t), \mathscr{G}''_t) : t \geq 0\}$ are two independent CBI-processes with the same branching mechanism ϕ and immigration mechanisms ψ' and ψ'', respectively. Let $y(t) = y'(t) + y''(t)$ and $\mathscr{G}_t = \sigma(\mathscr{G}'_t \cup \mathscr{G}''_t)$ for $t \geq 0$. Then $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanism $\psi := \psi' + \psi''$.*

Corollary 3.1.5 *Suppose that $\{(y'(t), \mathscr{G}'_t) : t \geq 0\}$ is a CB-processes with branching mechanism ϕ and $\{(y''(t), \mathscr{G}''_t) : t \geq 0\}$ is a CBI-processes with branching mechanism ϕ and immigration mechanism ψ. In addition, we assume the two process are independent. Let $y(t) = y'(t) + y''(t)$ and $\mathscr{G}_t = \sigma(\mathscr{G}'_t \cup \mathscr{G}''_t)$ for $t \geq 0$. Then $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanism ψ.*

Let us give a useful moment formula for the transition semigroup $(Q^\gamma_t)_{t\geq 0}$. Recall that the function $\psi \in \mathscr{I}$ has the representation

$$\psi(z) = \beta z + \int_0^\infty \big(1 - \mathrm{e}^{-zu}\big) n(\mathrm{d}u), \qquad z \geq 0, \tag{3.1.10}$$

where $\beta \geq 0$ is a constant and $(1 \wedge u)n(\mathrm{d}u)$ is a finite measure on $(0, \infty)$. In particular, if $un(\mathrm{d}u)$ is a finite measure on $(0, \infty)$, by (3.1.9) and (2.2.4) one can show

$$\int_0^\infty y Q^\gamma_t(x, \mathrm{d}y) = x\mathrm{e}^{-bt} + \psi'(0) \int_0^t \mathrm{e}^{-bs} \mathrm{d}s, \tag{3.1.11}$$

where

$$\psi'(0) = \beta + \int_0^\infty u n(\mathrm{d}u). \tag{3.1.12}$$

From (3.1.11) we have

$$\int_0^\infty y Q^\gamma_t(x, \mathrm{d}y) = x\mathrm{e}^{-bt} + \psi'(0) b^{-1}(1 - \mathrm{e}^{-bt}) \tag{3.1.13}$$

with the convention $b^{-1}(1 - \mathrm{e}^{-bt}) = t$ for $b = 0$.

The following theorem gives a necessary and sufficient condition for the ergodicity of the semigroup $(Q^\gamma_t)_{t\geq 0}$.

Theorem 3.1.6 *Suppose that $b \geq 0$ and $\phi(z) \neq 0$ for every $z > 0$. Then $Q_t^\gamma(x, \cdot)$ converges to a probability measure η on $\mathbb{R}_+$ as $t \to \infty$ if and only if*

$$\int_0^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z < \infty \text{ for some } \lambda > 0. \tag{3.1.14}$$

If (3.1.14) *holds, the Laplace transform of η is given by*

$$L_\eta(\lambda) = \exp\Big\{ -\int_0^\infty \psi(v_s(\lambda))\mathrm{d}s \Big\}, \qquad \lambda \geq 0. \tag{3.1.15}$$

Proof. Since $\phi(z) \geq 0$ for all $z \geq 0$, from (2.2.2) we see $t \mapsto v_t(\lambda)$ is decreasing. Then (2.2.6) implies $\lim_{t\to\infty} v_t(\lambda) = 0$. By (3.1.9) we have

$$\lim_{t\to\infty} \int_0^\infty \mathrm{e}^{-\lambda y} Q_t^\gamma(x, \mathrm{d}y) = \exp\Big\{ -\int_0^\infty \psi(v_s(\lambda))\mathrm{d}s \Big\} \tag{3.1.16}$$

for every $\lambda \geq 0$. A further application of (2.2.2) gives

$$\int_0^t \psi(v_s(\lambda))\mathrm{d}s = \int_{v_t(\lambda)}^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z.$$

It follows that

$$\int_0^\infty \psi(v_s(\lambda))\mathrm{d}s = \int_0^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z,$$

which is a continuous function of $\lambda \geq 0$ if and only if (3.1.14) holds. Then the result follows by (3.1.16) and Theorem 1.1.2. □

Corollary 3.1.7 *Suppose that $b > 0$. Then $Q_t^\gamma(x, \cdot)$ converges to a probability measure η on $\mathbb{R}_+$ as $t \to \infty$ if and only if $\int_1^\infty \log u\, n(\mathrm{d}u) < \infty$. In this case, the Laplace transform of η is given by* (3.1.15).

Proof. We have $\phi(z) = bz + o(z)$ as $z \to 0$. Thus (3.1.14) holds if and only if

$$\int_0^\lambda \frac{\psi(z)}{z} \mathrm{d}z < \infty \text{ for some } \lambda > 0,$$

which is equivalent to

$$\int_0^\lambda \frac{\mathrm{d}z}{z} \int_0^\infty \big(1 - \mathrm{e}^{-zu}\big) n(\mathrm{d}u) = \int_0^\infty n(\mathrm{d}u) \int_0^{\lambda u} \frac{1 - \mathrm{e}^{-y}}{y} \mathrm{d}y < \infty$$

for some $\lambda > 0$. The latter holds if and only if $\int_1^\infty \log u n(\mathrm{d}u) < \infty$. Then we have the result by Theorem 3.1.6. $\square$

In the situation of Theorem 3.1.6, it is easy to show that η is a stationary distribution for $(Q_t^\gamma)_{t\geq 0}$. The fact that the CBI-process may have a non-trivial stationary distribution makes it a more interesting model in many respects than the CB-process without immigration. Note also that the transition semigroup $(Q_t^b)_{t\geq 0}$ given by (2.3.3) is a special case of the one defined by (3.1.9).

Theorem 3.1.8 *Suppose that* $b > 0$ *and let* $q_t'(\lambda)$ *be defined by* (2.3.1). *Then for every* $\lambda \geq 0$ *the limit* $q'(\lambda) :=\downarrow\lim_{t\to\infty} q_t'(\lambda)$ *exists and is given by*

$$q'(\lambda) = \exp\Big\{ - \int_0^\infty \phi_0'(v_s(\lambda))\mathrm{d}s\Big\}, \qquad \lambda \geq 0. \tag{3.1.17}$$

Moreover, we have $q'(0+) = q'(0) = 1$ *if and only if* $\int_1^\infty u\log u m(\mathrm{d}u) < \infty$. *The last condition is also equivalent to* $q'(\lambda) > 0$ *for some and hence all* $\lambda > 0$.

Proof. The first assertion is easy in view of (2.3.1). By Corollary 3.1.7, we have $\int_1^\infty u\log u m(\mathrm{d}u) < \infty$ if and only if $\lambda \mapsto q'(\lambda)$ is the Laplace transform of a probability η on $\mathbb{R}_+$. Then the other two assertions hold obviously. $\square$

Theorem 3.1.9 *Suppose that* $b > 0$ *and* $\phi'(z) \to \infty$ *as* $z \to \infty$. *Then* $Q_t^b(x,\cdot)$ *converges as* $t \to \infty$ *to a probability* η *on* $(0,\infty)$ *if and only if* $\int_1^\infty u\log u m(\mathrm{d}u) < \infty$. *If the condition holds, then* η *has Laplace transform* $L_\eta = q'$ *given by* (3.1.17).

Proof. By Corollary 3.1.7 and Theorem 3.1.8 we have the results with η being a probability measure on $\mathbb{R}_+$. By Theorem 2.3.1 the measure $Q_t^b(0,\cdot)$ is supported by $(0,\infty)$, hence $Q_t^b(x,\cdot)$ is supported by $(0,\infty)$ for every $x \geq 0$. From (2.3.5) we have $L_\eta(\infty) \leq q_t'(\infty) = 0$ for $t > 0$. That implies $\eta(\{0\}) = 0$. $\square$

Example 3.1.1 Suppose that $c > 0$, $0 < \alpha \leq 1$ and b are constants and let $\phi(z) = cz^{1+\alpha} + bz$ for $z \geq 0$. In this case the cumulant semigroup $(v_t)_{t\geq 0}$ is given by (2.2.19). Let $\beta \geq 0$ and let $\psi(z) = \beta z^\alpha$ for $z \geq 0$. We can use (3.1.9) to define the transition semigroup $(Q_t^\gamma)_{t\geq 0}$. It is easy to show that

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_t^\gamma(x,\mathrm{d}y) = \frac{1}{\big[1 + cq_\alpha^b(t)\lambda^\alpha\big]^{\beta/c\alpha}}\,\mathrm{e}^{-xv_t(\lambda)}, \qquad \lambda \geq 0. \tag{3.1.18}$$

The concept of SC-semigroup associated with branching processes was introduced in Li (1995/6, 1996). Theorem 3.1.2 can be regarded as a special form of main theorem of Li (1995/6). Theorem 3.1.6 and Corollary 3.1.7 were given in Pinsky (1972). Other results in this section can be found in Li (2000).

3.2 Stationary immigration distributions

In this section, we give a brief discussion of the structures of stationary distributions of the CBI-processes. The results here were first given in Li (2002) in the setting of measure-valued processes. Given two probability measures η_1 and η_2 on $\mathbb{R}_+$, we write $\eta_1 \preceq \eta_2$ if $\eta_1 * \gamma = \eta_2$ for another probability measure γ on $\mathbb{R}_+$. Clearly, the measure γ is unique if it exists. Let $(Q_t)_{t\geq 0}$ be the transition semigroup defined by (2.1.15) and (2.1.20), where $(v_t)_{t\geq 0}$ has the representation (2.2.1). Let $(Q_t^\circ)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ to $(0,\infty)$. Let $\mathscr{E}^*(Q)$ denote the set of probabilities η on $\mathbb{R}_+$ satisfying $\eta Q_t \preceq \eta$ for all $t \geq 0$.

Theorem 3.2.1 *For each $\eta \in \mathscr{E}^*(Q)$ there is a unique SC-semigroup $(\gamma_t)_{t\geq 0}$ associated with $(Q_t)_{t\geq 0}$ such that $\eta Q_t * \gamma_t = \eta$ for $t \geq 0$ and $\eta = \lim_{t\to\infty} \gamma_t$.*

Proof. By the definition of $\mathscr{E}^*(Q)$, for each $t \geq 0$ there is a unique probability measure γ_t on $\mathbb{R}_+$ satisfying $\eta = (\eta Q_t) * \gamma_t$. By the branching property of $(Q_t)_{t\geq 0}$ one can show $(\mu_1 * \mu_2)Q_t = (\mu_1 Q_t) * (\mu_2 Q_t)$ for any $t \geq 0$ and any probability measures μ_1 and μ_2 on $\mathbb{R}_+$. Then for $r, t \geq 0$ we have

$$(\eta Q_{r+t}) * \gamma_{r+t} = (\eta Q_t) * \gamma_t = \{[(\eta Q_r) * \gamma_r]Q_t\} * \gamma_t = (\eta Q_{r+t}) * (\gamma_r Q_t) * \gamma_t.$$

A cancelation gives (3.1.1), so $(\gamma_t)_{t\geq 0}$ is an SC-semigroup associated with $(Q_t)_{t\geq 0}$. Now for every $\lambda \geq 0$ the function $t \mapsto L_{\gamma_t}(\lambda)$ is decreasing. By the relation $\eta = (\eta Q_t) * \gamma_t$ one can see $t \mapsto L_{\eta Q_t}(\lambda)$ is increasing. Then there are probability measures η_i and η_p on $\mathbb{R}_+$ so that $\eta_i * \eta_p = \eta$ and

$$L_{\eta_i}(\lambda) = \lim_{t\to\infty} L_{\eta Q_t}(\lambda), \quad L_{\eta_p}(\lambda) = \lim_{t\to\infty} L_{\gamma_t}(\lambda).$$

These imply $\eta_i = \lim_{t\to\infty} \eta Q_t$ and $\eta_p = \lim_{t\to\infty} \gamma_t$. It follows that η_i is a stationary distribution of $(Q_t)_{t\geq 0}$, so we must have $\eta_i = \delta_0$ and $\eta_p = \eta$. $\square$

In the situation of Theorem 3.2.1, it is easy to see the measure $\eta \in \mathscr{E}^*(Q)$ is the unique stationary distribution of the transition semigroup $(Q_t^\gamma)_{t\geq 0}$ defined by (3.1.4). Then we can identify $\mathscr{E}^*(Q)$ with the set of stationary distributions of immigration processes associated with $(Q_t)_{t\geq 0}$. As a consequence of Theorem 3.1.6 and 3.2.1, every $\mu \in \mathscr{E}^*(Q)$ is infinitely divisible. Recall that we write $\mu = I(h,l)$ if μ is an infinitely divisible probability measure on $\mathbb{R}_+$ with $\psi := -\log L_\mu$ given by (1.2.3). Let $\mathscr{E}(Q^\circ)$ denote the set of excessive measures ν for $(Q_t^\circ)_{t\geq 0}$ satisfying

$$\int_0^\infty (1 \wedge u)\nu(\mathrm{d}u) < \infty.$$

The following result gives some characterizations of the set $\mathscr{E}^*(Q)$.

Theorem 3.2.2 *Let $\eta = I(\beta, \nu)$ be an infinitely divisible probability measure on $\mathbb{R}_+$. Then $\eta \in \mathscr{E}^*(Q)$ if and only if (β, ν) satisfy*

$$\beta h_t \leq \beta \quad \textit{and} \quad \beta l_t + \nu Q_t^\circ \leq \nu, \qquad t \geq 0. \tag{3.2.1}$$

In particular, if $\nu \in \mathscr{E}(Q^\circ)$, then $\eta = I(0, \nu) \in \mathscr{E}^*(Q)$.

Proof. It is easy to show that $\eta Q_t = I(\beta_t, \nu_t)$, where $\beta_t = \beta h_t$ and $\nu_t = \beta l_t + \nu Q_t^\circ$. Then $\eta Q_t \preceq \eta$ holds if and only if (3.2.1) is satisfied. The second assertion is immediate. □

3.3 Scaling limits of discrete immigration models

In this section, we prove a limit theorem of rescaled Galton–Watson branching processes with immigration, which leads to the CBI-processes. This kind of limit theorems were studied in Aliev and Shchurenkov (1982), Kawazu and Watanabe (1971) and Li (2006) among many others.

Let g and h be two probability generating functions. Suppose that $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ and $\{\eta_n : n = 1, 2, \ldots\}$ are independent families of positive integer-valued i.i.d. random variables with distributions given by g and h, respectively. Given another positive integer-valued random variable $y(0)$ independent of $\{\xi_{n,i}\}$ and $\{\eta_n\}$, we define inductively

$$y(n) = \sum_{i=1}^{y(n-1)} \xi_{n,i} + \eta_n, \qquad n = 1, 2, \ldots. \tag{3.3.1}$$

Then $\{y(n) : n = 0, 1, 2, \ldots\}$ is a discrete-time positive integer-valued Markov chain with transition matrix $Q(i, j)$ determined by

$$\sum_{j=0}^{\infty} Q(i, j) z^j = g(z)^i h(z), \qquad |z| \leq 1. \tag{3.3.2}$$

The random variable $y(n)$ can be thought of as the number of individuals in generation $n \geq 0$ of an evolving particle system. After one unit time, each of the $y(n)$ particles splits independently of others into a random number of offspring according to the distribution given by g and a random number of immigrants are added to the system according to the probability law given by h. The n-step transition matrix $Q^n(i, j)$ of $\{y(n) : n = 0, 1, 2, \ldots\}$ is given by

$$\sum_{j=0}^{\infty} Q^n(i, j) z^j = g^n(z)^i \prod_{j=1}^{n} h(g^{j-1}(z)), \qquad |z| \leq 1, \tag{3.3.3}$$

where $g^n(z)$ is defined by $g^n(z) = g(g^{n-1}(z))$ successively with $g^0(z) = z$. We call any positive integer-valued Markov chain with transition probabilities given by (3.3.2) or (3.3.3) a *Galton–Watson branching process with immigration* (GWI-process) with parameters (g, h). When $h \equiv 1$, this reduces to the GW-process defined in the first section.

Suppose that for each integer $k \geq 1$ we have a GWI-process $\{y_k(n) : n \geq 0\}$ with parameters (g_k, h_k). Let $z_k(n) = y_k(n)/k$. Then $\{z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k := \{0, 1/k, 2/k, \ldots\}$ and n-step transition probability $Q_k^n(x, \mathrm{d}y)$ determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y} Q_k^n(x, \mathrm{d}y) = g_k^n(\mathrm{e}^{-\lambda/k})^{kx} \prod_{j=1}^{n} h(g_k^{j-1}(\mathrm{e}^{-\lambda/k})), \quad \lambda \geq 0. \tag{3.3.4}$$

Suppose that $\{\gamma_k\}$ is a positive real sequence so that $\gamma_k \to \infty$ increasingly as $k \to \infty$. Let $[\gamma_k t]$ denote the integer part of $\gamma_k t \geq 0$. In view of (3.3.4), given $z_k(0) = x$ the conditional distribution $Q_k^{[\gamma_k t]}(x, \cdot)$ of $z_k([\gamma_k t])$ on E_k is determined by

$$\begin{aligned} \int_{E_k} \mathrm{e}^{-\lambda y} Q_k^{[\gamma_k t]}(x, \mathrm{d}y) \\ = \exp\Big\{ - x v_k(t, \lambda) - \int_0^{\frac{[\gamma_k t]}{\gamma_k}} \bar{H}_k(v_k(s, \lambda)) \mathrm{d}s \Big\}, \end{aligned} \tag{3.3.5}$$

where $v_k(t, \lambda)$ is given by (2.1.7) and

$$\bar{H}_k(\lambda) = -\gamma_k \log h_k(\mathrm{e}^{-\lambda/k}), \qquad \lambda \geq 0.$$

For any $z \geq 0$ let $G_k(z)$ be defined by (2.1.8) and let

$$H_k(z) = \gamma_k[1 - h_k(\mathrm{e}^{-z/k})]. \tag{3.3.6}$$

Condition 3.3.1 *There is a function ψ on $[0, \infty)$ such that $H_k(z) \to \psi(z)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.*

It is simple to see that $H_k \in \mathscr{I}$. By Theorem 1.2.2, if the above condition is satisfied, the limit function ψ has the representation (3.1.10). A different of proof of the following theorem was given in Li (2006).

Theorem 3.3.2 *Suppose that Conditions 2.1.2 and 3.3.1 are satisfied. Let $\{y(t) : t \geq 0\}$ be a CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ defined by* (3.1.9). *If $z_k(0)$ converges to $y(0)$ in distribution, then $\{z_k([\gamma_k t]) : t \geq 0\}$ converges to $\{y(t) : t \geq 0\}$ in distribution on $D([0, \infty), \mathbb{R}_+)$.*

Proof. By Theorem 2.1.6 for every $a \geq 0$ we have $v_k(t, \lambda) \to v_t(\lambda)$ uniformly on $[0, a]^2$ as $k \to \infty$. For $\lambda > 0$ and $x \geq 0$ set $e_\lambda(x) = \mathrm{e}^{-\lambda x}$. In view of (3.3.5) we have

$$\lim_{k\to\infty} \sup_{x\in E_k} \left|Q_k^{[\gamma_k t]} e_\lambda(x) - Q_t^\gamma e_\lambda(x)\right| = 0$$

for every $t \geq 0$. Then the result follows as in the proof of Theorem 2.1.9. □

Example 3.3.1 In a special case, we can give a characterization for the CBI-process in terms of a stochastic differential equation. Let $m = \mathbf{E}[\xi_{1,1}]$. From (3.3.1) we have

$$y(n) - y(n-1) = \sqrt{y(n-1)} \sum_{i=1}^{y(n-1)} \frac{\xi_{n,i} - m}{\sqrt{y(n-1)}} - (1-m)y(n-1) + \eta_n.$$

Then it is natural to expect that a typical CBI-process would solve the stochastic differentia equation

$$\mathrm{d}y(t) = \sqrt{2cy(t)}\mathrm{d}B(t) - by(t)\mathrm{d}t + \beta\mathrm{d}t, \qquad t \geq 0, \tag{3.3.7}$$

where $\{B(t) : t \geq 0\}$ is a Brownian motion. The above equation has a unique positive strong solution; see Ikeda and Watanabe (1989, pp.235–236). In fact, the solution $\{y(t) : t \geq 0\}$ has transition semigroup given by (3.1.18) with $\alpha = 1$. Let $C^2(\mathbb{R}_+)$ denote the set of bounded continuous real functions on $\mathbb{R}_+$ with bounded continuous derivatives up to the second order. Then $\{y(t) : t \geq 0\}$ has generator A given by

$$Af(x) = c\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + (\beta - bx)\frac{\mathrm{d}}{\mathrm{d}x} f(x), \qquad f \in C^2(\mathbb{R}_+).$$

In particular, for $\beta = 0$ the solution of (3.3.7) is called *Feller's branching diffusion.*

3.4 A reconstruction of the sample path

In this section, we give a reconstruction of the sample path of the CBI-process using a Poisson random measure. Suppose that $(H_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t\geq 0}$ and $\mathbf{Q}_H$ is the σ-finite measure on $D(0,\infty)$ determined by (2.4.1). Let $\{X_t : t \geq 0\}$ be a CB-process with transition semigroup $(Q_t)_{t\geq 0}$ and $N(\mathrm{d}s, \mathrm{d}w)$ a Poisson random measure on $(0,\infty) \times D(0,\infty)$ with intensity $\mathrm{d}s\mathbf{Q}_H(\mathrm{d}w)$. Suppose that $\{X_t : t \geq 0\}$ and $N(\mathrm{d}s, \mathrm{d}w)$ are independent. We define the measure-valued process

$$Y_t = X_t + \int_{(0,t)} \int_{D(0,\infty)} w_{t-s} N(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0. \tag{3.4.1}$$

The following theorem generalizes a result of Pitman and Yor (1982).

Theorem 3.4.1 *For $t \geq 0$ let $\mathscr{G}_t$ be the σ-algebra generated by the collection of random variables $\{N((0,u] \times A) : A \in \mathscr{A}^0_{t-u}, 0 \leq u < t\}$. Then $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ is an immigration process with transition semigroup $(Q^H_t)_{t\geq 0}$ given by*

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q^H_t(x, \mathrm{d}y) = \exp\Big\{ - x v_t(\lambda) - \int_0^t \mathrm{d}s \int_0^\infty (1 - \mathrm{e}^{-\lambda y}) H_s(\mathrm{d}y)\Big\}. \qquad (3.4.2)$$

Proof. By Corollary 3.1.5, we only need to consider the special case with $X_t = 0$ for all $t \geq 0$. In this case, it is easy to show that Y_t has distribution $Q^H_t(0, \cdot)$ on $\mathbb{R}_+$. Let $t \geq r > u \geq 0$ and let h be a bounded positive function on $D(0, \infty)$ measurable relative to $\mathscr{A}^0_{r-u}$. For $\lambda \geq 0$ we can see as in the proof of Theorem 2.4.2 that

$$\begin{aligned}
&\mathbf{P}\Big[\exp\Big\{ - \int_0^u \int_{D(0,\infty)} h(w) N(\mathrm{d}s, \mathrm{d}w) - \lambda Y_t\Big\}\Big] \\
&\quad = \mathbf{P}\Big[\exp\Big\{ - \int_0^t \int_{D(0,\infty)} \big[h(w) 1_{\{s\leq u\}} + \lambda w_{t-s}\big] N(\mathrm{d}s, \mathrm{d}w)\Big\}\Big] \\
&\quad = \exp\Big\{ - \int_0^t \mathrm{d}s \int_{D(0,\infty)} \big(1 - \mathrm{e}^{-h(w)1_{\{s\leq u\}}} \mathrm{e}^{-\lambda w_{t-s}}\big) \mathbf{Q}_H(\mathrm{d}w)\Big\} \\
&\quad = \exp\Big\{ - \int_0^u \mathrm{d}s \int_{D(0,\infty)} \big(1 - \mathrm{e}^{-h(w)} \mathrm{e}^{-v_{t-r}(\lambda) w_{r-s}}\big) \mathbf{Q}_H(\mathrm{d}w)\Big\} \\
&\qquad \cdot \exp\Big\{ - \int_u^r \mathrm{d}s \int_{D(0,\infty)} \big(1 - \mathrm{e}^{-v_{t-r}(\lambda) w_{r-s}}\big) \mathbf{Q}_H(\mathrm{d}w)\Big\} \\
&\qquad \cdot \exp\Big\{ - \int_r^t \mathrm{d}s \int_{D(0,\infty)} \big(1 - \mathrm{e}^{-\lambda w_{t-s}}\big) \mathbf{Q}_H(\mathrm{d}w)\Big\} \\
&\quad = \mathbf{P}\Big[\exp\Big\{ - \int_0^u \int_{D(0,\infty)} h(w) N(\mathrm{d}s, \mathrm{d}w) - Y_r v_{t-r}(\lambda)\Big\}\Big] \\
&\qquad \cdot \exp\Big\{ - \int_r^t \mathrm{d}s \int_0^\infty (1 - \mathrm{e}^{-\lambda y}) H_{t-s}(\mathrm{d}y)\Big\},
\end{aligned}$$

where we have used the Markov property (2.4.1) for the third equality. That shows $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ is a Markov process in $\mathbb{R}_+$ with transition semigroup $(Q^H_t)_{t\geq 0}$. $\square$

Let $\mathbf{P}_x(\mathrm{d}w)$ denote the distribution on $D[0, \infty)$ of the CB-process $\{x(t) : t \geq 0\}$ with $x(0) = x$. Suppose that $\psi \in \mathscr{I}$ be given by (3.1.10). If the condition (2.2.14) is satisfied, we can define a σ-finite measure $\mathbf{Q}_H(\mathrm{d}w)$ on $D[0, \infty)$ by

$$\mathbf{Q}_H(\mathrm{d}w) = \beta \mathbf{Q}_{(0)}(\mathrm{d}w) + \int_0^\infty n(\mathrm{d}x) \mathbf{P}_x(\mathrm{d}w). \qquad (3.4.3)$$

This corresponds to the entrance law $(H_t)_{t>0}$ for $(Q^\circ_t)_{t\geq 0}$ defined by

$$H_t = \beta l_t + \int_0^\infty n(\mathrm{d}x) Q_t(x, \cdot), \qquad t > 0. \qquad (3.4.4)$$

In this case, it is easy to show that

$$\int_0^\infty (1 - \mathrm{e}^{-\lambda y}) H_t(\mathrm{d}y) = \psi(v_t(\lambda)), \quad t > 0, \lambda \geq 0. \tag{3.4.5}$$

Then from Theorem 3.4.1 we obtain

Corollary 3.4.2 *Suppose that* (2.2.14) *is satisfied and* $(H_t)_{t>0}$ *is given by* (3.4.4). *Then* $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ *is an immigration process with transition semigroup* $(Q_t^\gamma)_{t\geq 0}$ *given by* (3.1.9).

The reconstruction (3.4.1) of the immigration process can be interpreted similarly as (2.4.3). Here the Poisson random measure $N(\mathrm{d}s, \mathrm{d}w)$ determines both the immigration times and the evolutions of the descendants of the immigrants.

Chapter 4

Martingale problems and stochastic equations

Martingale problems play a very important role in the study of Markov processes. In this chapter we prove the equivalence of a number of martingale problems for CBI-processes. From the martingale problems we derive some stochastic equations. Using the stochastic equations, we give simple proofs of Lamperti's transformations on CB-processes and spectrally positive Lévy processes.

4.1 Martingale problem formulations

In this section we give some formulations of the CBI-process in terms of martingale problems and prove their equivalence. Suppose that (ϕ, ψ) are given respectively by (2.1.13) and (3.1.10) with $un(\mathrm{d}u)$ being a finite measure on $(0, \infty)$. For $f \in C^2(\mathbb{R}_+)$ define

$$\begin{aligned} Lf(x) = cxf''(x) + x\int_0^\infty \big[f(x+z) - f(x) - zf'(x)\big]m(\mathrm{d}z) \\ + (\beta - bx)f'(x) + \int_0^\infty \big[f(x+z) - f(x)\big]n(\mathrm{d}z). \end{aligned} \tag{4.1.1}$$

We shall identify the operator L as the generator of the CBI-process. For this purpose we need the following:

Proposition 4.1.1 *Let* $(Q_t^\gamma)_{t\geq 0}$ *be the transition semigroup defined by* (2.1.20) *and* (3.1.9). *Then for any* $t \geq 0$ *and* $\lambda \geq 0$ *we have*

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_t^\gamma(x, \mathrm{d}y) = \mathrm{e}^{-x\lambda} + \int_0^t \mathrm{d}s \int_0^\infty [y\phi(\lambda) - \psi(\lambda)]\mathrm{e}^{-y\lambda} Q_s^\gamma(x, \mathrm{d}y). \tag{4.1.2}$$

Proof. Recall that $v_t'(\lambda) = (\partial/\partial\lambda)v_t(\lambda)$. By differentiating both sides of (3.1.9) we get

$$\int_0^\infty y\mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y)\Big[xv_t'(\lambda) + \int_0^t \psi'(v_s(\lambda))v_s'(\lambda)\mathrm{d}s\Big].$$

From this and (2.2.3) it follows that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) &= -\Big[x\frac{\partial}{\partial t}v_t(\lambda) + \psi(v_t(\lambda))\Big]\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) \\
&= \Big[x\phi(\lambda)v_t'(\lambda) - \psi(\lambda)\Big]\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) \\
&\quad - \int_0^t \psi'(v_s(\lambda))\frac{\partial}{\partial s}v_s(\lambda)\mathrm{d}s\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) \\
&= \Big[x\phi(\lambda)v_t'(\lambda) - \psi(\lambda)\Big]\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) \\
&\quad + \phi(\lambda)\int_0^t \psi'(v_s(\lambda))v_s'(\lambda)\mathrm{d}s\int_0^\infty \mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y) \\
&= \int_0^\infty [x\phi(\lambda) - \psi(\lambda)]\mathrm{e}^{-y\lambda}Q_t^\gamma(x,\mathrm{d}y).
\end{aligned}$$

That gives (4.1.2). □

Suppose that $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses and $\{y(t) : t \geq 0\}$ is a càdlàg process in $\mathbb{R}_+$ that is adapted to $(\mathscr{G}_t)_{t\geq 0}$ and satisfies $\mathbf{P}[y(0)] < \infty$. Let us consider the following properties:

(1) For every $T \geq 0$ and $\lambda \geq 0$,

$$\exp\Big\{-v_{T-t}(\lambda)y(t) - \int_0^{T-t}\psi(v_s(\lambda))\mathrm{d}s\Big\}, \qquad 0 \leq t \leq T,$$

is a martingale.

(2) For every $\lambda \geq 0$,

$$H_t(\lambda) := \exp\Big\{-\lambda y(t) + \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{d}s\Big\}, \quad t \geq 0,$$

is a local martingale.

(3) (a) The process $\{y(t) : t \geq 0\}$ has no negative jumps. Let $N(\mathrm{d}s, \mathrm{d}z)$ be the optional random measure on $(0,\infty)^2$ defined by

$$N(\mathrm{d}s,\mathrm{d}z) = \sum_{s>0} 1_{\{\Delta y(s)\neq 0\}}\delta_{(s,\Delta y(s))}(\mathrm{d}s,\mathrm{d}z),$$

where $\Delta y(s) = y(s) - y(s-)$, and let $\hat{N}(\mathrm{d}s, \mathrm{d}z)$ denote the predictable compensator of $N(\mathrm{d}s, \mathrm{d}z)$. Then $\hat{N}(\mathrm{d}s, \mathrm{d}z) = y(s-)\mathrm{d}sm(\mathrm{d}z) + \mathrm{d}sn(\mathrm{d}z)$.

(b) If we let $\tilde{N}(\mathrm{d}s, \mathrm{d}z) = N(\mathrm{d}s, \mathrm{d}z) - \hat{N}(\mathrm{d}s, \mathrm{d}z)$, then

$$y(t) = y(0) + M_t^c + M_t^d + \int_0^t \Big[\beta + \int_0^\infty zn(\mathrm{d}z) - by(s)\Big]\mathrm{d}s,$$

where $t \mapsto M_t^c$ is a continuous local martingale with quadratic variation $2cy(t-)\mathrm{d}t$ $= 2cy(t)\mathrm{d}t$ and

$$t \mapsto M_t^d = \int_0^t \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}z)$$

is a purely discontinuous local martingale.

(4) For every $f \in C^2(\mathbb{R}_+)$ we have

$$f(y(t)) = f(y(0)) + \int_0^t Lf(y(s))\mathrm{d}s + \text{local mart.}$$

Theorem 4.1.2 *The above properties* (1), (2), (3) *and* (4) *are equivalent to each other. Those properties hold if and only if* $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ *is a CBI-process with parameters* (ϕ, ψ).

Proof. Clearly, (1) holds if and only if $\{y(t) : t \geq 0\}$ is a Markov process relative to $(\mathscr{G}_t)_{t\geq 0}$ with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ defined by (3.1.9). Then we only need to prove the equivalence of the four properties.

(1)⇒(2): Suppose that (1) holds. Then $\{y(t) : t \geq 0\}$ is a CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ given by (3.1.9). By (4.1.2) and the Markov property it is easy to see that

$$Y_t(\lambda) := \mathrm{e}^{-\lambda y(t)} + \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{e}^{-\lambda y(s)}\mathrm{d}s$$

is a martingale. By integration by parts applied to

$$Z_t(\lambda) := \mathrm{e}^{-\lambda y(t)} \;\text{ and }\; W_t(\lambda) := \exp\Big\{\int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{d}s\Big\} \tag{4.1.3}$$

we obtain

$$\mathrm{d}H_t(\lambda) = \mathrm{e}^{-\lambda y(t-)}\mathrm{d}W_t(\lambda) + W_t(\lambda)\mathrm{d}\mathrm{e}^{-\lambda y(t)} = W_t(\lambda)\mathrm{d}Y_t(\lambda).$$

Then $\{H_t(\lambda)\}$ is a local martingale.

(2)⇒(3): For any $\lambda \geq 0$ define $Z_t(\lambda)$ and $W_t(\lambda)$ by (4.1.3). We have $Z_t(\lambda) = H_t(\lambda)W_t(\lambda)^{-1}$ and so

$$\mathrm{d}Z_t(\lambda) = W_t(\lambda)^{-1}\mathrm{d}H_t(\lambda) - Z_{t-}(\lambda)[\psi(\lambda) - y(t-)\phi(\lambda)]\mathrm{d}t \tag{4.1.4}$$

by integration by parts. Then $\{Z_t(\lambda)\}$ is a special semi-martingale; see, e.g., Dellacherie and Meyer (1982, p.213). By Itô's formula we find the $\{y(t)\}$ is also a special semi-martingale. We define the optional random measure $N(\mathrm{d}s, \mathrm{d}z)$ on $[0, \infty) \times \mathbb{R}$ by

$$N(\mathrm{d}s, \mathrm{d}z) = \sum_{s>0} 1_{\{\Delta y(s) \neq 0\}} \delta_{(s, \Delta y(s))}(\mathrm{d}s, \mathrm{d}z),$$

where $\Delta y(s) = y(s) - y(s-)$. Let $\hat{N}(\mathrm{d}s, \mathrm{d}z)$ denote the predictable compensator of $N(\mathrm{d}s, \mathrm{d}z)$ and let $\tilde{N}(\mathrm{d}s, \mathrm{d}z)$ denote the compensated random measure; see Dellacherie and Meyer (1982, pp.371–374). It follows that

$$y(t) = y(0) + U_t + M_t^c + M_t^d, \tag{4.1.5}$$

where $\{U_t\}$ is a predictable process with locally bounded variations, $\{M_t^c\}$ is a continuous local martingale and

$$M_t^d = \int_0^t \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0, \tag{4.1.6}$$

is a purely discontinuous local martingale; see Dellacherie and Meyer (1982, p.353 and p.376) or Jacod and Shiryaev (2003, p.84). Let $\{C_t\}$ denote the quadratic variation process of $\{M_t^c\}$. By Itô's formula,

$$\begin{aligned} Z_t(\lambda) = {} & Z_0(\lambda) - \lambda \int_0^t Z_{s-}(\lambda)\mathrm{d}U_s + \frac{1}{2}\lambda^2 \int_0^t Z_{s-}(\lambda)\mathrm{d}C_s \\ & + \int_0^t \int_{\mathbb{R}} Z_{s-}(\lambda)\big(\mathrm{e}^{-z\lambda} - 1 + z\lambda\big)\hat{N}(\mathrm{d}s, \mathrm{d}z) + \text{local mart.} \end{aligned} \tag{4.1.7}$$

In view of (4.1.4) and (4.1.7) we get

$$[y(t)\phi(\lambda) - \psi(\lambda)]\mathrm{d}t = \frac{1}{2}\lambda^2 \mathrm{d}C_t - \lambda \mathrm{d}U_t + \int_{\mathbb{R}} \big(\mathrm{e}^{-z\lambda} - 1 + z\lambda\big)\hat{N}(\mathrm{d}t, \mathrm{d}z)$$

by the uniqueness of canonical decompositions of special semi-martingales; see Dellacherie and Meyer (1982, p.213). By substituting the representation (2.1.13) of ϕ into the above equation and comparing both sides it is easy to find that (3.a) and (3.b) hold.

(3)⇒(4): This follows by Itô's formula.

(4)⇒(1): Let $G = G(t, x) \in C^{1,2}([0, T] \times \mathbb{R}_+)$. For $0 \leq t \leq T$ and $k \geq 1$ we have

$$G(t, y(t)) = G(0, y(0)) + \sum_{j=0}^{\infty} \big[G(t \wedge j/k, y(t \wedge (j+1)/k)) - G(t \wedge j/k, y(t \wedge j/k))\big]$$

$$+\sum_{j=0}^{\infty}\big[G(t\wedge(j+1)/k, y(t\wedge(j+1)/k)) - G(t\wedge j/k, y(t\wedge(j+1)/k))\big],$$

where the summations only consist of finitely many non-trivial terms. By applying (4) term by term we obtain

$$\begin{aligned}G(t,y(t)) = {}& G(0,y(0)) + \sum_{j=0}^{\infty}\int_{t\wedge j/k}^{t\wedge(j+1)/k}\Big\{[\beta - by(s)]G'_y(t\wedge j/k, y(s)) \\ & + cy(s)G''_{xx}(t\wedge j/k, y(s)) + y(s)\int_0^\infty\Big[G(t\wedge j/k, y(s)+z) \\ & - G(t\wedge j/k, y(s)) - zG'_y(t\wedge j/k, y(s))\Big]m(\mathrm{d}z) \\ & + \int_0^\infty\Big[G(t\wedge j/k, y(s)+z) - G(t\wedge j/k, y(s))\Big]n(\mathrm{d}z)\Big\}\mathrm{d}s \\ & + \sum_{j=0}^{\infty}\int_{t\wedge j/k}^{t\wedge(j+1)/k} G'_t(s, y(t\wedge(j+1)/k))\mathrm{d}s + M_k(t),\end{aligned}$$

where $\{M_k(t)\}$ is a local martingale. Since $\{y(t)\}$ is a càdlàg process, letting $k\to\infty$ in the equation above gives

$$\begin{aligned}G(t,y(t)) = {}& G(0,y(0)) + \int_0^t\Big\{G'_t(s,y(s)) - by(s)G'_y(s,y(s)) \\ & + cy(s)G''_{xx}(s,y(s)) + y(s)\int_0^\infty\Big[G(s,y(s)+z) \\ & - G(s,y(s)) - zG'_x(s,y(s))\Big]m(\mathrm{d}z) \\ & + \int_0^\infty\Big[G(s,y(s)+z) - G(s,y(s))\Big]n(\mathrm{d}z)\Big\}\mathrm{d}s + M(t),\end{aligned}$$

where $\{M(t)\}$ is a local martingale. For any $T\ge 0$ and $\lambda\ge 0$ we may apply the above to

$$G(t,x) = \exp\Big\{-v_{T-t}(\lambda)x - \int_0^{T-t}\psi(v_s(\lambda))\mathrm{d}s\Big\}$$

to see $t\mapsto G(t,y(t))$ is a local martingale. □

The above property (4) implies that the generator of the CBI-process is the closure of the generator L in the sense of Ethier and Kurtz (1986). This explicit form of the generator was first given in Kawazu and Watanabe (1971).

4.2 Stochastic equations of CBI-processes

In this section we establish some stochastic equations for the CBI-processes. The reader may refer to Dawson and Li (2006, 2010), Fu and Li (2010) and Li and Ma (2008) for more results on this topic. Suppose that (ϕ, ψ) are given respectively by (2.1.13) and (3.1.10) with $un(\mathrm{d}u)$ being a finite measure on $(0, \infty)$. Let $(Q_t^\gamma)_{t\geq 0}$ be the transition semigroup defined by (2.1.20) and (3.1.9). In this section, we derive some stochastic equations for the CBI-processes.

Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\{B(t) : t \geq 0\}$ be an $(\mathscr{F}_t)$-Brownian motion and let $\{p_0(t) : t \geq 0\}$ and $\{p_1(t) : t \geq 0\}$ be $(\mathscr{F}_t)$-Poisson point processes on $(0, \infty)^2$ with characteristic measures $m(dz)du$ and $n(dz)du$, respectively. We assume that the white noise and the Poisson processes are independent of each other. Let $N_0(ds, dz, du)$ and $N_1(ds, dz, du)$ denote the Poisson random measures on $(0, \infty)^3$ associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let $\tilde{N}_0(ds, dz, du)$ denote the compensated measure of $N_0(ds, dz, du)$. Let us consider the stochastic integral equation

$$\begin{aligned} y(t) = y(0) &+ \int_0^t \sqrt{2cy(s)}\mathrm{d}B(s) + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t (\beta - by(s))\mathrm{d}s + \int_0^t \int_0^\infty zN_1(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{4.2.1}$$

where $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$. We understand the last term on the right-hand side as an integral over the set $\{(s, z, u) : 0 < s \leq t, 0 < z < \infty, 0 < u \leq y(s-)\}$ and give similar interpretations for other integrals with respect to Poisson random measures in this section.

Theorem 4.2.1 *There is a unique positive weak solution to* (4.2.1) *and the solution is a CBI-process with transition semigroup* $(Q_t^\gamma)_{t\geq 0}$.

Proof. Suppose that $\{y(t)\}$ is a càdlàg realization of the CBI-process with transition semigroup given by (2.1.20) and (3.1.9). By Theorem 4.1.2 the process has no negative jumps and the random measure

$$N(\mathrm{d}s, \mathrm{d}z) := \sum_{s>0} 1_{\{y(s)\neq y(s-)\}}\delta_{(s, y(s)-y(s-))}(\mathrm{d}s, \mathrm{d}z)$$

has predictable compensator

$$\hat{N}(\mathrm{d}s, \mathrm{d}z) = y(s-)\mathrm{d}sm(\mathrm{d}z) + \mathrm{d}sn(\mathrm{d}z)$$

and

$$\begin{aligned} y(t) = y(0) &+ t\Big[\beta + \int_0^\infty un(\mathrm{d}u)\Big] - \int_0^t by(s-)\mathrm{d}s \\ &+ M^c(t) + \int_0^t \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{4.2.2}$$

where $\tilde{N}(\mathrm{d}s, \mathrm{d}z) = N(\mathrm{d}s, \mathrm{d}z) - \hat{N}(\mathrm{d}s, \mathrm{d}z)$ and $t \mapsto M^c(t)$ is a continuous local martingale with quadratic variation $2cy(t-)\mathrm{d}t$. By representation theorems for semimartingales, we have equation (4.2.1) on an extension of the original probability space; see, e.g., Ikeda and Watanabe (1989, p.90 and p.93). That proves the existence of a weak solution to (4.2.1). Conversely, if $\{y(t)\}$ is a positive solution to (4.2.1), one can use Itô's formula to see the process is a solution of the martingale problem associated with the generator L defined by (4.1.1). By Theorem 4.2.1 we see $\{y(t)\}$ is a CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$. That implies the weak uniqueness of the solution to (4.2.1). $\square$

Theorem 4.2.2 *Suppose that $m(\mathrm{d}z) = qz^{-1-\alpha}\mathrm{d}z$ for constants $q \geq 0$ and $1 < \alpha < 2$. Then the CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ is the unique positive weak solution of*

$$\mathrm{d}y(t) = \sqrt{2cy(t)}\mathrm{d}B(t) + \sqrt[\alpha]{qy(t-)}\mathrm{d}z_0(t) - by(t)\mathrm{d}t + \mathrm{d}z_1(t), \tag{4.2.3}$$

where $\{B(t)\}$ is a Brownian motion, $\{z_0(t)\}$ is a one-sided α-stable process with Lévy measure $z^{-1-\alpha}\mathrm{d}z$, $\{z_1(t)\}$ is an increasing Lévy process defined by (β, n), and $\{B(t)\}$, $\{z_0(t)\}$ and $\{z_1(t)\}$ are independent of each other.

Proof. We assume $q > 0$, for otherwise the proof is easier. Let us consider the CBI-process $\{y(t)\}$ given by (4.2.1) with $\{N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ being a Poisson random measure on $(0, \infty)^3$ with intensity $qz^{-1-\alpha}\mathrm{d}s\mathrm{d}z\mathrm{d}u$. We define the random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$ on $(0, \infty)^2$ by

$$\begin{aligned} N((0,t] \times B) = & \int_0^t \int_0^\infty \int_0^{y(s-)} 1_{\{y(s-)>0\}} 1_B\Big(\frac{z}{\sqrt[\alpha]{qy(s-)}}\Big) N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ & + \int_0^t \int_0^\infty \int_0^{1/q} 1_{\{y(s-)=0\}} 1_B(z) N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

It is easy to compute that $\{N(\mathrm{d}s, \mathrm{d}z)\}$ has predictable compensator

$$\begin{aligned} \hat{N}((0,t] \times B) &= \int_0^t \int_0^\infty 1_{\{y(s-)>0\}} 1_B\Big(\frac{z}{\sqrt[\alpha]{qy(s-)}}\Big) \frac{qy(s-)\mathrm{d}s\mathrm{d}z}{z^{1+\alpha}} \\ &\quad + \int_0^t \int_0^\infty 1_{\{y(s-)=0\}} 1_B(z) \frac{\mathrm{d}s\mathrm{d}z}{z^{1+\alpha}} \\ &= \int_0^t \int_0^\infty 1_B(z) \frac{\mathrm{d}s\mathrm{d}z}{z^{1+\alpha}}. \end{aligned}$$

Thus $\{N(\mathrm{d}s, \mathrm{d}z)\}$ is a Poisson random measure with intensity $z^{-1-\alpha}\mathrm{d}s\mathrm{d}z$; see, e.g., Ikeda and Watanabe (1989, p.93). Now define the Lévy processes

$$z_0(t) = \int_0^t \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}z) \ \text{ and } \ z_1(t) = \beta t + \int_0^t \int_0^\infty zN_1(\mathrm{d}s, \mathrm{d}z),$$

where $\tilde{N}(\mathrm{d}s, \mathrm{d}z) = N(\mathrm{d}s, \mathrm{d}z) - \hat{N}(\mathrm{d}s, \mathrm{d}z)$. It is easy to see that

$$\begin{aligned}\int_0^t \sqrt[\alpha]{qy(s-)}\mathrm{d}z_0(s) &= \int_0^t \int_0^\infty \sqrt[\alpha]{qy(s-)}\, z\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &= \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).\end{aligned}$$

Then we get (4.2.3) from (4.2.1). Conversely, if $\{y(t)\}$ is a solution of (4.2.3), one can use Itô's formula to see that $\{y(t)\}$ solves the martingale problem associated with the generator L defined by (4.1.1) with $m(\mathrm{d}z) = qz^{-1-\alpha}\mathrm{d}z$. Then $\{y(t)\}$ is a CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ and the solution of (4.2.3) is unique in law. □

Theorem 4.2.3 *The pathwise uniqueness holds for positive solutions to* (4.2.1).

Proof. For each integer $n \geq 0$ define $a_n = \exp\{-n(n+1)/2\}$. Then $a_n \to 0$ decreasingly as $n \to \infty$ and

$$\int_{a_n}^{a_{n-1}} z^{-1} dz = n, \qquad n \geq 1.$$

Let $x \mapsto g_n(x)$ be a positive continuous function supported by (a_n, a_{n-1}) so that

$$\int_{a_n}^{a_{n-1}} g_n(x) dx = 1$$

and $g_n(x) \leq 2(nx)^{-1}$ for every $x > 0$. For $n \geq 0$ let

$$f_n(z) = \int_0^{|z|} dy \int_0^y g_n(x) dx, \qquad z \in \mathbb{R}.$$

It is easy to see that $|f_n'(z)| \leq 1$ and

$$0 \leq |z| f_n''(z) = |z| g_n(|z|) \leq 2n^{-1}, \qquad z \in \mathbb{R}.$$

Moreover, we have $f_n(z) \to |z|$ increasingly as $n \to \infty$. Suppose that $\{y(t) : t \geq 0\}$ and $\{z(t) : t \geq 0\}$ are both positive solutions of (4.2.1). Let $\alpha_t = z(t) - y(t)$ for $t \geq 0$. From (4.2.1) we have

$$\begin{aligned}\alpha_t = \alpha_0 - b\int_0^t \alpha_{s-} ds + \sqrt{2c}\int_0^t \big(\sqrt{z(s)} - \sqrt{y(s)}\big) dB(s) \\ + \int_0^t \int_0^\infty \int_{y(s-)}^{z(s-)} z\tilde{N}_0(ds, dz, du).\end{aligned}$$

By this and Itô's formula,

$$\begin{aligned} f_n(\alpha_t) = {} & f_n(\alpha_0) - b\int_0^t f_n'(\alpha_s)\alpha_s ds + c\int_0^t f_n''(\alpha_s)\big(\sqrt{z(s)} - \sqrt{y(s)}\big)^2 ds \\ & + \int_0^t \alpha_s 1_{\{\alpha_s>0\}} ds \int_0^\infty [f_n(\alpha_s+z) - f_n(\alpha_s) - zf_n'(\alpha_s)]m(dz) \\ & - \int_0^t \alpha_s 1_{\{\alpha_s<0\}} ds \int_0^\infty [f_n(\alpha_s-z) - f_n(\alpha_s) + zf_n'(\alpha_s)]m(dz) \\ & + \text{martingale}. \end{aligned} \tag{4.2.4}$$

It is easy to see that $|f_n(a+x) - f_n(a)| \leq |x|$ for any $a, x \in \mathbb{R}$. If $ax \geq 0$, we have

$$|f_n(a+x) - f_n(a) - xf_n'(a)| \leq (2|ax|) \wedge (n^{-1}|x|^2).$$

Taking the expectation in both sides of (4.2.4) gives

$$\begin{aligned} \mathbf{P}[f_n(\alpha_t)] \leq {} & \mathbf{P}[f_n(\alpha_0)] + |b|\int_0^t \mathbf{P}[|\alpha_s|]ds + c\int_0^t \mathbf{P}[f_n''(\alpha_s)|\alpha_s|]ds \\ & + \int_0^t ds\int_0^\infty \{(2z\mathbf{P}[|\alpha_s|]) \wedge (n^{-1}z^2)\}m(dz). \end{aligned}$$

Then letting $n \to \infty$ we get

$$\mathbf{P}[|z(t) - y(t)|] \leq \mathbf{P}[|z(0) - y(0)|] + |b|\int_0^t \mathbf{P}[|z(s) - y(s)|]ds.$$

By this and Gronwall's inequality one can see the pathwise uniqueness holds for (4.2.1). □

Theorem 4.2.3 was first proved in Dawson and Li (2006), see also Fu and Li (2010) and Li and Ma (2008). By Theorems 4.2.1 and 4.2.3 there is a unique positive strong solution to (4.2.1); see, e.g., Situ (2005, p.76 and p.104). The pathwise uniqueness of (4.2.3) was proved in Fu and Li (2010).

4.3 Lamperti's transformations by time changes

The results of Lamperti (1967b) assert that CB-processes are in one-to-one correspondence with spectrally positive Lévy processes via simple random time changes. Caballero et al. (2009) recently gave proofs of those results using the approach of stochastic equations; see also Helland (1978) and Silverstein (1968). Suppose that ϕ is a branching mechanism given by (2.1.13). Let $\{x(t) : t \geq 0\}$ be a CB-process with $x(0) = x \geq 0$

and with branching mechanism ϕ is given by (2.1.13). Let $\{Y_t : t \geq 0\}$ be a spectrally positive Lévy process such that $Y_0 = x$ and

$$\log \mathbf{P} \exp\{i\lambda(Y_t - Y_r)\} = (t-r)\phi(-i\lambda), \qquad \lambda \in \mathbb{R}, t \geq r \geq 0. \tag{4.3.1}$$

Let $\tau = \inf\{s \geq 0 : Y_s = 0\}$ be its first hitting time at zero and let $Z_t = Y_{t\wedge\tau}$ for $t \geq 0$. The proofs of the following two theorems were essentially adopted from Caballero et al. (2009).

Theorem 4.3.1 *For any $t \geq 0$ let $z(t) = x(\kappa(t))$, where*

$$\kappa(t) = \inf\Big\{u \geq 0 : \int_0^u x(s-)\mathrm{d}s = \int_0^u x(s)\mathrm{d}s \geq t\Big\}. \tag{4.3.2}$$

Then $\{z(t) : t \geq 0\}$ is distributed identically on $D[0,\infty)$ with $\{Z_t : t \geq 0\}$.

Proof. By the result of Theorem 4.2.1, we may assume $\{x(t)\}$ solves the stochastic integral equation

$$\begin{aligned} x(t) = x &+ \int_0^t \sqrt{2cx(s-)}\mathrm{d}B(s) - \int_0^t bx(s-)\mathrm{d}s \\ &+ \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{4.3.3}$$

where $\{B(t)\}$ is a Brownian motion, $\{N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ is a Poisson random measures on $(0,\infty)^3$ with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ and $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$. It follows that

$$\begin{aligned} z(t) &= x + \int_0^{\kappa(t)} \sqrt{2cx(s-)}\mathrm{d}B(s) - \int_0^{\kappa(t)} bx(s-)\mathrm{d}s \\ &\quad + \int_0^{\kappa(t)} \int_0^\infty \int_0^{x(s-)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &= x + \sqrt{2c}W(t) - b\int_0^t z(s-)\mathrm{d}\kappa(s) \\ &\quad + \int_0^t \int_0^\infty \int_0^{x(\kappa(s)-)} z\tilde{N}_0(\mathrm{d}\kappa(s), \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{4.3.4}$$

where

$$W(t) = \int_0^{\kappa(t)} \sqrt{x(s-)}\mathrm{d}B(s) = \int_0^t \sqrt{z(s-)}\mathrm{d}B(\kappa(s))$$

is a continuous martingale. From (4.3.2) we have

$$1_{\{z(s-)>0\}}\mathrm{d}\kappa(s) = 1_{\{z(s-)>0\}}z(s-)^{-1}\mathrm{d}s.$$

Let $\tau_0 = \inf\{t \geq 0 : z(t) = 0\}$. Since zero is a trap for $\{z(t)\}$, we have

$$\int_0^t z(s-)\mathrm{d}\kappa(s) = \int_0^t 1_{\{z(s-)>0\}}\mathrm{d}s = t \wedge \tau_0.$$

Then $\{W(t)\}$ has quadratic variation process $\langle W\rangle(t) = t \wedge \tau_0$, so it is a Brownian motion stopped at τ_0. It is easy to extend $\{W(t)\}$ to a Brownian motion with infinite time. Now define the random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$ on $(0,\infty)^2$ by

$$N((0,t] \times (a,b]) = \int_0^t \int_a^b \int_0^{z(s-)} 1_{\{z(s-)>0\}} N_0(\mathrm{d}\kappa(s), \mathrm{d}z, \mathrm{d}u),$$

where $t \geq 0$ and $b \geq a > 0$. It is easy to compute that $\{N((0,t] \times (a,b]) : t \geq 0\}$ has predictable compensator

$$\hat{N}((0,t] \times (a,b]) = \int_0^t m(a,b]z(s-)\mathrm{d}\kappa(s) = \int_0^t m(a,b]1_{\{s\leq\tau_0\}}\mathrm{d}s.$$

Then we can extend $\{N(\mathrm{d}s, \mathrm{d}z)\}$ is a Poisson random measure on $(0,\infty)^2$ with intensity $\mathrm{d}s m(\mathrm{d}z)$; see, e.g., Ikeda and Watanabe (1989, p.93). From (4.3.4) we conclude that $\{z(t)\}$ is distributed on $D[0,\infty)$ identically with $\{Z_t : t \geq 0\}$. □

Theorem 4.3.2 *For any $t \geq 0$ let $X_t = Z_{\theta(t)}$, where*

$$\theta(t) = \inf\Big\{u \geq 0 : \int_0^u Z_{s-}^{-1}\mathrm{d}s = \int_0^u Z_s^{-1}\mathrm{d}s \geq t\Big\}. \tag{4.3.5}$$

Then $\{X_t : t \geq 0\}$ is distributed identically on $D[0,\infty)$ with $\{x(t) : t \geq 0\}$.

Proof. By the Lévy–Itô decomposition, up to an extension of the original probability space we may assume $\{Y_t\}$ is given by

$$Y_t = x + \sqrt{2c}W(t) - bt + \int_0^t \int_0^\infty \int_0^1 z\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $\{W(t)\}$ is a Brownian motion, $\{M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ is a Poisson random measures on $(0,\infty)^3$ with intensity $\mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$ and $\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$. It follows that

$$X_t = x + \sqrt{2c}W(\theta(t)) - b\theta(t) + \int_0^t \int_0^\infty \int_0^1 z\tilde{M}_0(\mathrm{d}\theta(s), \mathrm{d}z, \mathrm{d}u). \tag{4.3.6}$$

From (4.3.5) we have

$$\theta(t) = \int_0^t Z_{\theta(s)}\mathrm{d}s = \int_0^t X_s\mathrm{d}s.$$

Then the continuous martingale $\{W(\theta(t))\}$ has the representation

$$W(\theta(t)) = \int_0^t \sqrt{X_s}\mathrm{d}B(s), \qquad t \geq 0$$

for another Brownian motion $\{B(t)\}$. Now we take an independent Poisson random measure $\{M_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ on $(0, \infty)^3$ with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ and define the random measure

$$N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = 1_{\{u \leq X_{s-}\}} M_0(\mathrm{d}\theta(s), \mathrm{d}z, X_{s-}^{-1}\mathrm{d}u) + 1_{\{u > X_{s-}\}} M_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

It is easy to see that $\{N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ has deterministic compensator $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$, so it is a Poisson random measures. From (4.3.6) we see that $\{X_t\}$ is a weak solution of (4.3.3). That gives the desired result. □

Chapter 5

State-dependent immigration structures

In this chapter we investigate the structures of state-dependent immigration associated with CB-processes. For simplicity, we only consider interactive immigration rates. The models are defined in terms of some stochastic integral equations generalizing (4.2.1). We prove the existence and pathwise uniqueness of solutions to the stochastic integral equations. Similar immigration structures were studied in Li (2011) in the setting of superprocesses by considering different types of stochastic equations. We shall deal with processes with càdlàg paths as in Li (2011). Let ϕ is a branching mechanism given by (2.1.13). For notational convenience, we defined the constant $\sigma = \sqrt{2c}$, which will be used throughout this chapter.

5.1 Time-dependent immigration

In this section, we introduce a generalization of the CBI-process. Let $(Q_t)_{t\geq 0}$ be the transition semigroup defined by (2.1.15) and (2.1.20). We consider a set of functions $\{\psi_s : s \geq 0\} \subset \mathscr{I}$ given by

$$\psi_s(z) = \beta_s z + \int_0^\infty \big(1 - \mathrm{e}^{-zu}\big) n_s(\mathrm{d}u), \qquad z \geq 0, \tag{5.1.1}$$

where $\beta_s \geq 0$ and $(1 \wedge u) n_s(\mathrm{d}u)$ is a finite measure on $(0, \infty)$. We assume $s \mapsto \psi_s(z)$ is locally bounded and measurable on $[0, \infty)$ for each $z \geq 0$. By Theorems 1.2.3 and 1.2.4, for any $t \geq r \geq 0$ there is an infinitely divisible probability measure $\gamma_{r,t}$ on $[0, \infty)$ defined by

$$\int_0^\infty \mathrm{e}^{-\lambda y} \gamma_{r,t}(\mathrm{d}y) = \exp\Big\{ - \int_r^t \psi_s(v_{t-s}(\lambda)) \rho(s) \mathrm{d}s \Big\}, \qquad \lambda \geq 0. \tag{5.1.2}$$

Then we can define the probability kernels $(Q^\gamma_{r,t} : t \geq r \geq 0)$ by

$$Q^\gamma_{r,t}(x, \cdot) := Q_{t-r}(x, \cdot) * \gamma_{r,t}(\cdot), \qquad x \geq 0. \tag{5.1.3}$$

It is easily seen that

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_{r,t}^\gamma(x, \mathrm{d}y) = \exp\Big\{ - x v_{t-r}(\lambda) - \int_r^t \psi_s(v_{t-s}(\lambda))\mathrm{d}s \Big\}. \tag{5.1.4}$$

Moreover, the kernels $(Q_{r,t}^\gamma : t \geq r \geq 0)$ form a transition semigroup on $\mathbb{R}_+$. A Markov process with transition semigroup $(Q_{r,t}^\gamma : t \geq r \geq 0)$ is called a *special inhomogeneous CBI-process* with *branching mechanism* ϕ and *time-dependent immigration mechanism* $\{\psi_s : s \geq 0\}$. One can see that the time-space homogeneous transition semigroup associated with $(Q_{r,t}^\gamma : t \geq r \geq 0)$ is a Feller semigroup. Then $(Q_{r,t}^\gamma : t \geq r \geq 0)$ has a càdlàg realization $X = (\Omega, \mathscr{F}, \mathscr{F}_t, y(t), \mathbf{Q}_{r,x}^\gamma)$. In particular, if $un_s(\mathrm{d}u)$ is a locally bounded kernel from $[0, \infty)$ to $(0, \infty)$, one can derive from (2.2.4) and (5.1.4) that

$$\int_0^\infty y Q_{r,t}^\gamma(x, \mathrm{d}y) = x\mathrm{e}^{-b(t-r)} + \int_r^t \mathrm{e}^{-b(t-s)} \psi_s'(0)\mathrm{d}s. \tag{5.1.5}$$

where

$$\psi_s'(0) = \beta_s + \int_0^\infty z n_s(\mathrm{d}z).$$

The reader may refer to Li (2002) for the discussions of general inhomogeneous immigration processes in the setting of measure-valued processes.

5.2 Predictable immigration rates

Let ϕ be a branching mechanism given by (2.1.13) and ψ an immigration mechanism given by (3.1.10). In this section, we give a construction of CBI-processes with random immigration rates given by predictable processes. Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\{B(t) : t \geq 0\}$ be an $(\mathscr{F}_t)$-Brownian motion and let $\{p_0(t) : t \geq 0\}$ and $\{p_1(t) : t \geq 0\}$ be $(\mathscr{F}_t)$-Poisson point processes on $(0, \infty)^2$ with characteristic measures $m(\mathrm{d}z)du$ and $n(\mathrm{d}z)du$, respectively. We assume that the white noise and the Poisson processes are independent of each other. Let $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote the Poisson random measures on $(0, \infty)^3$ associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote the compensated measure of $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$. Suppose that $\rho = \{\rho(t) : t \geq 0\}$ is a positive $(\mathscr{F}_t)$-predictable process such that $t \mapsto \mathbf{P}[\rho(t)]$ is locally bounded. We are interested in positive càdlàg solutions of the stochastic equation

$$\begin{aligned} Y_t = Y_0 + \sigma \int_0^t \sqrt{Y_{s-}}\mathrm{d}B(s) + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ + \int_0^t (\beta\rho(s) - bY_{s-})\mathrm{d}s + \int_0^t \int_0^\infty \int_0^{\rho(s)} zN_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{5.2.1}$$

Clearly, the above equation is a generalization of (4.2.1). For any positive càdlàg solution $\{Y_t : t \geq 0\}$ of (5.2.1) satisfying $\mathbf{P}[Y_0] < \infty$, one can use a standard stopping time argument to show that $t \mapsto \mathbf{P}[Y_t]$ is locally bounded and

$$\mathbf{P}[Y_t] = \mathbf{P}[Y_0] + \psi'(0) \int_0^t \mathbf{P}[\rho(s)]\mathrm{d}s - b \int_0^t \mathbf{P}[Y_s]\mathrm{d}s, \tag{5.2.2}$$

where $\psi'(0)$ is defined by (3.1.12). By Itô's formula, it is easy to see that $\{Y_t : t \geq 0\}$ solves the following martingale problem: For every $f \in C^2(\mathbb{R}_+)$,

$$\begin{aligned} f(Y_t) = &\; f(Y_0) + \text{local mart.} - b \int_0^t f'(Y_s)Y_s\mathrm{d}s + \frac{1}{2}\sigma^2 \int_0^t f''(Y_s)Y_s\mathrm{d}s \\ &+ \int_0^t Y_s\mathrm{d}s \int_0^\infty [f(Y_s + z) - f(Y_s) - zf'(Y_s)]m(\mathrm{d}z) \\ &+ \int_0^t \rho(s)\Big\{\beta f'(Y_s) + \int_0^\infty [f(Y_s + z) - f(Y_s)]n(\mathrm{d}z)\Big\}\mathrm{d}s. \end{aligned} \tag{5.2.3}$$

Proposition 5.2.1 *Suppose that* $\{Y_t : t \geq 0\}$ *is a positive càdlàg solution of* (5.2.1) *and* $\{Z_t : t \geq 0\}$ *is a positive càdlàg solution of the equation with* $\rho = \{\rho(t) : t \geq 0\}$ *replaced by* $\eta = \{\eta(t) : t \geq 0\}$*. Then for any* $t \geq 0$ *we have*

$$\mathbf{P}[|Z_t - Y_t|] \leq \mathrm{e}^{|b|t}\Big\{\mathbf{P}[|Z_0 - Y_0|] + \psi'(0) \int_0^t \mathbf{P}[|\eta(s) - \rho(s)|]\mathrm{d}s\Big\}, \tag{5.2.4}$$

where $\psi'(0)$ *is defined by* (3.1.12).

Proof. The following arguments are modifications of those in the proof of Theorem 4.2.3. Let $\{f_n\}$ be the function sequence defined there. Write $\alpha_t = Z_t - Y_t$ for $t \geq 0$. From (5.2.1) we have

$$\begin{aligned} \alpha_t = &\; \alpha_0 + \beta \int_0^t [\eta(s) - \rho(s)]\mathrm{d}s + \sigma \int_0^t (\sqrt{Z_{s-}} - \sqrt{Y_{s-}})\mathrm{d}B(s) \\ &- b \int_0^t \alpha_{s-}\mathrm{d}s + \int_0^t \int_0^\infty \int_{Y_{s-}}^{Z_{s-}} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t \int_0^\infty \int_{\rho(s)}^{\eta(s)} zN_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{5.2.5}$$

By this and Itô's formula,

$$\begin{aligned} f_n(\alpha_t) = &\; f_n(\alpha_0) + \beta \int_0^t f_n'(\alpha_s)[\eta(s) - \rho(s)]\mathrm{d}s - b \int_0^t f_n'(\alpha_s)\alpha_s\mathrm{d}s \\ &+ \frac{1}{2}\sigma^2 \int_0^t f_n''(\alpha_s)(\sqrt{Z_{s-}} - \sqrt{Y_{s-}})^2\mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&+\int_0^t \alpha_s 1_{\{\alpha_s>0\}}\mathrm{d}s\int_0^\infty [f_n(\alpha_s+z)-f_n(\alpha_s)-zf_n'(\alpha_s)]m(\mathrm{d}z)\\
&-\int_0^t \alpha_s 1_{\{\alpha_s<0\}}\mathrm{d}s\int_0^\infty [f_n(\alpha_s-z)-f_n(\alpha_s)+zf_n'(\alpha_s)]m(\mathrm{d}z)\\
&+\int_0^t [\eta(s)-\rho(s)]1_{\{\eta(s)>\rho(s)\}}\mathrm{d}s\int_0^\infty [f_n(\alpha_s+z)-f_n(\alpha_s)]n(\mathrm{d}z)\\
&-\int_0^t [\rho(s)-\eta(s)]1_{\{\rho(s)>\eta(s)\}}\mathrm{d}s\int_0^\infty [f_n(\alpha_s-z)-f_n(\alpha_s)]n(\mathrm{d}z)\\
&+\text{martingale.} \qquad (5.2.6)
\end{aligned}$$

It is easy to see that $|f_n(a+x)-f_n(a)|\le |x|$ for any $a,x\in\mathbb{R}$. If $ax\ge 0$, we have

$$|f_n(a+x)-f_n(a)-xf_n'(a)|\le (2|ax|)\wedge(n^{-1}|x|^2).$$

Taking the expectation in both sides of (5.2.6) gives

$$\begin{aligned}
\mathbf{P}[f_n(\alpha_t)] \le\ & \mathbf{P}[f_n(\alpha_0)]+\beta\int_0^t \mathbf{P}[|\eta(s)-\rho(s)|]\mathrm{d}s+|b|\int_0^t \mathbf{P}[|\alpha_s|]\mathrm{d}s\\
&+\int_0^t \mathbf{P}[|\eta(s)-\rho(s)|]\mathrm{d}s\int_0^\infty zn(\mathrm{d}z)+n^{-1}\sigma^2 t\\
&+\int_0^t \mathrm{d}s\int_0^\infty \{(2z\mathbf{P}[|\alpha_s|])\wedge(n^{-1}z^2)\}m(\mathrm{d}z).
\end{aligned}$$

By letting $n\to\infty$ we get

$$\begin{aligned}
\mathbf{P}[|Z_t-Y_t|] \le\ & \mathbf{P}[|Z_0-Y_0|]+|b|\int_0^t \mathbf{P}[|Z_s-Y_s|]\mathrm{d}s\\
&+\psi'(0)\int_0^t \mathbf{P}[|\eta(s)-\rho(s)|]\mathrm{d}s. \qquad (5.2.7)
\end{aligned}$$

Then we get the desired estimate follows by Gronwall's inequality. □

Proposition 5.2.2 *Suppose that* $\{Y_t: t\ge 0\}$ *is a positive càdlàg solution of* (5.2.1) *and* $\{Z_t: t\ge 0\}$ *is a positive càdlàg solution of the equation with* (b,ρ) *replaced by* (c,η)*. Then for any* $t\ge 0$ *we have*

$$\begin{aligned}
\mathbf{P}\Big[\sup_{0\le s\le t}|Z_s-Y_s|\Big] \le\ & \mathbf{P}[|Z_0-Y_0|]+\psi'(0)\int_0^t \mathbf{P}[|\eta(s)-\rho(s)|]\mathrm{d}s\\
&+\Big(|b|+2\int_1^\infty zm(\mathrm{d}z)\Big)\int_0^t \mathbf{P}[|Z_s-Y_s|]\mathrm{d}s\\
&+2\sigma\Big(\int_0^t \mathbf{P}[|Z_s-Y_s|]\mathrm{d}s\Big)^{\frac12}\\
&+2\Big(\int_0^t \mathbf{P}[|Z_s-Y_s|]\mathrm{d}s\int_0^1 z^2m(\mathrm{d}z)\Big)^{\frac12},
\end{aligned}$$

where $\psi'(0)$ *is defined by* (3.1.12).

Proof. This follows by applying Doob's martingale inequality to (5.2.5). □

Theorem 5.2.3 *For any* $Y_0 \geq 0$ *there is a pathwise unique positive càdlàg solution* $\{Y_t : t \geq 0\}$ *of* (5.2.1).

Proof. The pathwise uniqueness of the solution follows by Proposition 5.2.1 and Gronwall's inequality. Without loss of generality, we may assume $Y_0 \geq 0$ is deterministic in proving the existence of the solution. We give the proof in two steps.

Step 1. Let $0 = r_0 < r_1 < r_2 < \cdots$ be an increasing sequence. For each $i \geq 1$ let η_i be a positive integrable random variable measurable with respect to $\mathscr{F}_{r_{i-1}}$. Let $\rho = \{\rho(t) : t \geq 0\}$ be the positive $(\mathscr{F}_t)$-predictable step process given by

$$\rho(t) = \sum_{i=1}^{\infty} \eta_i 1_{(r_{i-1}, r_i]}(t), \qquad t \geq 0.$$

By Theorem 4.2.1, on each interval $(r_{i-1}, r_i]$ there is a pathwise unique solution $\{Y_t : r_{i-1} < t \leq r_i\}$ to

$$\begin{aligned} Y_t = Y_{r_{i-1}} &+ \sigma \int_{r_{i-1}}^t \sqrt{Y_{s-}} \mathrm{d}B(s) + \int_{r_{i-1}}^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_{r_{i-1}}^t (\beta\eta_i - bY_{s-})\mathrm{d}s + \int_{r_{i-1}}^t \int_0^\infty \int_0^{\eta_i} zN_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

Then $\{Y_t : t \geq 0\}$ is a solution to (5.2.1).

Step 2. Suppose that $\rho = \{\rho(t) : t \geq 0\}$ is general positive $(\mathscr{F}_t)$-predictable process such that $t \mapsto \mathbf{P}[\rho(t)]$ is locally bounded. Take a sequence of positive predictable step processes $\rho_k = \{\rho_k(t) : t \geq 0\}$ so that

$$\mathbf{P}\Big[\int_0^t |\rho_k(s) - \rho(s)|\mathrm{d}s\Big] \to 0 \tag{5.2.8}$$

for every $t \geq 0$ as $k \to \infty$. Let $\{Y_k(t) : t \geq 0\}$ be the solution to (5.2.1) with $\rho = \rho_k$. By Proposition 5.2.1, Gronwall's inequality and (5.2.8) one sees

$$\sup_{0 \leq s \leq t} \mathbf{P}[|Y_k(s) - Y_i(s)|] \to 0$$

for every $t \geq 0$ as $i, k \to \infty$. Then Proposition 5.2.2 implies

$$\mathbf{P}\Big[\sup_{0 \leq s \leq t} |Y_k(s) - Y_i(s)|\Big] \to 0$$

for every $t \geq 0$ as $i, k \to \infty$. Thus there is a subsequence $\{k_i\} \subset \{k\}$ and a càdlàg process $\{Y_t : t \geq 0\}$ so that

$$\sup_{0 \leq s \leq t} |Y_{k_i}(s) - Y_s| \to 0$$

almost surely for every $t \geq 0$ as $i \to \infty$. It is routine to show that $\{Y_t : t \geq 0\}$ is a solution to (5.2.1). □

Theorem 5.2.4 *If $\rho = \{\rho(t) : t \geq 0\}$ is a deterministic locally bounded positive Borel function, the solution $\{Y_t : t \geq 0\}$ of* (5.2.1) *is a special inhomogeneous CBI-process with branching mechanism ϕ and time-dependent immigration mechanisms $\{\rho(t)\psi : t \geq 0\}$.*

Proof. By Theorem 4.2.1, when $\rho(t) = \rho$ is a deterministic constant function, the process $\{Y_t : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanisms $\rho\psi$. If $\rho = \{\rho(t) : t \geq 0\}$ is a general deterministic locally bounded positive Borel function, we can take each step function $\rho_k = \{\rho_k(t) : t \geq 0\}$ in the last proof to be deterministic. Then the solution $\{Y_k(t) : t \geq 0\}$ of (5.2.1) with $\rho = \rho_k$ is a special inhomogeneous CBI-process with branching mechanism ϕ and time-dependent immigration mechanisms $\{\rho_k(t)\psi : t \geq 0\}$. In other words, for any $\lambda \geq 0$, $t \geq r \geq 0$ and $G \in \mathscr{F}_r$ we have

$$\mathbf{P}[1_G \mathrm{e}^{-\lambda Y_k(t)}] = \mathbf{P}\Big[1_G \exp\Big\{ - Y_k(r) v_{t-r}(\lambda) - \int_r^t \rho_k(s)\psi(v_{t-s}(\lambda))\mathrm{d}s\Big\}\Big].$$

Letting $k \to \infty$ along the sequence $\{k_i\}$ mentioned in the last proof gives

$$\mathbf{P}[1_G \mathrm{e}^{-\lambda Y_t}] = \mathbf{P}\Big[1_G \exp\Big\{ - Y_r v_{t-r}(\lambda) - \int_r^t \rho(s)\psi(v_{t-s}(\lambda))\mathrm{d}s\Big\}\Big].$$

Then $\{Y_t : t \geq 0\}$ is a CBI-process with immigration rate $\rho = \{\rho(t) : t \geq 0\}$. □

In view of the result of Theorem 5.2.4, the solution $\{Y_t : t \geq 0\}$ to (5.2.1) can be called an inhomogeneous CBI-process with branching mechanism ϕ, immigration mechanism ψ and *predictable immigration rate* $\rho = \{\rho(t) : t \geq 0\}$. The results in this section are slight modifications of those in Li (2011+), where some path-valued branching processes were introduced.

5.3 Interactive immigration rates

In this section, we give a construction of CBI-processes with interactive immigration rates. We shall use the set up of the second section. Suppose that $z \mapsto q(z)$ is a positive

Lipschitz function on $[0, \infty)$. We consider the stochastic equation

$$\begin{aligned} Y_t = Y_0 + \sigma \int_0^t \sqrt{Y_{s-}} \mathrm{d}B(s) + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z \tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ + \int_0^t [\beta q(Y_{s-}) - bY_{s-}] \mathrm{d}s + \int_0^t \int_0^\infty \int_0^{q(Y_{s-})} z N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{5.3.1}$$

This reduces to (4.2.1) when q is a constant function. We may interpret the solution $\{Y_t : t \geq 0\}$ of (5.3.1) as a CBI-process with *interactive immigration rate* given by the process $s \mapsto q(Y_{s-})$.

Theorem 5.3.1 *There is a pathwise unique solution* $\{Y_t : t \geq 0\}$ *of* (5.3.1).

Proof. Suppose that $\{Y_t : t \geq 0\}$ and $\{Z_t : t \geq 0\}$ are two solutions to this equation. Let $K \geq 0$ be a Lipschitz constant for the function $z \mapsto q(z)$. By (5.2.4) we have

$$\mathbf{P}[|Z_t - Y_t|] \leq \psi'(0) \mathrm{e}^{|b|t} \int_0^t \mathbf{P}[|q(Z_s) - q(Y_s)|] \mathrm{d}s \leq K \psi'(0) \mathrm{e}^{|b|t} \int_0^t \mathbf{P}[|Z_s - Y_s|] \mathrm{d}s.$$

Then the pathwise uniqueness for (5.3.1) follows by Gronwall's inequality. We next prove the existence of the solution using an approximating argument. Let $Y_0(t) \equiv 0$. By Theorem 5.2.3 we can define inductively the sequence of processes $\{Y_k(t) : t \geq 0\}$, $k = 1, 2, \ldots$ as pathwise unique solutions of the stochastic equations

$$\begin{aligned} Y_k(t) = Y_0 - b \int_0^t Y_k(s-) \mathrm{d}s + \sigma \int_0^t \sqrt{Y_k(s-)} \mathrm{d}B(s) \\ + \beta \int_0^t q(Y_{k-1}(s-)) \mathrm{d}s + \int_0^t \int_0^\infty \int_0^{Y_k(s-)} z \tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ + \int_0^t \int_0^\infty \int_0^{q(Y_{k-1}(s-))} z N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{5.3.2}$$

Let $Z_k(t) = Y_k(t) - Y_{k-1}(t)$. By (5.2.4) we have

$$\begin{aligned} \mathbf{P}[|Z_k(t)|] &\leq \psi'(0) \mathrm{e}^{|b|t} \int_0^t \mathbf{P}[|q(Y_{k-1}(s)) - q(Y_{k-2}(s))|] \mathrm{d}s \\ &\leq K \psi'(0) \mathrm{e}^{|b|t} \int_0^t \mathbf{P}[|Z_{k-1}(s)|] \mathrm{d}s. \end{aligned}$$

By (5.3.2) one sees that $\{Z_1(t) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanism $q(0)\psi$. In view of (3.1.13), we have

$$\mathbf{P}[|Z_1(t)|] = \mathrm{e}^{-bt} \mathbf{P}[Y_0] + \psi'(0) b^{-1} (1 - \mathrm{e}^{-bt}).$$

By a standard argument, one shows

$$\sum_{k=1}^{\infty} \sup_{0\le s\le t} \mathbf{P}[|Y_k(s) - Y_{k-1}(s)|] < \infty,$$

so the Lipschitz property of $z \mapsto q(z)$ implies

$$\sum_{k=1}^{\infty} \sup_{0\le s\le t} \mathbf{P}[|q(Y_k(s)) - q(Y_{k-1}(s))|] < \infty.$$

It follows that

$$\lim_{k,l\to\infty} \int_0^t \mathbf{P}[|q(Y_k(s)) - q(Y_l(s))|]\mathrm{d}s = 0.$$

Then there exists a predictable process $\rho = \{\rho(s) : s \ge 0\}$ so that

$$\lim_{k\to\infty} \int_0^t \mathbf{P}[|q(Y_k(s)) - \rho(s)|]\mathrm{d}s = 0. \tag{5.3.3}$$

Let $\{Y_t : t \ge 0\}$ be the positive càdlàg process defined by (5.2.1). By Proposition 5.2.2, there is a subsequence $\{k_n\} \subset \{k\}$ so that a.s.

$$\lim_{n\to\infty} \sup_{0\le s\le t} |Y_{k_n}(s) - Y_s| = 0, \qquad t \ge 0.$$

By the continuity of $z \mapsto q(z)$ we get a.s.

$$\lim_{n\to\infty} q(Y_{k_n}(s-)) = q(Y(s-)), \qquad t \ge 0.$$

This and (5.3.3) imply that

$$\int_0^t \mathbf{P}[|q(Y(s-)) - \rho(s)|]\mathrm{d}s = 0.$$

Then letting $k \to \infty$ along $\{k_n + 1\}$ in (5.3.2) we see $\{Y_t : t \ge 0\}$ is a solution of (5.3.1). □

By Itô's formula, it is easy to see that the solution $\{Y_t : t \ge 0\}$ of (5.3.1) solves the following martingale problem: For every $f \in C^2(\mathbb{R}_+)$,

$$f(Y_t) = f(Y_0) + \text{local mart.} - b\int_0^t f'(Y_s)Y_s\mathrm{d}s + \frac{1}{2}\sigma^2 \int_0^t f''(Y_s)Y_s\mathrm{d}s$$
$$+ \int_0^t Y_s\mathrm{d}s \int_0^{\infty} [f(Y_s + z) - f(Y_s) - zf'(Y_s)]m(\mathrm{d}z)$$

$$+\int_0^t q(Y_s)\Big\{\beta f'(Y_s) + \int_0^\infty [f(Y_s+z) - f(Y_s)]n(\mathrm{d}z)\Big\}\mathrm{d}s. \qquad (5.3.4)$$

By Theorem 5.3.1, the solution is a strong Markov process with generator given by

$$\begin{aligned} Af(x) = {} & \frac{1}{2}\sigma^2 x f''(x) + x\int_0^\infty [f(x+z) - f(x) - zf'(x)]m(\mathrm{d}z) \\ & - bxf'(x) + q(x)\Big\{\beta f'(x) + \int_0^\infty [f(x+z) - f(x)]n(\mathrm{d}z)\Big\}. \qquad (5.3.5) \end{aligned}$$

We can also consider two Lipschitz functions $z \mapsto q_1(z)$ and $z \mapsto q_2(z)$ on $[0,\infty)$. By slightly modifying the arguments, one can show there is a pathwise unique solution to

$$\begin{aligned} Y_t = {} & Y_0 + \sigma\int_0^t \sqrt{Y_{s-}}\mathrm{d}B(s) + \int_0^t\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ & + \int_0^t [\beta q_1(Y_{s-}) - bY_{s-}]\mathrm{d}s + \int_0^t\int_0^\infty\int_0^{q_2(Y_{s-})} zN_1(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \qquad (5.3.6) \end{aligned}$$

The solution of this equation can be understood as a CBI-process with *interactive immigration rates* given by the processes $s \mapsto q_1(Y_{s-})$ and $s \mapsto q_2(Y_{s-})$. This type of immigration structures were studied in Li (2011) in the setting of superprocesses by considering a different type of stochastic equations.

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