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CONTINUOUS-STATE BRANCHING PROCESSES WITH IMMIGRATION

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Abstract

This work provides a brief introduction to continuous-state branching processes with or without immigration. The processes are constructed by taking rescaling limits of classical discrete-state branching models. We give quick developments of the martingale problems and stochastic equations of the continuous-state processes. The proofs here are more elementary than those appearing in the literature before. We have made them readable without requiring too much preliminary knowledge on branching processes and stochastic analysis. Using the stochastic equations, we give characterizations of the local and global maximal jumps of the processes. Under suitable conditions, their strong Feller property and exponential ergodicity are studied by a coupling method based on one of the stochastic equations.

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Introduction

Continuous-state branching processes (CB-processes) and continuous-state branching processes with immigration (CBI-processes) constitute important classes of Markov processes taking values in the positive (= nonnegative) half line. They were introduced as probabilistic models describing the evolution of large populations with small individuals. The study of CB-processes was initiated by Feller (1951), who noticed that a diffusion process may arise in a limit theorem of Galton–Watson discrete branching processes; see also Aliev and Shchurenkov (1982), Grimvall (1974) and Lamperti (1967a). A characterization of CB-processes by random time changes of Lévy processes was given by Lamperti (1967b). The convergence of rescaled discrete branching processes with immigration to CBI-processes was studied in Aliev (1985), Kawazu and Watanabe (1971) and Li (2006). From a mathematical point of view, the continuous-state processes are usually easier to deal with because both their time and state spaces are smooth, and the distributions that appear are infinitely divisible. For general treatments and backgrounds of CB- and CBI-processes, the reader may refer to Kyprianou (2014) and Li (2011). In the recent work of Pardoux (2016), more complicated probabilistic population models involving competition were studied, which extend the stochastic logistic growth model of Lambert (2005).

A continuous CBI-process with subcritical branching mechanism was used by Cox et al. (1985) to describe the evolution of interest rates and has been known in mathematical finance as the *Cox–Ingersoll–Ross model* (CIR-model). Compared with other financial models introduced before, the CIR-model is more appealing as it is positive and mean-reverting. The asymptotic behavior of the estimators of the parameters in this model was studied by Overbeck and Rydén (1997); see also Li and Ma (2015). Applications of stochastic calculus to finance including those of the CIR-model were discussed systematically in Lamberton and Lapeyre (1996). A natural generalization of the CBI-process is the so-called affine Markov process, which has also been used a lot in mathematical finance; see, e.g., Duffie et al. (2003) and the references therein.

A strong stochastic equation for general CBI-processes was first established in Dawson and Li (2006). A flow of discontinuous CB-processes was constructed in Bertoin and Le Gall (2006) by weak solutions to a stochastic equation. Their results were extended to flows of CBI-processes in Dawson and Li (2012) using strong solutions; see also Li (2014) and Li and Ma (2008). For the stable branching CBI-process, a strong stochastic differential equation driven by Lévy processes was established in Fu and Li (2010). The approach of stochastic equations has played an important role in recent developments of the theory and applications of CB- and CBI-processes.

The purpose of these notes is to provide a brief introduction to CB- and CBI-processes accessible to graduate students with reasonable background in probability theory and stochastic processes. In particular, we give a quick development of the stochastic equations of the processes and some immediate applications. The proofs given here are more elementary than those appearing in the literature before. We have made them readable without requiring too much preliminary knowledge on branching processes and stochastic analysis.

In Section 1, we review some properties of Laplace transforms of finite measures on the positive half line. In Section 2, a construction of CB-processes is given as rescaling limits of Galton–Watson branching processes. This approach also gives the physical interpretation of the CB-processes. Some basic properties of the processes are developed in Section 3. The Laplace transforms of some positive integral functionals are calculated explicitly in Section 4. In Section 5, the CBI-processes are constructed as rescaling limits of Galton–Watson branching processes with immigration. In Section 6, we present reconstructions of the CB- and CBI-processes by Poisson random measures determined by entrance laws, which reveal the structures of the trajectories of the processes. Several equivalent formulations of martingale problems for CBI-processes are given in Section 7. From those we derive the stochastic equations of the processes in Section 8. Using the stochastic equations, some characterizations of local and global maximal jumps of the CB- and CBI-processes are given in Section 9. In Section 10, we prove the strong Feller property and the exponential ergodicity of the CBI-process under suitable conditions using a coupling based on one of the stochastic equations.

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1 Laplace transforms of measures

Let $\mathscr{B}[0,\infty)$ be the Borel σ-algebra on the positive half line $[0,\infty)$. Let $B[0,\infty) = \mathrm{b}\mathscr{B}[0,\infty)$ be the set of bounded Borel functions on $[0,\infty)$. Given a finite measure μ on $[0,\infty)$, we define the *Laplace transform* L_μ of μ by

$$L_\mu(\lambda) = \int_{[0,\infty)} \mathrm{e}^{-\lambda x}\,\mu(\mathrm{d}x), \qquad \lambda \geq 0. \tag{1.1}$$

Theorem 1.1 *A finite measure on $[0,\infty)$ is uniquely determined by its Laplace transform.*

Proof. Suppose that μ_1 and μ_2 are finite measures on $[0,\infty)$ and $L_{\mu_1}(\lambda) = L_{\mu_2}(\lambda)$ for all $\lambda \geq 0$. Let $\mathscr{K} = \{x \mapsto \mathrm{e}^{-\lambda x} : \lambda \geq 0\}$ and let $\mathscr{L}$ be the class of functions $F \in B[0,\infty)$ so that

$$\int_{[0,\infty)} F(x)\mu_1(\mathrm{d}x) = \int_{[0,\infty)} F(x)\mu_2(\mathrm{d}x).$$

Then $\mathscr{K}$ is closed under multiplication and $\mathscr{L}$ is a monotone vector space containing $\mathscr{K}$. It is easy to see $\sigma(\mathscr{K}) = \mathscr{B}[0,\infty)$. Then the monotone class theorem implies $\mathscr{L} \supset \mathrm{b}\sigma(\mathscr{K}) = B[0,\infty)$. That proves the desired result. $\square$

Theorem 1.2 *Let $\{\mu_n\}$ be a sequence of finite measures on $[0,\infty)$ and $\lambda \mapsto L(\lambda)$ a continuous function on $[0,\infty)$. If $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L(\lambda)$ for every $\lambda \geq 0$, then there is a finite measure μ on $[0,\infty)$ such that $L_\mu = L$ and $\lim_{n\to\infty}\mu_n = \mu$ by weak convergence.*

Proof. We can regard each μ_n as a finite measure on $[0,\infty]$, the one-point compactification of $[0,\infty)$. Let F_n denote the distribution function of μ_n. By Helly's theorem we infer that $\{F_n\}$ contains a subsequence $\{F_{n_k}\}$ that converges weakly on $[0,\infty]$ to some distribution function F. Then the corresponding subsequence $\{\mu_{n_k}\}$ converges weakly on $[0,\infty]$ to the finite measure μ determined by F. It follows that

$$\mu[0,\infty] = \lim_{k\to\infty}\mu_{n_k}[0,\infty] = \lim_{k\to\infty}\mu_{n_k}[0,\infty) = \lim_{k\to\infty} L_{\mu_{n_k}}(0) = L(0). \tag{1.2}$$

Moreover, for $\lambda > 0$ we have

$$\begin{aligned}\int_{[0,\infty]} \mathrm{e}^{-\lambda x}\,\mu(\mathrm{d}x) &= \lim_{k\to\infty}\int_{[0,\infty]} \mathrm{e}^{-\lambda x}\,\mu_{n_k}(\mathrm{d}x) \\ &= \lim_{k\to\infty}\int_{[0,\infty)} \mathrm{e}^{-\lambda x}\,\mu_{n_k}(\mathrm{d}x) = L(\lambda), \end{aligned}\tag{1.3}$$

where $\mathrm{e}^{-\lambda\cdot\infty} = 0$ by convention. By letting $\lambda \to 0+$ in (1.3) and using the continuity of L at $\lambda = 0$ we find $\mu[0,\infty) = L(0)$. From this and (1.2) we see μ is supported by $[0,\infty)$. By Theorem 1.7 of Li (2011, p.4) we have $\lim_{n\to\infty}\mu_{n_k} = \mu$ weakly on $[0,\infty)$. It follows that, for $\lambda \geq 0$,

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda x}\,\mu(\mathrm{d}x) = \lim_{k\to\infty}\int_{[0,\infty)} \mathrm{e}^{-\lambda x}\,\mu_{n_k}(\mathrm{d}x) = L(\lambda).$$

Then $L_\mu = L$. If μ_n does not converge weakly to μ, then F_n does not converge weakly to F, so there is a subsequence $\{F_{n'_k}\} \subset \{F_n\}$ that converges weakly to a limit $G \neq F$. The above arguments show that G corresponds to a finite measure on $[0, \infty)$ with Laplace transform $L = L_\mu$, yielding a contradiction. Then $\lim_{n\to\infty} \mu_n = \mu$ weakly on $[0, \infty)$. $\square$

Corollary 1.3 *Let $\mu_1, \mu_2, \ldots$ and μ be finite measures on $[0, \infty)$. Then $\mu_n \to \mu$ weakly if and only if $L_{\mu_n}(\lambda) \to L_\mu(\lambda)$ for every $\lambda \geq 0$.*

Proof. If $\mu_n \to \mu$ weakly, we have $\lim_{n\to\infty} L_{\mu_n}(\lambda) = L_\mu(\lambda)$ for every $\lambda \geq 0$. The converse assertion is a consequence of Theorem 1.2. $\square$

Given two probability measures μ_1 and μ_2 on $[0, \infty)$, we denote by $\mu_1 \times \mu_2$ their product measure on $[0, \infty)^2$. The image of $\mu_1 \times \mu_2$ under the mapping $(x_1, x_2) \mapsto x_1 + x_2$ is called the *convolution* of μ_1 and μ_2 and is denoted by $\mu_1 * \mu_2$, which is a probability measure on $[0, \infty)$. According to the definition, for any $F \in B[0, \infty)$ we have

$$\int_{[0,\infty)} F(x)(\mu_1 * \mu_2)(\mathrm{d}x) = \int_{[0,\infty)} \mu_1(\mathrm{d}x_1) \int_{[0,\infty)} F(x_1 + x_2)\mu_2(\mathrm{d}x_2). \tag{1.4}$$

Clearly, if ξ_1 and ξ_2 are independent random variables with distributions μ_1 and μ_2 on $[0, \infty)$, respectively, then the random variable $\xi_1 + \xi_2$ has distribution $\mu_1 * \mu_2$. It is easy to show that

$$L_{\mu_1 * \mu_2}(\lambda) = L_{\mu_1}(\lambda) L_{\mu_2}(\lambda), \qquad \lambda \geq 0. \tag{1.5}$$

Let $\mu^{*0} = \delta_0$ and define $\mu^{*n} = \mu^{*(n-1)} * \mu$ inductively for integers $n \geq 1$.

We say a probability distribution μ on $[0, \infty)$ is *infinitely divisible* if for each integer $n \geq 1$, there is a probability μ_n such that $\mu = \mu_n^{*n}$. In this case, we call μ_n the *n-th root* of μ. A positive random variable ξ is said to be *infinitely divisible* if it has infinitely divisible distribution on $[0, \infty)$. Write $\psi \in \mathscr{I}$ if $\lambda \mapsto \psi(\lambda)$ is a positive function on $[0, \infty)$ with the *Lévy–Khintchine representation*:

$$\psi(\lambda) = h\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u}) l(\mathrm{d}u), \tag{1.6}$$

where $h \geq 0$ and $l(\mathrm{d}u)$ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge u) l(\mathrm{d}u) < \infty.$$

The relation $\psi = -\log L_\mu$ establishes a one-to-one correspondence between the functions $\psi \in \mathscr{I}$ and infinitely divisible probability measures μ on $[0, \infty)$; see, e.g., Theorem 1.39 in Li (2011, p.20).

2 Construction of CB-processes

Let $\{p(j) : j \in \mathbb{N}\}$ be a probability distribution on the space of positive integers $\mathbb{N} := \{0, 1, 2, \ldots\}$. It is well-known that $\{p(j) : j \in \mathbb{N}\}$ is uniquely determined by its generating function g defined by

$$g(z) = \sum_{j=0}^{\infty} p(j) z^j, \qquad |z| \leq 1.$$

Suppose that $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ is a family of $\mathbb{N}$-valued i.i.d. random variables with distribution $\{p(j) : j \in \mathbb{N}\}$. Given an $\mathbb{N}$-valued random variable $x(0)$ independent of $\{\xi_{n,i}\}$, we define inductively

$$x(n) = \sum_{i=1}^{x(n-1)} \xi_{n,i}, \qquad n = 1, 2, \ldots. \tag{2.1}$$

Here we understand $\sum_{i=1}^{0} = 0$. For $i \in \mathbb{N}$ let $\{Q(i,j) : j \in \mathbb{N}\}$ denote the i-fold convolution of $\{p(j) : j \in \mathbb{N}\}$, that is, $Q(i,j) = p^{*i}(j)$ for $i, j \in \mathbb{N}$. For any $n \geq 1$ and $\{i_0, i_1, \cdots, i_{n-1} = i, j\} \subset \mathbb{N}$ it is easy to see that

$$\begin{aligned}
&\mathbf{P}\Big(x(n) = j \Big| x(0) = i_0, x(1) = i_1, \cdots, x(n-1) = i_{n-1}\Big) \\
&\quad = \mathbf{P}\bigg(\sum_{i=1}^{x(n-1)} \xi_{n,i} = j \bigg| x(n-1) = i_{n-1}\bigg) \\
&\quad = \mathbf{P}\bigg(\sum_{k=1}^{i} \xi_{n,k} = j\bigg) = Q(i,j).
\end{aligned}$$

Then $\{x(n) : n \geq 0\}$ is an $\mathbb{N}$-valued Markov chain with one-step transition matrix $Q = (Q(i,j) : i, j \in \mathbb{N})$. The random variable $x(n)$ can be thought of as the number of individuals in generation n of an evolving population system. After one unit time, each individual in the population splits independently of others into a random number of offspring according to the distribution $\{p(j) : j \in \mathbb{N}\}$. Clearly, we have, for $i \in \mathbb{N}$ and $|z| \leq 1$,

$$\sum_{j=0}^{\infty} Q(i,j) z^j = \sum_{j=0}^{\infty} p^{*i}(j) z^j = g(z)^i. \tag{2.2}$$

Clearly, the transition matrix Q satisfies the *branching property*:

$$Q(i_1 + i_2, \cdot) = Q(i_1, \cdot) * Q(i_2, \cdot), \qquad i_1, i_2 \in \mathbb{N}. \tag{2.3}$$

This means that different individuals in the population propagate independently each other.

A Markov chain in $\mathbb{N}$ with one-step transition matrix defined by (2.2) is called a *Galton–Watson branching process* (GW-process) or a *Bienaymé–Galton–Watson branching process* (BGW-process) with *branching distribution* given by g; see, e.g., Athreya and

Ney (1972) and Harris (1963). The study of the model goes back to Bienaymé (1845) and Galton and Watson (1874).

By a general result in the theory of Markov chains, for any $n \geq 1$ the n-step transition matrix of the GW-process is just the n-fold product matrix $Q^n = (Q^n(i,j) : i,j \in \mathbb{N})$.

Proposition 2.1 *For any $n \geq 1$ and $i \in \mathbb{N}$ we have*

$$\sum_{j=0}^{\infty} Q^n(i,j)z^j = g^{\circ n}(z)^i, \qquad |z| \leq 1, \tag{2.4}$$

where $g^{\circ n}(z)$ is defined by $g^{\circ n}(z) = g \circ g^{\circ(n-1)}(z) = g(g^{\circ(n-1)}(z))$ successively with $g^{\circ 0}(z) = z$ by convention.

Proof. From (2.2) we know (2.4) holds for $n = 1$. Now suppose that (2.4) holds for some $n \geq 1$. We have

$$\begin{aligned}\sum_{j=0}^{\infty} Q^{n+1}(i,j)z^j &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} Q(i,k)Q^n(k,j)z^j \\ &= \sum_{k=0}^{\infty} Q(i,k)g^{\circ n}(z)^k = g^{\circ(n+1)}(z)^i.\end{aligned}$$

Then (2.4) also holds when n is replaced by $n+1$. That proves the result by induction. □

It is easy to see that zero is a trap for the GW-process. If $g'(1-) < \infty$, by differentiating both sides of (2.4) we see the first moment of the distribution $\{Q^n(i,j) : j \in \mathbb{N}\}$ is given by

$$\sum_{j=1}^{\infty} jQ^n(i,j) = ig'(1-)^n. \tag{2.5}$$

Example 2.1 Given a GW-process $\{x(n) : n \geq 0\}$, we can define its *extinction time* $\tau_0 = \inf\{n \geq 0 : x(n) = 0\}$. In view of (2.1), we have $x(n) = 0$ on the event $\{n \geq \tau_0\}$. Let $q = \mathbf{P}(\tau_0 < \infty | x(0) = 1)$ be the *extinction probability*. By the independence of the propagation of different individuals we have $\mathbf{P}(\tau_0 < \infty | x(0) = i) = q^i$ for any $i = 0, 1, 2, \ldots$. By the total probability formula,

$$\begin{aligned} q &= \sum_{j=0}^{\infty} \mathbf{P}(x(1) = j | x(0) = 1)\mathbf{P}(\tau_0 < \infty | x(0) = 1, x(1) = j) \\ &= \sum_{j=0}^{\infty} \mathbf{P}(\xi_{1,1} = j)\mathbf{P}(\tau_0 < \infty | x(1) = j) = \sum_{j=0}^{\infty} p(j)q^j = g(q).\end{aligned}$$

Then the extinction probability q is a solution to the equation $z = g(z)$ on $[0,1]$. Clearly, in the case of $p(1) < 1$ we have $q = 1$ if and only if $\sum_{j=1}^{\infty} jp(j) \leq 1$.

Now suppose we have a sequence of GW-processes $\{x_k(n) : n \geq 0\}$, $k = 1, 2, \dots$ with branching distributions given by the probability generating functions g_k, $k = 1, 2, \dots$. Let $z_k(n) = k^{-1}x_k(n)$. Then $\{z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k := \{0, k^{-1}, 2k^{-1}, \dots\}$ and n-step transition probability $Q_k^n(x, \mathrm{d}y)$ determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y}\, Q_k^n(x, \mathrm{d}y) = g_k^{\circ n}(\mathrm{e}^{-\lambda/k})^{kx}, \qquad \lambda \geq 0. \tag{2.6}$$

Suppose that $\{\gamma_k\}$ is a positive sequence so that $\gamma_k \to \infty$ increasingly as $k \to \infty$. Let $\lfloor \gamma_k t \rfloor$ denote the integer part of $\gamma_k t$. Clearly, given $z_k(0) = x \in E_k$, for any $t \geq 0$ the random variable $z_k(\lfloor \gamma_k t \rfloor) = k^{-1}x_k(\lfloor \gamma_k t \rfloor)$ has distribution $Q_k^{\lfloor \gamma_k t \rfloor}(x, \cdot)$ on E_k determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y}\, Q_k^{\lfloor \gamma_k t \rfloor}(x, \mathrm{d}y) = \exp\{-x v_k(t, \lambda)\}, \tag{2.7}$$

where

$$v_k(t, \lambda) = -k \log g_k^{\circ \lfloor \gamma_k t \rfloor}(\mathrm{e}^{-\lambda/k}). \tag{2.8}$$

We are interested in the asymptotic behavior of the sequence of continuous time processes $\{z_k(\lfloor \gamma_k t \rfloor) : t \geq 0\}$ as $k \to \infty$. By (2.8), for $\gamma_k^{-1}(i-1) \leq t < \gamma_k^{-1} i$ we have

$$v_k(t, \lambda) = v_k(\gamma_k^{-1}\lfloor \gamma_k t \rfloor, \lambda) = v_k(\gamma_k^{-1}(i-1), \lambda).$$

It follows that

$$\begin{aligned}
v_k(t, \lambda) &= v_k(0, \lambda) + \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [v_k(\gamma_k^{-1} j, \lambda) - v_k(\gamma_k^{-1}(j-1), \lambda)] \\
&= \lambda - k \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [\log g_k^{\circ j}(\mathrm{e}^{-\lambda/k}) - \log g_k^{\circ (j-1)}(\mathrm{e}^{-\lambda/k})] \\
&= \lambda - k \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \log \big[g_k(g_k^{\circ (j-1)}(\mathrm{e}^{-\lambda/k})) g_k^{\circ (j-1)}(\mathrm{e}^{-\lambda/k})^{-1} \big] \\
&= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \bar{\phi}_k(-k \log g_k^{\circ (j-1)}(\mathrm{e}^{-\lambda/k})) \\
&= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \bar{\phi}_k(v_k(\gamma_k^{-1}(j-1), \lambda)) \\
&= \lambda - \int_0^{\gamma_k^{-1}\lfloor \gamma_k t \rfloor} \bar{\phi}_k(v_k(s, \lambda)) \mathrm{d}s,
\end{aligned} \tag{2.9}$$

where

$$\bar{\phi}_k(z) = k\gamma_k \log \big[g_k(\mathrm{e}^{-z/k})\, \mathrm{e}^{z/k} \big], \qquad z \geq 0. \tag{2.10}$$

It is easy to see that

$$\bar{\phi}_k(z) = k\gamma_k \log\left[1 + (k\gamma_k)^{-1}\tilde{\phi}_k(z)\,\mathrm{e}^{z/k}\right], \tag{2.11}$$

where

$$\tilde{\phi}_k(z) = k\gamma_k[g_k(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}]. \tag{2.12}$$

The sequence $\{\tilde{\phi}_k\}$ is sometimes easier to handle than the original sequence $\{\bar{\phi}_k\}$. The following lemma shows that the two sequences are really not very different.

Lemma 2.2 *Suppose that either $\{\bar{\phi}_k\}$ or $\{\tilde{\phi}_k\}$ is uniformly bounded on each bounded interval. Then we have:* (i) $\lim_{k\to\infty}|\bar{\phi}_k(z) - \tilde{\phi}_k(z)| = 0$ *uniformly on each bounded interval;* (ii) $\{\bar{\phi}_k\}$ *is uniformly Lipschitz on each bounded interval if and only if so is* $\{\tilde{\phi}_k\}$.

Proof. The first assertion follows immediately from (2.11). By the same relation we have

$$\bar{\phi}_k'(z) = \frac{[\tilde{\phi}_k'(z) + k^{-1}\tilde{\phi}_k(z)]\,\mathrm{e}^{z/k}}{1 + (k\gamma_k)^{-1}\tilde{\phi}_k(z)\,\mathrm{e}^{z/k}}, \qquad z \geq 0.$$

Then $\{\bar{\phi}_k'\}$ is uniformly bounded on each bounded interval if and only if so is $\{\tilde{\phi}_k'\}$. That gives the second assertion. □

By the above lemma, if either $\{\tilde{\phi}_k\}$ or $\{\bar{\phi}_k\}$ is uniformly Lipschitz on each bounded interval, then they converge or diverge simultaneously and in the convergent case they have the same limit. For the convenience of statement of the results, we formulate the following condition:

Condition 2.3 *The sequence $\{\tilde{\phi}_k\}$ is uniformly Lipschitz on $[0, a]$ for every $a \geq 0$ and there is a function ϕ on $[0, \infty)$ so that $\tilde{\phi}_k(z) \to \phi(z)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.*

Proposition 2.4 *Suppose that Condition 2.3 is satisfied. Then the limit function ϕ has representation*

$$\phi(z) = bz + cz^2 + \int_{(0,\infty)} \big(\mathrm{e}^{-zu} - 1 + zu\big)m(\mathrm{d}u), \quad z \geq 0, \tag{2.13}$$

where $c \geq 0$ and b are constants and $m(\mathrm{d}u)$ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (u \wedge u^2)m(\mathrm{d}u) < \infty.$$

Proof. For each $k \geq 1$ let us define the function ϕ_k on $[0, k]$ by

$$\phi_k(z) = k\gamma_k[g_k(1 - z/k) - (1 - z/k)]. \tag{2.14}$$

From (2.12) and (2.14) we have

$$\tilde{\phi}_k'(z) = \gamma_k \,\mathrm{e}^{-z/k}[1 - g_k'(\mathrm{e}^{-z/k})], \qquad z \geq 0,$$

and

$$\phi_k'(z) = \gamma_k[1 - g_k'(1 - z/k)], \qquad 0 \leq z \leq k.$$

Since $\{\tilde{\phi}_k\}$ is uniformly Lipschitz on each bounded interval, the sequence $\{\tilde{\phi}_k'\}$ is uniformly bounded on each bounded interval. Then $\{\phi_k'\}$ is also uniformly bounded on each bounded interval, and so the sequence $\{\phi_k\}$ is uniformly Lipschitz on each bounded interval. Let $a \geq 0$. By the mean-value theorem, for $k \geq a$ and $0 \leq z \leq a$ we have

$$\begin{aligned} \tilde{\phi}_k(z) - \phi_k(z) &= k\gamma_k\big[g_k(\mathrm{e}^{-z/k}) - g_k(1 - z/k) - \mathrm{e}^{-z/k} + (1 - z/k)\big] \\ &= k\gamma_k[g_k'(\eta_k) - 1](\mathrm{e}^{-z/k} - 1 + z/k), \end{aligned}$$

where

$$1 - a/k \leq 1 - z/k \leq \eta_k \leq \mathrm{e}^{-z/k} \leq 1.$$

Choose $k_0 \geq a$ so that $\mathrm{e}^{-2a/k_0} \leq 1 - a/k_0$. Then $\mathrm{e}^{-2a/k} \leq 1 - a/k$ for $k \geq k_0$ and hence

$$\gamma_k|g_k'(\eta_k) - 1| \leq \sup_{0\leq z\leq 2a} \gamma_k|g_k'(\mathrm{e}^{-z/k}) - 1| = \sup_{0\leq z\leq 2a} \mathrm{e}^{z/k}\,|\tilde{\phi}_k'(z)|.$$

Since $\{\tilde{\phi}_k'\}$ is uniformly bounded on $[0, 2a]$, the sequence $\{\gamma_k|g_k'(\eta_k) - 1| : k \geq k_0\}$ is bounded. Then $\lim_{k\to\infty} |\phi_k(z) - \tilde{\phi}_k(z)| = 0$ uniformly on each bounded interval. It follows that $\lim_{k\to\infty} \phi_k(z) = \phi(z)$ uniformly on each bounded interval. Then the result follows by Corollary 1.46 in Li (2011, p.26). □

Proposition 2.5 *For any function ϕ with representation* (2.13) *there is a sequence $\{\tilde{\phi}_k\}$ in the form of* (2.12) *satisfying Condition 2.3.*

Proof. By the proof of Proposition 2.4 it suffices to construct a sequence $\{\phi_k\}$ with the expression (2.14) that is uniformly Lipschitz on $[0, a]$ and $\phi_k(z) \to \phi(z)$ uniformly on $[0, a]$ for every $a \geq 0$. To simplify the formulations we decompose the function ϕ into two parts. Let $\phi_0(z) = \phi(z) - bz$. We first define

$$\gamma_{0,k} = (1 + 2c)k + \int_{(0,\infty)} u(1 - \mathrm{e}^{-ku})m(\mathrm{d}u)$$

and

$$g_{0,k}(z) = z + k^{-1}\gamma_{0,k}^{-1}\phi_0(k(1 - z)), \qquad |z| \leq 1.$$

It is easy to see that $z \mapsto g_{0,k}(z)$ is an analytic function satisfying $g_{0,k}(1) = 1$ and

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} g_{0,k}(0) \geq 0, \qquad n \geq 0.$$

Therefore $g_{0,k}(\cdot)$ is a probability generating function. Let $\phi_{0,k}$ be defined by (2.14) with (γ_k, g_k) replaced by $(\gamma_{0,k}, g_{0,k})$. Then $\phi_{0,k}(z) = \phi_0(z)$ for $0 \leq z \leq k$. That completes the proof if $b = 0$. In the case $b \neq 0$, we set

$$g_{1,k}(z) = \frac{1}{2}\left(1 + \frac{b}{|b|}\right) + \frac{1}{2}\left(1 - \frac{b}{|b|}\right)z^2.$$

Let $\gamma_{1,k} = |b|$ and let $\phi_{1,k}(z)$ be defined by (2.14) with (γ_k, g_k) replaced by $(\gamma_{1,k}, g_{1,k})$. Then

$$\phi_{1,k}(z) = bz + \frac{1}{2k}(|b| - b)z^2.$$

Finally, let $\gamma_k = \gamma_{0,k} + \gamma_{1,k}$ and $g_k = \gamma_k^{-1}(\gamma_{0,k}g_{0,k} + \gamma_{1,k}g_{1,k})$. Then the sequence $\phi_k(z)$ defined by (2.14) is equal to $\phi_{0,k}(z) + \phi_{1,k}(z)$ which satisfies the required condition. $\Box$

Lemma 2.6 *Suppose that the sequence* $\{\tilde{\phi}_k\}$ *defined by* (2.12) *is uniformly Lipschitz on* $[0, 1]$. *Then there are constants* $B, N \geq 0$ *such that* $v_k(t, \lambda) \leq \lambda\, \mathrm{e}^{Bt}$ *for every* $t, \lambda \geq 0$ *and* $k \geq N$.

Proof. Let $b_k := \tilde{\phi}_k'(0+)$ for $k \geq 1$. Since $\{\tilde{\phi}_k\}$ is uniformly Lipschitz on $[0, 1]$, the sequence $\{b_k\}$ is bounded. From (2.12) we have $b_k = \gamma_k[1 - g_k'(1-)]$. By (2.5) and (2.12) it is not hard to obtain

$$\int_{E_k} yQ_k^{\lfloor\gamma_k t\rfloor}(x, \mathrm{d}y) = xg_k'(1-)^{\lfloor\gamma_k t\rfloor} = x\left(1 - \frac{b_k}{\gamma_k}\right)^{\lfloor\gamma_k t\rfloor}.$$

Let $B \geq 0$ be a constant such that $2|b_k| \leq B$ for all $k \geq 1$. Since $\gamma_k \to \infty$ as $k \to \infty$, there is $N \geq 1$ so that

$$0 \leq \left(1 - \frac{b_k}{\gamma_k}\right)^{\gamma_k/B} \leq \left(1 + \frac{B}{2\gamma_k}\right)^{\gamma_k/B} \leq \mathrm{e}, \qquad k \geq N.$$

It follows that, for $t \geq 0$ and $k \geq N$,

$$\int_{E_k} yQ_k^{\lfloor\gamma_k t\rfloor}(x, \mathrm{d}y) \leq x \exp\left\{B\lfloor\gamma_k t\rfloor/\gamma_k\right\} \leq x\, \mathrm{e}^{Bt}.$$

Then the desired estimate follows from (2.5) and Jensen's inequality. $\Box$

Theorem 2.7 *Suppose that Condition 2.3 holds. Then for every* $a \geq 0$ *we have* $v_k(t, \lambda) \to$ *some* $v_t(\lambda)$ *uniformly on* $[0, a]^2$ *as* $k \to \infty$ *and the limit function solves the integral equation*

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))\mathrm{d}s, \qquad \lambda, t \geq 0. \tag{2.15}$$

Proof. The following argument is a modification of that of Aliev and Shchurenkov (1982) and Aliev (1985). In view of (2.9), we can write

$$v_k(t,\lambda) = \lambda + \varepsilon_k(t,\lambda) - \int_0^t \bar{\phi}_k(v_k(s,\lambda))\mathrm{d}s, \tag{2.16}$$

where

$$\varepsilon_k(t,\lambda) = \big(t - \gamma_k^{-1}\lfloor\gamma_k t\rfloor\big)\bar{\phi}_k\big(v_k(\gamma_k^{-1}\lfloor\gamma_k t\rfloor,\lambda)\big).$$

By Lemma 2.2 and Condition 2.3, for any $0 < \varepsilon \leq 1$ we can choose $N \geq 1$ so that $|\bar{\phi}_k(z) - \phi(z)| \leq \varepsilon$ for $k \geq N$ and $0 \leq z \leq a\,\mathrm{e}^{Ba}$. It follows that, for $0 \leq t \leq a$ and $0 \leq \lambda \leq a$,

$$|\varepsilon_k(t,\lambda)| \leq \gamma_k^{-1}\big|\bar{\phi}_k\big(v_k(\gamma_k^{-1}\lfloor\gamma_k t\rfloor,\lambda)\big)\big| \leq \gamma_k^{-1}M, \tag{2.17}$$

where

$$M = 1 + \sup_{0\leq z\leq a\,\mathrm{e}^{Ba}} |\phi(z)|.$$

For $n \geq k \geq N$ let

$$K_{k,n}(t,\lambda) = \sup_{0\leq s\leq t} |v_n(s,\lambda) - v_k(s,\lambda)|.$$

By (2.16) and (2.17) we obtain, for $0 \leq t \leq a$ and $0 \leq \lambda \leq a$,

$$\begin{aligned} K_{k,n}(t,\lambda) &\leq (\gamma_k^{-1}+\gamma_n^{-1})M + \int_0^t |\bar{\phi}_k(v_k(s,\lambda)) - \bar{\phi}_n(v_n(s,\lambda))|\mathrm{d}s \\ &\leq (\gamma_k^{-1}+\gamma_n^{-1})M + 2\varepsilon a + \int_0^t |\phi_k(v_k(s,\lambda)) - \phi_n(v_n(s,\lambda))|\mathrm{d}s \\ &\leq (\gamma_k^{-1}+\gamma_n^{-1})M + 2\varepsilon a + L\int_0^t K_{k,n}(s,\lambda)\mathrm{d}s, \end{aligned}$$

where $L = \sup_{0\leq z\leq a\,\mathrm{e}^{Ba}} |\phi'(z)|$. By Gronwall's inequality,

$$K_{k,n}(t,\lambda) \leq [(\gamma_k^{-1}+\gamma_n^{-1})M + 2\varepsilon a]\exp\{Lt\}, \qquad 0 \leq t,\lambda \leq a.$$

Then $v_k(t,\lambda) \to$ some $v_t(\lambda)$ uniformly on $[0,a]^2$ as $k \to \infty$ for every $a \geq 0$. From (2.16) we get (2.15). □

Theorem 2.8 *Suppose that ϕ is a function given by* (2.13). *Then for any $\lambda \geq 0$ there is a unique positive solution $t \mapsto v_t(\lambda)$ to* (2.15). *Moreover, the solution satisfies the semigroup property:*

$$v_{r+t}(\lambda) = v_r \circ v_t(\lambda) = v_r(v_t(\lambda)), \qquad r,t,\lambda \geq 0. \tag{2.18}$$

Proof. By Proposition 2.5 there is a sequence $\{\tilde{\phi}_k\}$ in form (2.12) satisfying Condition 2.3. Let $v_k(t,\lambda)$ be given by (2.7) and (2.8). By Theorem 2.7 the limit $v_t(\lambda) = \lim_{k\to\infty} v_k(t,\lambda)$ exists and solves (2.15). Clearly, any positive solution $t \mapsto v_t(\lambda)$ to (2.15) is locally bounded. The uniqueness of the solution follows by Gronwall's inequality. The relation (2.18) is a consequence of the uniqueness of the solution. □

Theorem 2.9 *Suppose that ϕ is a function given by* (2.13). *For any $\lambda \geq 0$ let $t \mapsto v_t(\lambda)$ be the unique positive solution to* (2.15). *Then we can define a transition semigroup $(Q_t)_{t\geq 0}$ on $[0,\infty)$ by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\, Q_t(x, \mathrm{d}y) = \mathrm{e}^{-xv_t(\lambda)}, \qquad \lambda \geq 0, x \geq 0. \tag{2.19}$$

Proof. By Proposition 2.5, there is a sequence $\{\tilde{\phi}_k\}$ in form (2.12) satisfying Condition 2.3. By Theorem 2.7 we have $v_k(t,\lambda) \to v_t(\lambda)$ uniformly on $[0,a]^2$ as $k \to \infty$ for every $a \geq 0$. Taking $x_k \in E_k$ satisfying $x_k \to x$ as $k \to \infty$, we see by Theorem 1.2 that (2.19) defines a probability measure $Q_t(x,\mathrm{d}y)$ on $[0,\infty)$ and $\lim_{k\to\infty} Q_k^{\lfloor \gamma_k t\rfloor}(x_k,\cdot) = Q_t(x,\cdot)$ by weak convergence. By a monotone class argument one can see that $Q_t(x,\mathrm{d}y)$ is a kernel on $[0,\infty)$. The semigroup property of the family of kernels $(Q_t)_{t\geq 0}$ follows from (2.18) and (2.19). □

Proposition 2.10 *For every $t \geq 0$ the function $\lambda \mapsto v_t(\lambda)$ is strictly increasing on $[0,\infty)$.*

Proof. By the continuity of $t \mapsto v_t(\lambda)$, for any $\lambda_0 > 0$ there is $t_0 > 0$ so that $v_t(\lambda_0) > 0$ for $0 \leq t \leq t_0$. Then (2.19) implies $Q_t(x,\{0\}) < 1$ for $x > 0$ and $0 \leq t \leq t_0$, and so $\lambda \mapsto v_t(\lambda)$ is strictly increasing for $0 \leq t \leq t_0$. By the semigroup property (2.18) we infer $\lambda \mapsto v_t(\lambda)$ is strictly increasing for all $t \geq 0$. □

Theorem 2.11 *The transition semigroup $(Q_t)_{t\geq 0}$ defined by* (2.19) *is a Feller semigroup.*

Proof. For $\lambda > 0$ and $x \geq 0$ set $e_\lambda(x) = \mathrm{e}^{-\lambda x}$. We denote by $\mathscr{D}_0$ the linear span of $\{e_\lambda : \lambda > 0\}$. By Proposition 2.10, the operator Q_t preserves $\mathscr{D}_0$ for every $t \geq 0$. By the continuity of $t \mapsto v_t(\lambda)$ it is easy to show that $t \mapsto Q_t e_\lambda(x)$ is continuous for $\lambda > 0$ and $x \geq 0$. Then $t \mapsto Q_t f(x)$ is continuous for every $f \in \mathscr{D}_0$ and $x \geq 0$. Let $C_0[0,\infty)$ be the space of continuous functions on $[0,\infty)$ vanishing at infinity. By the Stone–Weierstrass theorem, the set $\mathscr{D}_0$ is uniformly dense in $C_0[0,\infty)$; see, e.g., Hewitt and Stromberg (1965, pp.98-99). Then each operator Q_t preserves $C_0[0,\infty)$ and $t \mapsto Q_t f(x)$ is continuous for $x \geq 0$ and $f \in C_0[0,\infty)$. That gives the Feller property of the semigroup $(Q_t)_{t\geq 0}$. □

A Markov process in $[0,\infty)$ is called a *continuous-state branching process* (CB-process) with *branching mechanism* ϕ if it has transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.19). It is simple to see that $(Q_t)_{t\geq 0}$ satisfies the *branching property*:

$$Q_t(x_1 + x_2, \cdot) = Q_t(x_1,\cdot) * Q_t(x_2,\cdot), \qquad t, x_1, x_2 \geq 0. \tag{2.20}$$

The family of functions $(v_t)_{t\geq 0}$ is called the *cumulant semigroup* of the CB-process. Since $(Q_t)_{t\geq 0}$ is a Feller semigroup, the process has a Hunt realization; see, e.g., Chung (1982, p.75). Clearly, zero is a trap for the CB-process.

Proposition 2.12 *Suppose that $\{(x_1(t), \mathscr{F}_t^1) : t \geq 0\}$ and $\{(x_2(t), \mathscr{F}_t^2) : t \geq 0\}$ are two independent CB-processes with branching mechanism ϕ. Let $x(t) = x_1(t) + x_2(t)$ and $\mathscr{F}_t = \sigma(\mathscr{F}_t^1 \cup \mathscr{F}_t^2)$. Then $\{(x(t), \mathscr{F}_t) : t \geq 0\}$ is also a CB-processes with branching mechanism ϕ.*

Proof. Let $t \geq r \geq 0$ and for $i = 1, 2$ let F_i be a bounded $\mathscr{F}_r^i$-measurable random variable. For any $\lambda \geq 0$ we have

$$\begin{aligned}
\mathbf{P}\big[F_1F_2\,\mathrm{e}^{-\lambda x(t)}\big] &= \mathbf{P}\big[F_1\,\mathrm{e}^{-\lambda x_1(t)}\big]\mathbf{P}\big[F_2\,\mathrm{e}^{-\lambda x_2(t)}\big] \\
&= \mathbf{P}\big[F_1\,\mathrm{e}^{-x_1(r)v_{t-r}(\lambda)}\big]\mathbf{P}\big[F_2\,\mathrm{e}^{-x_2(r)v_{t-r}(\lambda)}\big] \\
&= \mathbf{P}\big[F_1F_2\,\mathrm{e}^{-x(r)v_{t-r}(\lambda)}\big].
\end{aligned}$$

A monotone class argument shows that

$$\mathbf{P}\big[F\,\mathrm{e}^{-\lambda x(t)}\big] = \mathbf{P}\big[F\,\mathrm{e}^{-x(r)v_{t-r}(\lambda)}\big]$$

for any bounded $\mathscr{F}_r$-measurable random variable F. Then $\{(x(t), \mathscr{F}_t) : t \geq 0\}$ is a Markov processes with transition semigroup $(Q_t)_{t\geq 0}$. □

Let $D[0,\infty)$ denote the space of positive càdlàg paths on $[0,\infty)$ furnished with the Skorokhod topology. The following theorem is a slight modification of Theorem 2.1 of Li (2006), which gives a physical interpretation of the CB-process as an approximation of the GW-process with small individuals.

Theorem 2.13 *Suppose that Condition 2.3 holds. Let $\{x(t) : t \geq 0\}$ be a càdlàg CB-process with transition semigroup $(Q_t)_{t\geq 0}$ defined by* (2.19)*. For $k \geq 1$ let $\{z_k(n) : n \geq 0\}$ be a Markov chain with state space $E_k := \{0, k^{-1}, 2k^{-1}, \ldots\}$ and n-step transition probability $Q_k^n(x, \mathrm{d}y)$ determined by* (2.6)*. If $z_k(0)$ converges to $x(0)$ in distribution, then $\{z_k(\lfloor\gamma_k t\rfloor) : t \geq 0\}$ converges as $k \to \infty$ to $\{x(t) : t \geq 0\}$ in distribution on $D[0,\infty)$.*

Proof. For $\lambda > 0$ and $x \geq 0$ set $e_\lambda(x) = \mathrm{e}^{-\lambda x}$. Let $C_0[0,\infty)$ be the space of continuous functions on $[0,\infty)$ vanishing at infinity. By (2.7), (2.19) and Theorem 2.7 it is easy to show

$$\lim_{k\to\infty} \sup_{x\in E_k} \big|Q_k^{\lfloor\gamma_k t\rfloor} e_\lambda(x) - Q_t e_\lambda(x)\big| = 0, \qquad \lambda > 0.$$

Then the Stone–Weierstrass theorem implies

$$\lim_{k\to\infty} \sup_{x\in E_k} \big|Q_k^{\lfloor\gamma_k t\rfloor} f(x) - Q_t f(x)\big| = 0, \qquad f \in C_0[0,\infty).$$

By Ethier and Kurtz (1986, p.226 and pp.233–234) we conclude that $\{z_k(\lfloor\gamma_k t\rfloor) : t \geq 0\}$ converges to the CB-process $\{x(t) : t \geq 0\}$ in distribution on $D[0,\infty)$. □

For any $w \in D[0,\infty)$ let $\tau_0(w) = \inf\{s > 0 : w(s) \text{ or } w(s-) = 0\}$. Let $D_0[0,\infty)$ be the set of paths $w \in D[0,\infty)$ such that $w(t) = 0$ for $t \geq \tau_0(w)$. Then $D_0[0,\infty)$ is a Borel subset of $D[0,\infty)$. It is not hard to show that the distributions of the processes $\{z_k(\lfloor\gamma_k t\rfloor) : t \geq 0\}$ and $\{x(t) : t \geq 0\}$ are all supported by $D_0[0,\infty)$. By Theorem 1.7 of Li (2011, p.4) we have the following:

Corollary 2.14 *Under the conditions of Theorem 2.13, the sequence* $\{z_k(\lfloor\gamma_k t\rfloor) : t \geq 0\}$ *converges as* $k \to \infty$ *to* $\{x(t) : t \geq 0\}$ *in distribution on* $D_0[0,\infty)$.

The convergence of rescaled GW-processes to diffusion processes was first studied by Feller (1951). Lamperti (1967a) showed that all CB-processes are weak limits of rescaled GW-processes. A characterization of CB-processes by random time changes of Lévy processes was given by Lamperti (1967b); see also Kyprianou (2014). We have followed Aliev and Shchurenkov (1982) and Li (2006, 2011) in some of the above calculations.

Example 2.2 For any $0 \leq \alpha \leq 1$ the function $\phi(\lambda) = \lambda^{1+\alpha}$ can be represented in the form of (2.13). In particular, for $0 < \alpha < 1$ we can use integration by parts to see

$$\begin{aligned}
&\int_{(0,\infty)} (\mathrm{e}^{-\lambda u} - 1 + \lambda u)\frac{\mathrm{d}u}{u^{2+\alpha}} \\
&\quad = \lambda^{1+\alpha}\int_{(0,\infty)} (\mathrm{e}^{-v} - 1 + v)\frac{\mathrm{d}v}{v^{2+\alpha}} \\
&\quad = \lambda^{1+\alpha}\bigg[-\frac{\mathrm{e}^{-v} - 1 + v}{(1+\alpha)v^{1+\alpha}}\bigg|_0^\infty + \int_{(0,\infty)} \frac{(1-\mathrm{e}^{-v})\mathrm{d}v}{(1+\alpha)v^{1+\alpha}}\bigg] \\
&\quad = \frac{\lambda^{1+\alpha}}{1+\alpha}\bigg[-(1-\mathrm{e}^{-v})\frac{1}{\alpha v^{\alpha}}\bigg|_0^\infty + \int_{(0,\infty)} \mathrm{e}^{-v}\frac{\mathrm{d}v}{\alpha v^{\alpha}}\bigg] \\
&\quad = \frac{\Gamma(1-\alpha)}{\alpha(1+\alpha)}\lambda^{1+\alpha}.
\end{aligned}$$

Thus we have

$$\lambda^{1+\alpha} = \frac{\alpha(1+\alpha)}{\Gamma(1-\alpha)}\int_{(0,\infty)} (\mathrm{e}^{-\lambda u} - 1 + \lambda u)\frac{\mathrm{d}u}{u^{2+\alpha}}, \qquad \lambda \geq 0. \tag{2.21}$$

Example 2.3 Suppose that there are constants $c > 0$, $0 < \alpha \leq 1$ and b so that $\phi(z) = cz^{1+\alpha} + bz$. Let $q_\alpha^0(t) = \alpha t$ and $q_\alpha^b(t) = b^{-1}(1 - \mathrm{e}^{-\alpha bt})$ for $b \neq 0$. By solving the equation

$$\frac{\partial}{\partial t}v_t(\lambda) = -cv_t(\lambda)^{1+\alpha} - bv_t(\lambda), \qquad v_0(\lambda) = \lambda$$

we get

$$v_t(\lambda) = \frac{\mathrm{e}^{-bt}\lambda}{\big[1 + cq_\alpha^b(t)\lambda^\alpha\big]^{1/\alpha}}, \qquad t \geq 0, \lambda \geq 0. \tag{2.22}$$

3 Some basic properties

In this section we prove some basic properties of CB-processes. Most of the results presented here can be found in Grey (1974) and Li (2000). We here use the treatments in Li (2011). Suppose that ϕ is a branching mechanism defined by (2.13). This is a convex function on $[0,\infty)$. In fact, it is easy to see that

$$\phi'(z) = b + 2cz + \int_{(0,\infty)} u\big(1 - \mathrm{e}^{-zu}\big)m(\mathrm{d}u), \qquad z \geq 0, \tag{3.1}$$

which is an increasing function. In particular, we have $\phi'(0) = b$. The limit $\phi(\infty) := \lim_{z\to\infty} \phi(z)$ exists in $[-\infty, \infty]$ and $\phi'(\infty) := \lim_{z\to\infty} \phi'(z)$ exists in $(-\infty, \infty]$. In particular, we have

$$\phi'(\infty) := b + 2c\cdot\infty + \int_{(0,\infty)} u\, m(\mathrm{d}u) \tag{3.2}$$

with $0\cdot\infty = 0$ by convention. Observe also that $-\infty \leq \phi(\infty) \leq 0$ if and only if $\phi'(\infty) \leq 0$, and $\phi(\infty) = \infty$ if and only if $\phi'(\infty) > 0$.

The transition semigroup $(Q_t)_{t\geq 0}$ of the CB-process is defined by (2.15) and (2.19). From the branching property (2.20), we see that the probability measure $Q_t(x,\cdot)$ is infinitely divisible. Then $(v_t)_{t\geq 0}$ has the *canonical representation*:

$$v_t(\lambda) = h_t\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u})l_t(\mathrm{d}u), \qquad t \geq 0, \lambda \geq 0, \tag{3.3}$$

where $h_t \geq 0$ and $l_t(\mathrm{d}u)$ is a σ-finite measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge u)l_t(\mathrm{d}u) < \infty.$$

The pair (h_t, l_t) is uniquely determined by (3.3); see, e.g., Proposition 1.30 in Li (2011, p.16). By differentiating both sides of the equation and using (2.15) it is easy to find

$$h_t + \int_{(0,\infty)} ul_t(\mathrm{d}u) = \frac{\partial}{\partial\lambda}v_t(0+) = \mathrm{e}^{-bt}, \qquad t \geq 0. \tag{3.4}$$

Then we infer that $l_t(\mathrm{d}u)$ satisfies

$$\int_{(0,\infty)} ul_t(\mathrm{d}u) < \infty.$$

From (2.19) and (3.4) we get

$$\int_{[0,\infty)} yQ_t(x,\mathrm{d}y) = x\,\mathrm{e}^{-bt}, \qquad t \geq 0, x \geq 0. \tag{3.5}$$

We say the branching mechanism ϕ is *critical*, *subcritical* or *supercritical* according as $b = 0$, $b \geq 0$ or $b \leq 0$, respectively.

From (2.15) we see that $t \mapsto v_t(\lambda)$ is first continuous and then continuously differentiable. Moreover, it is easy to show that

$$\frac{\partial}{\partial t} v_t(\lambda)\Big|_{t=0} = -\phi(\lambda), \qquad \lambda \geq 0.$$

By the semigroup property $v_{t+s} = v_s \circ v_t = v_t \circ v_s$ we get the backward differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda, \tag{3.6}$$

and forward differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(\lambda)\frac{\partial}{\partial \lambda} v_t(\lambda), \quad v_0(\lambda) = \lambda. \tag{3.7}$$

The corresponding equations for a branching process with continuous time and discrete state were given in Athreya and Ney (1972, p.106).

Proposition 3.1 *Suppose that $\lambda > 0$ and $\phi(\lambda) \neq 0$. Then the equation $\phi(z) = 0$ has no root between λ and $v_t(\lambda)$ for every $t \geq 0$. Moreover, we have*

$$\int_{v_t(\lambda)}^{\lambda} \phi(z)^{-1} \mathrm{d}z = t, \qquad t \geq 0. \tag{3.8}$$

Proof. By (2.13) we see $\phi(0) = 0$ and $z \mapsto \phi(z)$ is a convex function. Since $\phi(\lambda) \neq 0$ for some $\lambda > 0$ according to the assumption, the equation $\phi(z) = 0$ has at most one root in $(0, \infty)$. Suppose that $\lambda_0 \geq 0$ is a root of $\phi(z) = 0$. Then (3.7) implies $v_t(\lambda_0) = \lambda_0$ for all $t \geq 0$. By Proposition 2.10 we have $v_t(\lambda) > \lambda_0$ for $\lambda > \lambda_0$ and $0 < v_t(\lambda) < \lambda_0$ for $0 < \lambda < \lambda_0$. Then $\lambda > 0$ and $\phi(\lambda) \neq 0$ imply there is no root of $\phi(z) = 0$ between λ and $v_t(\lambda)$. From (3.6) we get (3.8). $\square$

Corollary 3.2 *Suppose that $\phi(z_0) \neq 0$ for some $z_0 > 0$. Let $\theta_0 = \inf\{z > 0 : \phi(z) \geq 0\}$ with the convention $\inf \emptyset = \infty$. Then $\lim_{t\to\infty} v_t(\lambda) = \theta_0$ increasingly for $0 < \lambda < \theta_0$ and decreasingly $\lambda > \theta_0$.*

Proof. In the case $\theta_0 = \infty$, we have $\phi(z) < 0$ for all $z > 0$. From (3.6) we see $\lambda \mapsto v_t(\lambda)$ is increasing. Then (3.6) implies $\lim_{t\to\infty} v_t(\lambda) = \infty$ for every $\lambda > 0$. In the case $\theta_0 < \infty$, we have clearly $\phi(\theta_0) = 0$. Furthermore, $\phi(z) < 0$ for $0 < z < \theta_0$ and $\phi(z) > 0$ for $z > \theta_0$. From (3.7) we see $v_t(\theta_0) = \theta_0$ for all $t \geq 0$. Then (3.8) implies that $\lim_{t\to\infty} v_t(\lambda) = \theta_0$ increasingly for $0 < \lambda < \theta_0$ and decreasingly $\lambda > \theta_0$. $\square$

Corollary 3.3 *Suppose that $\phi(z_0) \neq 0$ for some $z_0 > 0$. Then for any $x > 0$ we have*

$$\lim_{t\to\infty} Q_t(x, \cdot) = \mathrm{e}^{-x\theta_0}\,\delta_0 + (1 - \mathrm{e}^{-x\theta_0})\delta_\infty$$

by weak convergence of probability measures on $[0, \infty]$.

Proof. The space of probability measures on $[0,\infty]$ endowed the topology of weak convergence is compact and metrizable; see, e.g., Parthasarathy (1967, p.45). Let $\{t_n\}$ be any positive sequence so that $t_n \to \infty$ and $Q_{t_n}(x,\cdot) \to$ some $Q_\infty(x,\cdot)$ weakly as $n\to\infty$. By (2.19) and Corollary 3.2, for every $\lambda > 0$ we have

$$\begin{aligned}\int_{[0,\infty]} \mathrm{e}^{-\lambda y}\, Q_\infty(x,\mathrm{d}y) &= \lim_{n\to\infty}\int_{[0,\infty]} \mathrm{e}^{-\lambda y}\, Q_{t_n}(x,\mathrm{d}y)\\ &= \lim_{n\to\infty} \mathrm{e}^{-xv_{t_n}(\lambda)} = \mathrm{e}^{-x\theta_0}\,.\end{aligned}$$

It follows that

$$Q_\infty(x,\{0\}) = \lim_{\lambda\to\infty}\int_{[0,\infty]} \mathrm{e}^{-\lambda y}\, Q_\infty(x,\mathrm{d}y) = \mathrm{e}^{-x\theta_0}$$

and

$$Q_\infty(x,\{\infty\}) = \lim_{\lambda\to 0}\int_{[0,\infty]} (1-\mathrm{e}^{-\lambda y}) Q_\infty(x,\mathrm{d}y) = 1-\mathrm{e}^{-x\theta_0}\,.$$

That shows $Q_\infty(x,\cdot) = \mathrm{e}^{-x\theta_0}\,\delta_0 + (1-\mathrm{e}^{-x\theta_0})\delta_\infty$, which is independent of the particular choice of the sequence $\{t_n\}$. Then we have $Q_t(x,\cdot)\to Q_\infty(x,\cdot)$ weakly as $t\to\infty$. □

A simple asymptotic behavior of the CB-process is described in Corollary 3.3. Clearly, we have: (i) $\theta_0 > 0$ if and only if $b < 0$; (ii) $\theta_0 = \infty$ if and only if $\phi'(\infty) \le 0$. The reader can refer to Grey (1974) and Li (2011, Section 3.2) for more asymptotic results on the CB-process.

Since $(Q_t)_{t\ge 0}$ is a Feller transition semigroup, the CB-process has a Hunt process realization $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$; see, e.g., Chung (1982, p.75). Let $\tau_0 := \inf\{s \ge 0 : x(s) = 0\}$ denote the *extinction time* of the CB-process.

Theorem 3.4 *For every $t\ge 0$ the limit $\bar v_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0,\infty]$. Moreover, the mapping $t\mapsto \bar v_t$ is decreasing and for any $t\ge 0$ and $x>0$ we have*

$$\mathbf{Q}_x\{\tau_0 \le t\} = \mathbf{Q}_x\{x(t) = 0\} = \exp\{-x\bar v_t\}. \tag{3.9}$$

Proof. By Proposition 2.10 the limit $\bar v_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0,\infty]$ for every $t\ge 0$. For $t\ge r\ge 0$ we have

$$\bar v_t = \uparrow\lim_{\lambda\to\infty} v_r(v_{t-r}(\lambda)) = v_r(\bar v_{t-r}) \le \bar v_r. \tag{3.10}$$

Since zero is a trap for the CB-process, we get (3.9) by letting $\lambda\to\infty$ in (2.19). □

For the convenience of statement of the results in the sequel, we formulate the following condition on the branching mechanism, which is known as *Grey's condition*:

Condition 3.5 *There is some constant $\theta > 0$ so that*

$$\phi(z) > 0 \textit{ for } z\ge\theta \textit{ and } \int_\theta^\infty \phi(z)^{-1}\mathrm{d}z < \infty.$$

Theorem 3.6 *We have $\bar{v}_t < \infty$ for some and hence all $t > 0$ if and only if Condition 3.5 holds.*

Proof. By (3.10) it is simple to see that $\bar{v}_t = \uparrow\lim_{\lambda\to\infty} v_t(\lambda) < \infty$ for all $t > 0$ if and only if this holds for some $t > 0$. If Condition 3.5 holds, we can let $\lambda \to \infty$ in (3.8) to obtain

$$\int_{\bar{v}_t}^{\infty} \phi(z)^{-1}\mathrm{d}z = t \tag{3.11}$$

and hence $\bar{v}_t < \infty$ for $t > 0$. For the converse, suppose that $\bar{v}_t < \infty$ for some $t > 0$. By (3.6) there exists some $\theta > 0$ so that $\phi(\theta) > 0$, for otherwise we would have $\bar{v}_t \geq v_t(\lambda) \geq \lambda$ for all $\lambda \geq 0$, yielding a contradiction. Then $\phi(z) > 0$ for all $z \geq \theta$ by the convexity of the branching mechanism. As in the above we see that (3.11) still holds, so Condition 3.5 is satisfied. □

Theorem 3.7 *Let $\bar{v} = \downarrow\lim_{t\to\infty} \bar{v}_t \in [0, \infty]$. Then for any $x > 0$ we have*

$$\mathbf{Q}_x\{\tau_0 < \infty\} = \exp\{-x\bar{v}\}. \tag{3.12}$$

Moreover, we have $\bar{v} < \infty$ if and only if Condition 3.5 holds, and in this case $\bar{v}$ is the largest root of $\phi(z) = 0$.

Proof. The first assertion follows immediately from Theorem 3.4. By Theorem 3.6 we have $\bar{v}_t < \infty$ for some and hence all $t > 0$ if and only if Condition 3.5 holds. This is clearly equivalent to $\bar{v} < \infty$. From (3.11) we see $\bar{v}$ is the largest root of $\phi(z) = 0$. □

Corollary 3.8 *Suppose that Condition 3.5 holds. Then for any $x > 0$ we have $\mathbf{Q}_x\{\tau_0 < \infty\} = 1$ if and only if $b \geq 0$.*

By Corollary 3.2 and Theorem 3.7 we see $0 \leq \theta_0 \leq \bar{v} \leq \infty$. In fact, we have $0 \leq \theta_0 = \bar{v} < \infty$ if Condition 3.5 holds and $0 \leq \theta_0 < \bar{v} = \infty$ if there is $\theta > 0$ so that

$$\phi(z) > 0 \text{ for } z \geq \theta \text{ and } \int_{\theta}^{\infty} \phi(z)^{-1}\mathrm{d}z = \infty.$$

Proposition 3.9 *For any $t \geq 0$ and $\lambda \geq 0$ let $v_t'(\lambda) = (\partial/\partial\lambda)v_t(\lambda)$. Then we have*

$$v_t'(\lambda) = \exp\bigg\{ -\int_0^t \phi'(v_s(\lambda))\mathrm{d}s\bigg\}, \tag{3.13}$$

where ϕ' is given by (3.1).

Proof. Based on (2.15) and (3.6) it is elementary to see that

$$\frac{\partial}{\partial t}v_t'(\lambda) = \frac{\partial}{\partial\lambda}\frac{\partial}{\partial t}v_t(\lambda) = -\phi'(v_t(\lambda))v_t'(\lambda).$$

It follows that

$$\frac{\partial}{\partial t}\big[\log v_t'(\lambda)\big] = v_t'(\lambda)^{-1}\frac{\partial}{\partial t}v_t'(\lambda) = -\phi'(v_t(\lambda)).$$

Since $v_0'(\lambda) = 1$, we get (3.13). □

Theorem 3.10 *Let* $\phi_0'(z) = \phi'(z) - b$ *for* $z \geq 0$, *where* ϕ' *is given by* (3.1). *We can define a Feller transition semigroup* $(Q_t^b)_{t\geq 0}$ *on* $[0, \infty)$ *by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\, Q_t^b(x, \mathrm{d}y) = \exp\bigg\{ -xv_t(\lambda) - \int_0^t \phi_0'(v_s(\lambda))\mathrm{d}s \bigg\}. \tag{3.14}$$

Moreover, we have $Q_t^b(x, \mathrm{d}y) = \mathrm{e}^{bt}\, x^{-1}yQ_t(x, \mathrm{d}y)$ *for* $x > 0$ *and*

$$Q_t^b(0, \mathrm{d}y) = \mathrm{e}^{bt}[h_t\delta_0(\mathrm{d}y) + yl_t(\mathrm{d}y)], \qquad t, y \geq 0. \tag{3.15}$$

Proof. In view of (3.5), it is simple to check that $Q_t^b(x, \mathrm{d}y) := \mathrm{e}^{bt}\, x^{-1}yQ_t(x, \mathrm{d}y)$ defines a Markov transition semigroup $(Q_t^b)_{t\geq 0}$ on $(0, \infty)$. Let $q_t(\lambda) = \mathrm{e}^{bt}\, v_t(\lambda)$ and let $q_t'(\lambda) = (\partial/\partial\lambda)q_t(\lambda)$. By differentiating both sides of (2.19) we see

$$\int_{(0,\infty)} \mathrm{e}^{-\lambda y}\, Q_t^b(x, \mathrm{d}y) = \exp\{-xv_t(\lambda)\}q_t'(\lambda), \qquad x > 0, \lambda \geq 0.$$

From (3.3) and (3.13) we have

$$q_t'(\lambda) = \mathrm{e}^{bt}\left[h_t + \int_{(0,\infty)} \mathrm{e}^{-\lambda u}\, ul_t(\mathrm{d}u)\right] = \exp\bigg\{ - \int_0^t \phi_0'(v_s(\lambda))\mathrm{d}s \bigg\}.$$

Then we can define $Q_t^b(0, \mathrm{d}y)$ by (3.15) and extend $(Q_t^b)_{t\geq 0}$ to a Markov transition semigroup on $[0, \infty)$. The Feller property of the semigroup is immediate by (3.14). □

Corollary 3.11 *Let* $(Q_t^b)_{t\geq 0}$ *be the transition semigroup define by* (3.14). *Then we have* $Q_t^b(0, \{0\}) = \mathrm{e}^{bt}\, h_t$ *and* $Q_t^b(x, \{0\}) = 0$ *for* $t \geq 0$ *and* $x > 0$.

Theorem 3.12 *Suppose that* $T > 0$ *and* $x > 0$. *Then* $\mathbf{P}_x^{b,T}(\mathrm{d}\omega) = x^{-1}\, \mathrm{e}^{bT}\, x(\omega, T)$ $\mathbf{Q}_x(\mathrm{d}\omega)$ *defines a probability measure on* $(\Omega, \mathscr{F}_T)$. *Moreover, the process* $\{(x(t), \mathscr{F}_t) : 0 \leq t \leq T\}$ *under this measure is a Markov process with transition semigroup* $(Q_t^b)_{t\geq 0}$ *given by* (3.14).

Proof. Clearly, the probability measure $\mathbf{P}_x^{b,T}$ is carried by $\{x(T) > 0\} \in \mathscr{F}_T$. Then we have $\mathbf{P}_x^{b,T}\{x(t) > 0\} = 1$ for every $0 \leq t \leq T$. Let $0 \leq r \leq t \leq T$. Let F be a bounded $\mathscr{F}_r$-measurable random variable and f a bounded Borel function on $[0, \infty)$. By (3.5) and the Markov property under $\mathbf{Q}_x$,

$$\begin{aligned}
\mathbf{P}_x^{b,T}[Ff(x(t))] &= x^{-1}\, \mathrm{e}^{bT}\, \mathbf{Q}_x\big[Ff(x(t))x(T)\big] \\
&= x^{-1}\, \mathrm{e}^{bt}\, \mathbf{Q}_x\big[Ff(x(t))x(t)\big] \\
&= x^{-1}\, \mathrm{e}^{br}\, \mathbf{Q}_x\big[Fx(r)Q_{t-r}^b f(x(r))\big] \\
&= \mathbf{P}_x^{b,T}\big[FQ_{t-r}^b f(x(r))\big],
\end{aligned}$$

where we have used the relation $Q_{t-r}^b(x, \mathrm{d}y) = \mathrm{e}^{bt}\, x^{-1}yQ_{t-r}(x, \mathrm{d}y)$ for the third equality. Then $\{(x(t), \mathscr{F}_t) : 0 \leq t \leq T\}$ under $\mathbf{P}_x^{b,T}$ is a Markov process with transition semigroup $(Q_t^b)_{t\geq 0}$. □

Recall that zero is a trap for the CB-process. Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of its transition semigroup $(Q_t)_{t\geq 0}$ to $(0,\infty)$. For a σ-finite measure μ on $(0,\infty)$ write

$$\mu Q_t^\circ(\mathrm{d}y) = \int_{(0,\infty)} \mu(\mathrm{d}x) Q_t^\circ(x,\mathrm{d}y), \qquad t \geq 0, y > 0.$$

A family of σ-finite measures $(\kappa_t)_{t>0}$ on $(0,\infty)$ is called an *entrance rule* for $(Q_t^\circ)_{t\geq 0}$ if $\kappa_r Q_{t-r}^\circ \leq \kappa_t$ for all $t > r > 0$ and $\kappa_r Q_{t-r}^\circ \to \kappa_t$ as $r \to t$. We call $(\kappa_t)_{t>0}$ an *entrance law* if $\kappa_r Q_{t-r}^\circ = \kappa_t$ for all $t > r > 0$.

The special case of the canonical representation (3.3) with $h_t = 0$ for all $t > 0$ is particularly interesting. In this case, we have

$$v_t(\lambda) = \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u}) l_t(\mathrm{d}u), \qquad t > 0, \lambda \geq 0. \tag{3.16}$$

From this and (2.19) we have, for $t > 0$ and $\lambda \geq 0$,

$$\int_{(0,\infty)} (1 - \mathrm{e}^{-y\lambda}) l_t(\mathrm{d}y) = \lim_{x\to 0} x^{-1} \int_{(0,\infty)} (1 - \mathrm{e}^{-y\lambda}) Q_t^\circ(x,\mathrm{d}y).$$

Then, formally,

$$l_t = \lim_{x\to 0} x^{-1} Q_t(x,\cdot). \tag{3.17}$$

Theorem 3.13 *The cumulant semigroup $(v_t)_{t\geq 0}$ admits representation* (3.16) *if and only if $\phi'(\infty) = \infty$. In this case, the family $(l_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t\geq 0}$.*

Proof. By differentiating both sides of the general representation (3.3) we get

$$v_t'(\lambda) = h_t + \int_{(0,\infty)} u\, \mathrm{e}^{-\lambda u}\, l_t(\mathrm{d}u), \qquad t \geq 0, \lambda \geq 0. \tag{3.18}$$

From this and (3.13) it follows that

$$h_t = v_t'(\infty) = \exp\Big\{ - \int_0^t \phi'(\bar{v}_s) \mathrm{d}s \Big\}.$$

Then we have $\phi'(\infty) = \infty$ if $h_t = 0$ for any $t > 0$. For the converse, assume that $\phi'(\infty) = \infty$. If Condition 3.5 holds, we have $\bar{v}_t < \infty$ for $t > 0$ by Theorem 3.6, so $h_t = 0$ by (3.3). If Condition 3.5 does not hold, we have $\bar{v}_t = \infty$ by Theorem 3.6. Since $\phi'(\infty) = \infty$, by (3.18) and (3.13) we see $h_t = v_t'(\infty) = 0$ for $t > 0$. If $(v_t)_{t\geq 0}$ admits the representation (3.16), we can use (2.18) to see, for $t > r > 0$ and $\lambda \geq 0$,

$$\begin{aligned}\int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u}) l_t(\mathrm{d}u) &= \int_{(0,\infty)} (1 - \mathrm{e}^{-u v_{t-r}(\lambda)}) l_r(\mathrm{d}u) \\ &= \int_{(0,\infty)} l_r(\mathrm{d}x) \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u}) Q_{t-r}^\circ(x,\mathrm{d}u).\end{aligned}$$

Then $(l_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t\geq 0}$. $\square$

Corollary 3.14 *If Condition 3.5 holds, the cumulant semigroup admits the representation* (3.16) *and* $t \mapsto \bar{v}_t = l_t(0, \infty)$ *is the unique solution to the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_t = -\phi(\bar{v}_t), \qquad t > 0 \tag{3.19}$$

with singular initial condition $\bar{v}_{0+} = \infty$.

Proof. Under Condition 3.5, for every $t > 0$ we have $\bar{v}_t < \infty$ by Theorem 3.6. Moreover, the condition and the convexity of $z \mapsto \phi(z)$ imply $\phi'(\infty) = \infty$. Then we have the representation (3.16) by Theorem 3.13. The semigroup property of $(v_t)_{t\geq 0}$ implies $\bar{v}_{s+t} = v_s(\bar{v}_t)$ for $s > 0$ and $t > 0$. Then $t \mapsto \bar{v}_t$ satisfies (3.19). From (3.11) it is easy to see $\bar{v}_{0+} = \infty$. Suppose that $t \mapsto u_t$ and $t \mapsto v_t$ are two solutions to (3.19) with $u_{0+} = v_{0+} = \infty$. For any $\varepsilon > 0$ there exits $\delta > 0$ so that $u_s \geq v_\varepsilon$ for every $0 < s \leq \delta$. Since both $t \mapsto u_{s+t}$ and $t \mapsto v_{\varepsilon+t}$ are solutions to (3.19), we have $u_{s+t} \geq v_{\varepsilon+t}$ for $t \geq 0$ and $0 < s \leq \delta$ by Proposition 2.10. Then we can let $s \to 0$ and $\varepsilon \to 0$ to see $u_t \geq v_t$ for $t > 0$. By symmetry we get the uniqueness of the solution. □

Theorem 3.15 *If* $\delta := \phi'(\infty) < \infty$, *then we have, for* $t \geq 0$ *and* $\lambda \geq 0$,

$$v_t(\lambda) = \mathrm{e}^{-\delta t}\lambda + \int_0^t \mathrm{e}^{-\delta s}\,\mathrm{d}s \int_{(0,\infty)} (1 - \mathrm{e}^{-u v_{t-s}(\lambda)}) m(\mathrm{d}u), \tag{3.20}$$

that is, we have (3.3) *with*

$$h_t = \mathrm{e}^{-\delta t}, \quad l_t = \int_0^t \mathrm{e}^{-\delta s} m Q^\circ_{t-s} \mathrm{d}s, \quad t \geq 0. \tag{3.21}$$

In this case, the family $(l_t)_{t>0}$ *is an entrance rule for* $(Q^\circ_t)_{t\geq 0}$.

Proof. If $\delta := \phi'(\infty) < \infty$, by (3.2) we must have $c = 0$. In this case, we can write the branching mechanism into

$$\phi(\lambda) = \delta\lambda + \int_{(0,\infty)} (\mathrm{e}^{-\lambda z} - 1) m(\mathrm{d}z), \qquad \lambda \geq 0. \tag{3.22}$$

By (2.15) and integration by parts,

$$\begin{aligned} v_t(\lambda)\,\mathrm{e}^{\delta t} &= \lambda + \int_0^t \delta v_s(\lambda)\,\mathrm{e}^{\delta s}\,\mathrm{d}s - \int_0^t \phi(v_s(\lambda))\,\mathrm{e}^{\delta s}\,\mathrm{d}s \\ &= \lambda + \int_0^t \mathrm{e}^{\delta s}\,\mathrm{d}s \int_{(0,\infty)} (1 - \mathrm{e}^{-u v_s(\lambda)}) m(\mathrm{d}u). \end{aligned}$$

That gives (3.20) and (3.21). It is easy to see that $(l_t)_{t>0}$ is an entrance rule for $(Q^\circ_t)_{t\geq 0}$. □

Example 3.1 Suppose that there are constants $c > 0$, $0 < \alpha \leq 1$ and b so that $\phi(z) = cz^{1+\alpha} + bz$. Then Condition 3.5 is satisfied. Let $q^0_\alpha(t)$ be defined as in Example 2.3. By letting $\lambda \to \infty$ in (2.22) we get $\bar{v}_t = c^{-1/\alpha}\,\mathrm{e}^{-bt} q^b_\alpha(t)^{-1/\alpha}$ for $t > 0$. In particular, if $\alpha = 1$, then (3.16) holds with

$$l_t(\mathrm{d}u) = \frac{\mathrm{e}^{-bt}}{c^2 q^b_1(t)^2} \exp\Big\{ - \frac{u}{c q^b_1(t)} \Big\} \mathrm{d}u, \qquad t > 0, u > 0.$$

4 Positive integral functionals

In this section, we give characterizations of a class of positive integral functionals of the CB-process in terms of Laplace transforms. The corresponding results in the measure-valued setting can be found in Li (2011). For our purpose, it is more convenient to start the process from an arbitrary initial time $r \geq 0$. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_{r,t}, x(t), \mathbf{Q}_{r,x})$ a càdlàg realization of the CB-process with transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.15) and (2.19). For any $t \geq r \geq 0$ and $\lambda \geq 0$ we have

$$\mathbf{Q}_{r,x} \exp\{-\lambda x(t)\} = \exp\{-x u_r(\lambda)\}, \tag{4.1}$$

where $r \mapsto u_r(\lambda) := v_{t-r}(\lambda)$ is the unique bounded positive solution to

$$u_r(\lambda) + \int_r^t \phi(u_s(\lambda)) \mathrm{d}s = \lambda, \qquad 0 \leq r \leq t. \tag{4.2}$$

Proposition 4.1 *For* $\{t_1 < \cdots < t_n\} \subset [0, \infty)$ *and* $\{\lambda_1, \ldots, \lambda_n\} \subset [0, \infty)$ *we have*

$$\mathbf{Q}_{r,x} \exp\bigg\{ -\sum_{j=1}^n \lambda_j x(t_j) 1_{\{r\leq t_j\}}\bigg\} = \exp\{-x u(r)\}, \quad 0 \leq r \leq t_n, \tag{4.3}$$

where $r \mapsto u(r)$ *is a bounded positive function on* $[0, t_n]$ *solving*

$$u(r) + \int_r^{t_n} \phi(u(s)) \mathrm{d}s = \sum_{j=1}^n \lambda_j 1_{\{r\leq t_j\}}. \tag{4.4}$$

Proof. We shall give the proof by induction in $n \geq 1$. For $n = 1$ the result follows from (4.1) and (4.2). Now supposing (4.3) and (4.4) are satisfied when n is replaced by $n-1$, we prove they are also true for n. It is clearly sufficient to consider the case with $0 \leq r \leq t_1 < \cdots < t_n$. By the Markov property,

$$\begin{aligned}
&\mathbf{Q}_{r,x} \exp\bigg\{ -\sum_{j=1}^n \lambda_j x(t_j)\bigg\} \\
&\quad = \mathbf{Q}_{r,x}\bigg[\mathbf{Q}_{r,x}\bigg(\exp\bigg\{ -\sum_{j=1}^n \lambda_j x(t_j)\bigg\}\bigg|\mathscr{F}_{r,t_1}\bigg)\bigg] \\
&\quad = \mathbf{Q}_{r,x}\bigg[\mathrm{e}^{-x(t_1)\lambda_1}\, \mathbf{Q}_{r,x}\bigg(\exp\bigg\{ -\sum_{j=2}^n \lambda_j x(t_j)\bigg\}\bigg|\mathscr{F}_{r,t_1}\bigg)\bigg] \\
&\quad = \mathbf{Q}_{r,x}\bigg[\mathrm{e}^{-x(t_1)\lambda_1}\, \mathbf{Q}_{t_1,x(t_1)}\bigg(\exp\bigg\{ -\sum_{j=2}^n \lambda_j x(t_j)\bigg\}\bigg)\bigg] \\
&\quad = \mathbf{Q}_{r,x} \exp\Big\{ -x(t_1)\lambda_1 - x(t_1) w(t_1)\Big\},
\end{aligned}$$

where $r \mapsto w(r)$ is a bounded positive Borel function on $[0, t_n]$ satisfying

$$w(r) + \int_r^{t_n} \phi(w(s)) \mathrm{d}s = \sum_{j=2}^n \lambda_j 1_{\{r\leq t_j\}}. \tag{4.5}$$

Then the result for $n = 1$ implies that

$$\mathbf{Q}_{r,x} \exp\bigg\{-\sum_{j=1}^{n} \lambda_j x(t_j)\bigg\} = \exp\{-xu(r)\}$$

with $r \mapsto u(r)$ being a bounded positive Borel function on $[0, t_1]$ satisfying

$$u(r) + \int_r^{t_1} \phi(u(s))\mathrm{d}s = \lambda_1 + w(t_1). \tag{4.6}$$

Setting $u(r) = w(r)$ for $t_1 < r \leq t_n$, from (4.5) and (4.6) one checks that $r \mapsto u(r)$ is a bounded positive solution to (4.4) on $[0, t_n]$. □

Theorem 4.2 *Suppose that $t \geq 0$ and μ is a finite measure supported by $[0, t]$. Let $s \mapsto \lambda(s)$ be a bounded positive Borel function on $[0, t]$. Then we have*

$$\mathbf{Q}_{r,x} \exp\bigg\{-\int_{[r,t]} \lambda(s)x(s)\mu(\mathrm{d}s)\bigg\} = \exp\{-xu(r)\}, \quad 0 \leq r \leq t, \tag{4.7}$$

where $r \mapsto u(r)$ is the unique bounded positive solution on $[0, t]$ to

$$u(r) + \int_r^t \phi(u(s))\mathrm{d}s = \int_{[r,t]} \lambda(s)\mu(\mathrm{d}s). \tag{4.8}$$

Proof. Step 1. We first consider a bounded positive continuous function $s \mapsto \lambda(s)$ on $[0, t]$. To avoid triviality we assume $t > 0$. For any integer $n \geq 1$ define the finite measure μ_n on $[0, t]$ by

$$\mu_n(\mathrm{d}s) = \sum_{k=1}^{2^n} \mu[(k-1)t/2^n, kt/2^n)\delta_{kt/2^n}(\mathrm{d}s) + \mu(\{t\})\delta_t(\mathrm{d}s).$$

By Proposition 4.1 we see that

$$\mathbf{Q}_{r,x} \exp\bigg\{-\int_{[r,t]} \lambda(s)x(s)\mu_n(\mathrm{d}s)\bigg\} = \exp\{-xu_n(r)\}, \tag{4.9}$$

where $r \mapsto u_n(r)$ is a bounded positive solution on $[0, t]$ to

$$u_n(r) + \int_r^t \phi(u_n(s))\mathrm{d}s = \int_{[r,t]} \lambda(s)\mu_n(\mathrm{d}s). \tag{4.10}$$

Let $v_n(r) = u_n(t - r)$ for $0 \leq r \leq t$. Observe that $\phi(z) \geq bz \geq -|b|z$ for every $z \geq 0$. From (4.10) we have

$$v_n(r) \;=\; \int_{[t-r,t]} \lambda(s)\mu_n(\mathrm{d}s) - \int_0^r \phi(v_n(s))\mathrm{d}s$$

$$\leq \sup_{0\leq s\leq t} \lambda(s)\mu[0,t] + |b| \int_0^r v_n(s)\mathrm{d}s.$$

By Gronwall's inequality it is easy to show that $\{v_n\}$ and hence $\{u_n\}$ is uniformly bounded on $[0,t]$. Let $q_n(t) = t$. For any $0 \leq s < t$ let $q_n(s) = (\lfloor 2^n s/t \rfloor + 1)t/2^n$, where $\lfloor 2^n s/t \rfloor$ denotes the integer part of $2^n s/t$. Then $s \leq q_n(s) \leq s + t/2^n$. It is easy to see that

$$\int_{[r,t]} f(s)\mu_n(\mathrm{d}s) = \int_{[r,t]} f(q_n(s))\mu(\mathrm{d}s)$$

for any bounded Borel function f on $[0,t]$. By the right-continuity of $s \mapsto \lambda(s)$ and $s \mapsto x(s)$ we have

$$\lim_{n\to\infty} \int_{[r,t]} \lambda(s)\mu_n(\mathrm{d}s) = \int_{[r,t]} \lambda(s)\mu(\mathrm{d}s)$$

and

$$\lim_{n\to\infty} \int_{[r,t]} \lambda(s)x(s)\mu_n(\mathrm{d}s) = \int_{[r,t]} \lambda(s)x(s)\mu(\mathrm{d}s).$$

From (4.9) we see the limit $u(r) = \lim_{n\to\infty} u_n(r)$ exists and (4.7) holds for $0 \leq r \leq t$. Then we get (4.8) by letting $n \to \infty$ in (4.10).

Step 2. Let $B_0[0,\infty)$ be the set of bounded Borel functions $s \mapsto \lambda(s)$ for which there exist bounded positive solutions $r \mapsto u(r)$ of (4.8) such that (4.7) holds. Then $B_0[0,\infty)$ is closed under bounded pointwise convergence. The result of the first step shows that $B_0[0,\infty)$ contains all positive continuous functions on $[0,t]$. By Proposition 1.3 in Li (2011, p.3) we infer that $B_0[0,\infty)$ contains all bounded positive Borel functions on $[0,t]$.

Step 3. To show the uniqueness of the solution to (4.8), suppose that $r \mapsto v(r)$ is another bounded positive Borel function on $[0,t]$ satisfying this equation. Since $z \mapsto \phi(z)$ is locally Lipschitz, it is easy to find a constant $K \geq 0$ such that

$$\begin{aligned} |u(r) - v(r)| &\leq \int_r^t |\phi(u(s)) - \phi(v(s))|\mathrm{d}s \\ &\leq K \int_r^t |u(s) - v(s)|\mathrm{d}s. \end{aligned}$$

Let $U(r) = |u(t-r) - v(t-r)|$ for $0 \leq r \leq t$. We have

$$U(r) \leq K \int_0^r U(s)\mathrm{d}s, \quad 0 \leq r \leq t.$$

Then Gronwall's inequality implies $U(r) = 0$ for every $0 \leq r \leq t$. □

Suppose that $\mu(\mathrm{d}s)$ is a locally bounded Borel measure on $[0,\infty)$ and $s \mapsto \lambda(s)$ is a locally bounded positive Borel function on $[0,\infty)$. We define the *positive integral functional*:

$$A[r,t] := \int_{[r,t]} \lambda(s)x(s)\mu(\mathrm{d}s), \qquad t \geq r \geq 0.$$

By replacing $\lambda(s)$ with $\theta\lambda(s)$ in Theorem 4.2 for $\theta \geq 0$ we get a characterization of the Laplace transform of the random variable $A[r,t]$.

Theorem 4.3 *Let $t \geq 0$ be given. Let $\lambda \geq 0$ and let $s \mapsto \theta(s)$ be a bounded positive Borel function on $[0,t]$. Then for $0 \leq r \leq t$ we have*

$$\mathbf{Q}_{r,x} \exp\bigg\{ -\lambda x(t) - \int_r^t \theta(s)x(s)\mathrm{d}s \bigg\} = \exp\{-xu(r)\}, \tag{4.11}$$

where $r \mapsto u(r)$ is the unique bounded positive solution on $[0,t]$ to

$$u(r) + \int_r^t \phi(u(s))\mathrm{d}s = \lambda + \int_r^t \theta(s)\mathrm{d}s. \tag{4.12}$$

Proof. This follows by an application of Theorem 4.2 to the measure $\mu(\mathrm{d}s) = \mathrm{d}s + \delta_t(\mathrm{d}s)$ and the function $\lambda(s) = 1_{\{s<t\}}\theta(s) + 1_{\{s=t\}}\lambda$. □

Corollary 4.4 *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$ be a Hunt realization of the CB-process started from time zero. Then we have, for $t, \lambda, \theta \geq 0$,*

$$\mathbf{Q}_x \exp\bigg\{ -\lambda x(t) - \theta \int_0^t x(s)\mathrm{d}s \bigg\} = \exp\{-xv(t)\}, \tag{4.13}$$

where $t \mapsto v(t) = v(t, \lambda, \theta)$ is the unique positive solution to

$$\frac{\partial}{\partial t} v(t) = \theta - \phi(v(t)), \quad v(0) = \lambda. \tag{4.14}$$

Proof. By Theorem 4.3 we have (4.13) with $v(t) = u_t(0)$, where $r \mapsto u_t(r)$ is the unique bounded positive solution on $[0,t]$ to

$$u(r) + \int_r^t \phi(u(s))\mathrm{d}s = \lambda + (t-r)\theta.$$

Then $r \mapsto v(r) := u_t(t-r)$ is the unique bounded positive solution on $[0,t]$ of

$$v(r) + \int_0^r \phi(v(s))\mathrm{d}s = \lambda + r\theta. \tag{4.15}$$

Clearly, we can extend (4.15) to all $r \geq 0$ and the extended equation is equivalent with the differential equation (4.14). The uniqueness of the solution follows by Gronwall's inequality. □

Corollary 4.5 *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$ be a Hunt realization of the CB-process started from time zero. Then we have, for $t, \theta \geq 0$,*

$$\mathbf{Q}_x \exp\bigg\{ -\theta \int_0^t x(s)\mathrm{d}s \bigg\} = \exp\{-xv(t)\}. \tag{4.16}$$

where $t \mapsto v(t) = v(t, \theta)$ is the unique positive solution to

$$\frac{\partial}{\partial t} v(t) = \theta - \phi(v(t)), \quad v(0) = 0. \tag{4.17}$$

Corollary 4.6 *Let $t \geq 0$ be given. Let ϕ_1 and ϕ_2 be two branching mechanisms in form* (2.13) *satisfying $\phi_1(z) \geq \phi_2(z)$ for all $z \geq 0$. Let $t \mapsto v_i(t)$ be the solution to* (4.14) *or* (4.15) *with $\phi = \phi_i$. Then $v_1(t) \leq v_2(t)$ for all $t \geq 0$.*

Proof. Fix $t \geq 0$ and let $u_i(r) = v_i(t-r)$ for $0 \leq r \leq t$. Then $r \mapsto u_1(r)$ is the unique bounded positive solution on $[0, t]$ of

$$u(r) + \int_r^t \phi_1(u(s))\mathrm{d}s = \lambda + (t-r)\theta$$

and $r \mapsto u_2(r)$ is the unique bounded positive solution on $[0, t]$ of

$$u(r) + \int_r^t \phi_1(u(s))\mathrm{d}s = \lambda + \int_r^t [\theta + g(s)]\mathrm{d}s,$$

where $g(s) = \phi_1(u_2(s)) - \phi_2(u_2(s)) \geq 0$. By Theorem 4.3 one can see $u_1(r) \leq u_2(r)$ for all $0 \leq r \leq t$. □

Recall that $\phi'(\infty)$ is given by (3.2). Under the condition $\phi'(\infty) > 0$, we have $\phi(z) \to \infty$ as $z \to \infty$, so the inverse $\phi^{-1}(\theta) := \inf\{z \geq 0 : \phi(z) > \theta\}$ is well-defined for $\theta \geq 0$.

Proposition 4.7 *For $\theta > 0$ let $t \mapsto v(t, \theta)$ be the unique positive solution to* (4.17). *Then $\lim_{t\to\infty} v(t, \theta) = \infty$ if $\phi'(\infty) \leq 0$, and $\lim_{t\to\infty} v(t, \theta) = \phi^{-1}(\theta)$ if $\phi'(\infty) > 0$.*

Proof. By Proposition 2.10 we have $\mathbf{Q}_x\{x(t) > 0\} > 0$ for every $x > 0$ and $t \geq 0$. From (4.16) we see $t \mapsto v(t, \theta)$ is strictly increasing, so $(\partial/\partial t)v(t, \theta) > 0$ for all $\theta > 0$. Let $v(\infty, \theta) = \lim_{t\to\infty} v(t, \theta) \in (0, \infty]$. In the case $\phi'(\infty) \leq 0$, we clearly have $\phi(z) \leq 0$ for all $z \geq 0$. Then $(\partial/\partial t)v(t, \theta) \geq \theta > 0$ and $v(\infty, \theta) = \infty$. In the case $\phi'(\infty) > 0$, we note

$$\phi(v(t, \theta)) = \theta - \frac{\partial}{\partial t}v(t, \theta) < \theta,$$

and hence $v(t, \theta) < \phi^{-1}(\theta)$, implying $v(\infty, \theta) \leq \phi^{-1}(\theta) < \infty$. It follows that

$$0 = \lim_{t\to\infty} \frac{\partial}{\partial t}v(t, \theta) = \theta - \lim_{t\to\infty} \phi(v(t, \theta)) = \theta - \phi(v(\infty, \theta)).$$

Then we have $v(\infty, \theta) = \phi^{-1}(\theta)$. □

Theorem 4.8 *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$ be a Hunt realization of the CB-process started from time zero. If $\phi'(\infty) > 0$, then for $x > 0$ and $\theta > 0$ we have*

$$\mathbf{Q}_x \exp\bigg\{ -\theta \int_0^\infty x(s)\mathrm{d}s \bigg\} = \exp\{-x\phi^{-1}(\theta)\}$$

and

$$\mathbf{Q}_x\bigg\{ \int_0^\infty x(s)\mathrm{d}s < \infty \bigg\} = \exp\{-x\phi^{-1}(0)\},$$

where $\phi^{-1}(0) = \inf\{z > 0 : \phi(z) \geq 0\}$. If $\phi'(\infty) \leq 0$, then for any $x > 0$ we have

$$\mathbf{Q}_x\bigg\{ \int_0^\infty x(s)\mathrm{d}s < \infty \bigg\} = 0.$$

Proof. In view of (4.16), we have

$$\mathbf{Q}_x \exp\bigg\{ -\theta \int_0^\infty x(s)\mathrm{d}s \bigg\} = \lim_{t\to\infty} \exp\{-xv(t,\theta)\}.$$

Then the result follows from Proposition 4.7. $\square$

5 Construction of CBI-processes

Let $\{p(j) : j \in \mathbb{N}\}$ and $\{q(j) : j \in \mathbb{N}\}$ be probability distributions on $\mathbb{N} := \{0, 1, 2, \ldots\}$ with generating functions g and h, respectively. Suppose that $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ is a family of $\mathbb{N}$-valued i.i.d. random variables with distribution $\{p(j) : j \in \mathbb{N}\}$ and $\{\eta_n : n = 1, 2, \ldots\}$ is a family of $\mathbb{N}$-valued i.i.d. random variables with distribution $\{q(j) : j \in \mathbb{N}\}$. We assume the two families are independent of each other. Given an $\mathbb{N}$-valued random variable $y(0)$ independent of $\{\xi_{n,i}\}$ and $\{\eta_n\}$, we define inductively

$$y(n) = \sum_{i=1}^{y(n-1)} \xi_{n,i} + \eta_n, \qquad n = 1, 2, \ldots. \tag{5.1}$$

This is clearly a generalization of (2.1). For $i \in \mathbb{N}$ let $\{Q(i,j) : j \in \mathbb{N}\}$ denote the i-fold convolution of $\{p(j) : j \in \mathbb{N}\}$. Let

$$P(i,j) = (Q(i,\cdot) * q)(j) = (p^{*i} * q)(j), \qquad i, j \in \mathbb{N}.$$

For any $n \geq 1$ and $\{i_0, \cdots, i_{n-1} = i, j\} \subset \mathbb{N}$ we have

$$\begin{aligned}
&\mathbf{P}\Big(y(n) = j \Big| y(0) = i_0, y(1) = i_1, \cdots, y(n-1) = i_{n-1}\Big) \\
&\quad = \mathbf{P}\bigg(\sum_{k=1}^{y(n-1)} \xi_{n,k} + \eta_n = j \bigg| y(n-1) = i_{n-1}\bigg) \\
&\quad = \mathbf{P}\bigg(\sum_{k=1}^{i} \xi_{n,k} + \eta_n = j\bigg) = P(i,j).
\end{aligned}$$

Then $\{y(n) : n \geq 0\}$ is a Markov chain with one-step transition matrix $P = (P(i,j) : i, j \in \mathbb{N})$. The random variable $y(n)$ can be thought of as the number of individuals in generation n of a population system with immigration. After one unit time, each of the $y(n)$ individuals splits independently of others into a random number of offspring according to the distribution $\{p(j) : j \in \mathbb{N}\}$ and a random number of immigrants are added to the system according to the distribution $\{q(j) : j \in \mathbb{N}\}$. It is easy to see that

$$\sum_{j=0}^{\infty} P(i,j) z^j = g(z)^i h(z), \qquad |z| \leq 1. \tag{5.2}$$

A Markov chain in $\mathbb{N}$ with one-step transition matrix defined by (5.2) is called a *Galton–Watson branching process with immigration* (GWI-process) or a *Bienaymé–Galton–Watson branching process with immigration* (BGWI-process) with *branching distribution* given by g and *immigration distribution* given by h. When $h \equiv 1$, this reduces to the GW-process defined before. For any $n \geq 1$ the n-step transition matrix of the GWI-process is just the n-fold product $P^n = (P^n(i,j) : i, j \in \mathbb{N})$.

Proposition 5.1 *For any $n \geq 1$ and $i \in \mathbb{N}$ we have*

$$\sum_{j=0}^{\infty} P^n(i,j) z^j = g^{\circ n}(z)^i \prod_{j=1}^{n} h(g^{\circ(j-1)}(z)), \qquad |z| \leq 1. \tag{5.3}$$

Proof. From (5.2) we see (5.3) holds for $n = 1$. Now suppose that (5.3) holds for some $n \geq 1$. We have

$$\begin{aligned}\sum_{j=0}^{\infty} P^{n+1}(i,j)z^j &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P(i,k)P^n(k,j)z^j \\ &= \sum_{k=0}^{\infty} P(i,k)g^{\circ n}(z)^k \prod_{j=1}^{n} h(g^{\circ j-1}(z)) \\ &= g(g^{\circ n}(z))^i h(g^{\circ n}(z)) \prod_{j=1}^{n} h(g^{\circ j-1}(z)) \\ &= g^{\circ(n+1)}(z)^i \prod_{j=1}^{n+1} h(g^{\circ j-1}(z)).\end{aligned}$$

Then (5.3) also holds when n is replaced by $n+1$. That gives the result by induction. □

Suppose that for each integer $k \geq 1$ we have a GWI-process $\{y_k(n) : n \geq 0\}$ with branching distribution given by the probability generating function g_k and immigration distribution given by the probability generating function h_k. Let $z_k(n) = y_k(n)/k$. Then $\{z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k := \{0, 1/k, 2/k, \ldots\}$ and n-step transition probability $P_k^n(x, \mathrm{d}y)$ determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y}\, P_k^n(x,\mathrm{d}y) = g_k^{\circ n}(\mathrm{e}^{-\lambda/k})^{kx} \prod_{j=1}^{n} h_k(g_k^{\circ(j-1)}(\mathrm{e}^{-\lambda/k})). \tag{5.4}$$

Suppose that $\{\gamma_k\}$ is a positive real sequence so that $\gamma_k \to \infty$ increasingly as $k \to \infty$. Let $\lfloor \gamma_k t \rfloor$ denote the integer part of $\gamma_k t$. In view of (5.4), given $z_k(0) = x \in E_k$, the random variable $z_k(\lfloor \gamma_k t \rfloor) = k^{-1}y_k(\lfloor \gamma_k t \rfloor)$ has distribution $P_k^{\lfloor \gamma_k t \rfloor}(x, \cdot)$ on E_k determined by

$$\begin{aligned}&\int_{E_k} \mathrm{e}^{-\lambda y}\, P_k^{\lfloor \gamma_k t \rfloor}(x,\mathrm{d}y) \\ &\quad = g_k^{\circ\lfloor \gamma_k t \rfloor}(\mathrm{e}^{-\lambda/k})^{kx} \prod_{j=1}^{\lfloor \gamma_k t \rfloor} h_k(g_k^{j-1}(\mathrm{e}^{-\lambda/k})) \\ &\quad = \exp\big\{xk \log g_k^{\circ\lfloor \gamma_k t \rfloor}(\mathrm{e}^{-\lambda/k})\big\} \exp\bigg\{\sum_{j=1}^{\lfloor \gamma_k t \rfloor} \log h_k(g_k^{j-1}(\mathrm{e}^{-\lambda/k}))\bigg\} \\ &\quad = \exp\bigg\{-xv_k(t,\lambda) - \int_0^{\gamma_k^{-1}\lfloor \gamma_k t \rfloor} \bar{\psi}_k(v_k(s,\lambda))\mathrm{d}s\bigg\},\end{aligned} \tag{5.5}$$

where $v_k(t,\lambda)$ is given by (2.8) and

$$\bar{\psi}_k(z) = -\gamma_k \log h_k(\mathrm{e}^{-z/k}). \tag{5.6}$$

For any $z \geq 0$ we have

$$\bar{\psi}_k(z) = -\gamma_k \log\big[1 - \gamma_k^{-1}\tilde{\psi}_k(z)\big], \tag{5.7}$$

where

$$\tilde{\psi}_k(z) = \gamma_k[1 - h_k(\mathrm{e}^{-z/k})]. \tag{5.8}$$

Lemma 5.2 *Suppose that the sequence $\{\tilde{\psi}_k\}$ is uniformly bounded on each bounded interval. Then we have $\lim_{k\to\infty} |\bar{\psi}_k(z) - \tilde{\psi}_k(z)| = 0$ uniformly on each bounded interval.*

Proof. This is immediate by the relation (5.7). □

Condition 5.3 *There is a function ψ on $[0,\infty)$ such that $\tilde{\psi}_k(z) \to \psi(z)$ uniformly on $[0,a]$ for every $a \geq 0$ as $k \to \infty$.*

Proposition 5.4 *Suppose that Condition 5.3 is satisfied. Then the limit function ψ has representation*

$$\psi(z) = \beta z + \int_{(0,\infty)} \big(1 - \mathrm{e}^{-zu}\big)\nu(\mathrm{d}u), \qquad z \geq 0, \tag{5.9}$$

where $\beta \geq 0$ is a constant and $\nu(\mathrm{d}u)$ is a σ-finite measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge u)\nu(\mathrm{d}u) < \infty.$$

Proof. It is well-known that ψ has representation (5.9) if and only if $\mathrm{e}^{-\psi} = L_\mu$ is the Laplace transform of an infinitely divisible probability distribution μ on $[0,\infty)$; see, e.g., Theorem 1.39 in Li (2011, p.20). In view of (5.8), the function $\tilde{\psi}_k$ can be represented by a special form (5.9), so $\mathrm{e}^{-\tilde{\psi}_k} = L_{\mu_k}$ is the Laplace transform of an infinitely divisible distribution μ_k on $[0,\infty)$. By Lemma 5.2 and Condition 5.3 we have $\tilde{\psi}_k(z) \to \psi(z)$ uniformly on $[0,a]$ for every $a \geq 0$ as $k \to \infty$. By Theorem 1.2 there is a probability distribution μ on $[0,\infty)$ so that $\mu = \lim_{k\to\infty} \mu_k$ weakly and $\mathrm{e}^{-\psi} = L_\mu$. Clearly μ is also infinitely divisible, so ψ has representation (5.9). □

Proposition 5.5 *For any function ψ with representation* (5.9) *there is a sequence $\{\tilde{\psi}_k\}$ in the form of* (5.6) *satisfying Condition 5.3.*

Proof. This is similar to the proof of Proposition 2.5 and is left to the reader as an exercise. □

Theorem 5.6 *Suppose that ϕ and ψ are given by* (2.13) *and* (5.9)*, respectively. For any $\lambda \geq 0$ let $t \mapsto v_t(\lambda)$ be the unique positive solution to* (2.15)*. Then there is a Feller transition semigroup $(P_t)_{t\geq 0}$ on $[0,\infty)$ defined by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\, P_t(x,\mathrm{d}y) = \exp\bigg\{ - x v_t(\lambda) - \int_0^t \psi(v_s(\lambda))\mathrm{d}s \bigg\}. \tag{5.10}$$

Proof. This follows by arguments similar to those in Section 2. □

If a Markov process in $[0, \infty)$ has transition semigroup $(P_t)_{t\geq 0}$ defined by (5.10), we call it a *continuous-state branching process with immigration* (CBI-process) with *branching mechanism* ϕ and *immigration mechanism* ψ. In particular, if

$$\int_{(0,\infty)} u\nu(\mathrm{d}u) < \infty, \tag{5.11}$$

one can differentiate both sides of (5.10) and use (3.4) to see

$$\int_{[0,\infty)} yP_t(x, \mathrm{d}y) = x\,\mathrm{e}^{-bt} + \psi'(0)\int_0^t \mathrm{e}^{-bs}\,\mathrm{d}s, \tag{5.12}$$

where

$$\psi'(0) = \beta + \int_{(0,\infty)} u\nu(\mathrm{d}u). \tag{5.13}$$

Proposition 5.7 *Suppose that* $\{(y_1(t), \mathscr{G}_t^1) : t \geq 0\}$ *and* $\{(y_2(t), \mathscr{G}_t^2) : t \geq 0\}$ *are two independent CBI-processes with branching mechanism* ϕ *and immigration mechanisms* ψ_1 *and* ψ_2*, respectively. Let* $y(t) = y_1(t) + y_2(t)$ *and* $\mathscr{G}_t = \sigma(\mathscr{G}_t^1 \cup \mathscr{G}_t^2)$*. Then* $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ *is a CBI-processes with branching mechanism* ϕ *and immigration mechanism* $\psi = \psi_1 + \psi_2$.

Proof. Let $t \geq r \geq 0$ and for $i = 1, 2$ let F_i be a bounded positive $\mathscr{G}_r^i$-measurable random variable. For any $\lambda \geq 0$ we have

$$\begin{aligned}
\mathbf{P}\big[F_1F_2\,\mathrm{e}^{-\lambda y(t)}\big] &= \mathbf{P}\big[F_1\,\mathrm{e}^{-\lambda y_1(t)}\big]\mathbf{P}\big[F_2\,\mathrm{e}^{-\lambda y_2(t)}\big] \\
&= \mathbf{P}\bigg[F_1\exp\bigg\{-y_1(r)v_{t-r}(\lambda) - \int_0^{t-r}\psi_1(v_s(\lambda))\mathrm{d}s\bigg\}\bigg] \\
&\quad\cdot\mathbf{P}\bigg[F_2\exp\bigg\{-y_2(r)v_{t-r}(\lambda) - \int_0^{t-r}\psi_2(v_s(\lambda))\mathrm{d}s\bigg\}\bigg] \\
&= \mathbf{P}\bigg[F_1F_2\exp\bigg\{-y(r)v_{t-r}(\lambda) - \int_0^{t-r}\psi(v_s(\lambda))\mathrm{d}s\bigg\}\bigg].
\end{aligned}$$

As in the proof of Proposition 2.12, one can see $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ is a CBI-processes with branching mechanism ϕ and immigration mechanism ψ. □

The next theorem follows by a modification of the proof of Theorem 2.13.

Theorem 5.8 *Suppose that Conditions 2.3 and 5.3 are satisfied. Let* $\{y(t) : t \geq 0\}$ *be a CBI-process with transition semigroup* $(P_t)_{t\geq 0}$ *defined by* (5.10)*. For* $k \geq 1$ *let* $\{z_k(n) : n \geq 0\}$ *be a Markov chain with state space* $E_k := \{0, k^{-1}, 2k^{-1}, \ldots\}$ *and* n*-step transition probability* $P_k^n(x, \mathrm{d}y)$ *determined by* (5.4)*. If* $z_k(0)$ *converges to* $y(0)$ *in distribution, then* $\{z_k(\lfloor\gamma_k t\rfloor) : t \geq 0\}$ *converges to* $\{y(t) : t \geq 0\}$ *in distribution on* $D[0, \infty)$.

The convergence of rescaled GWI-processes to CBI-processes have been studied by many authors; see, e.g., Aliev (1985), Kawazu and Watanabe (1971) and Li (2006, 2011).

Example 5.1 The transition semigroup $(Q_t^b)_{t\geq 0}$ defined by (3.14) corresponds to a CBI-process with branching mechanism ϕ and immigration mechanism ϕ_0'.

Example 5.2 Suppose that $c > 0$, $0 < \alpha \leq 1$ and b are constants and let $\phi(z) = cz^{1+\alpha} + bz$. Let $v_t(\lambda)$ and $q_\alpha^0(t)$ be defined as in Example 2.3. Let $\beta \geq 0$ and let $\psi(z) = \beta z^\alpha$. We can use (5.10) to define the transition semigroup $(P_t)_{t\geq 0}$. It is easy to show that

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\, P_t(x, \mathrm{d}y) = \frac{1}{\left[1 + cq_\alpha^b(t)\lambda^\alpha\right]^{\beta/c\alpha}}\, \mathrm{e}^{-xv_t(\lambda)}, \qquad \lambda \geq 0.$$

6 Structures of sample paths

In this section, we give some reconstructions of the type of Pitman and Yor (1982) for the CB- and CBI-processes, which reveal the structures of their sample paths. Let $(Q_t)_{t\geq 0}$ be the transition semigroup of the CB-process with branching mechanism ϕ given by (2.13). Let $(Q_t^\circ)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ on $(0,\infty)$. Let $D[0,\infty)$ denote the space of positive càdlàg paths on $[0,\infty)$. On this space, we define the σ-algebras $\mathscr{A} = \sigma(\{w(s) : 0 \leq s < \infty\})$ and $\mathscr{A}_t = \sigma(\{w(s) : 0 \leq s \leq t\})$ for $t \geq 0$. For any $w \in D[0,\infty)$ let $\tau_0(w) = \inf\{s > 0 : w(s) \text{ or } w(s-) = 0\}$. Let $D_0[0,\infty)$ be the set of paths $w \in D[0,\infty)$ such that $w(t) = 0$ for $t \geq \tau_0(w)$. Let $D_1[0,\infty)$ be the set of paths $w \in D_0[0,\infty)$ satisfying $w(0) = 0$. Then both $D_0[0,\infty)$ and $D_1[0,\infty)$ are $\mathscr{A}$-measurable subsets of $D[0,\infty)$.

Theorem 6.1 *Suppose that $\phi'(\infty) = \infty$ and let $(l_t)_{t>0}$ be the entrance law for $(Q_t^\circ)_{t\geq 0}$ determined by* (3.16). *Then there is a unique σ-finite measure $\mathbf{N}_0$ on $(D[0,\infty), \mathscr{A})$ supported by $D_1[0,\infty)$ such that, for $0 < t_1 < t_2 < \cdots < t_n$ and $x_1, x_2, \ldots, x_n \in (0,\infty)$,*

$$\begin{aligned}&\mathbf{N}_0(w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n)\\ &\quad= l_{t_1}(\mathrm{d}x_1) Q^\circ_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n).\end{aligned} \tag{6.1}$$

Proof. Recall that $(Q_t^b)_{t\geq 0}$ is the transition semigroup on $[0,\infty)$ given by (3.14). Let $X = (D[0,\infty), \mathscr{A}, \mathscr{A}_t, w(t), \mathbf{Q}_x)$ be the canonical realization of $(Q_t)_{t\geq 0}$ and $Y = (D[0,\infty), \mathscr{A}, \mathscr{A}_t, w(t), \mathbf{Q}_x^b)$ the canonical realization of $(Q_t^b)_{t\geq 0}$. For any $T > 0$ there is a probability measure $\mathbf{P}_0^{b,T}$ on $(D[0,\infty), \mathscr{A})$ supported by $D_1[0,\infty)$ so that

$$\begin{aligned}&\mathbf{P}_0^{b,T}[F(\{w(s) : s \geq 0\}) G(\{w(T+s) : s \geq 0\})]\\ &\quad= \mathbf{Q}_0^b[F(\{w(s) : s \geq 0\}) \mathbf{Q}_{w(T)} G(\{w(s) : s \geq 0\})],\end{aligned}$$

where F is a bounded $\mathscr{A}_T$-measurable function and G is a bounded $\mathscr{A}$-measurable function. The formula above means that under $\mathbf{P}_0^{b,T}$ the random path $\{w(s) : s \in [0,T]\}$ is a Markov process with initial state $w(0) = 0$ and transition semigroup $(Q_t^b)_{t\geq 0}$ and $\{w(s) : s \in [T,\infty)\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. Then Corollary 3.11 implies $\mathbf{P}_0^{b,T}(w(s) = 0) = Q_s^b(0, \{0\}) = 0$ for every $0 < s \leq T$. Let $\mathbf{N}_0^{b,T}(\mathrm{d}w) = \mathrm{e}^{-bT} w(T)^{-1} 1_{\{w(T)>0\}} \mathbf{P}_0^{b,T}(\mathrm{d}w)$. We have

$$\begin{aligned}&\mathbf{N}_0^{b,T}(w(s_1) \in \mathrm{d}x_1, w(s_2) \in \mathrm{d}x_2, \ldots, w(s_m) \in \mathrm{d}x_m, w(T) \in \mathrm{d}z,\\ &\qquad\qquad w(t_1) \in \mathrm{d}y_1, w(t_2) \in \mathrm{d}y_2, \ldots, w(t_n) \in \mathrm{d}y_n)\\ &\quad= Q^b_{s_1}(0, \mathrm{d}x_1) Q^b_{s_2-s_1}(x_1, \mathrm{d}x_2) \cdots Q^b_{s_m-s_{m-1}}(x_{m-1}, \mathrm{d}x_m) Q^b_{T-s_m}(x_m, \mathrm{d}z)\\ &\qquad \mathrm{e}^{-bT} z^{-1} Q_{t_1-T}(z, \mathrm{d}y_1) Q_{t_2-t_1}(y_1, \mathrm{d}y_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, \mathrm{d}y_n)\\ &\quad= Q^b_{s_1}(0, \mathrm{d}x_1) x_1^{-1} x_1 \cdots Q^b_{s_m-s_{m-1}}(x_{m-1}, \mathrm{d}x_m) x_m^{-1} x_m Q^b_{T-s_m}(x_m, \mathrm{d}z)\\ &\qquad \mathrm{e}^{-bT} z^{-1} Q_{t_1-T}(z, \mathrm{d}y_1) Q_{t_2-t_1}(y_1, \mathrm{d}y_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, \mathrm{d}y_n)\\ &\quad= l_{s_1}(\mathrm{d}x_1) Q^\circ_{s_2-s_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{s_m-s_{m-1}}(x_{m-1}, \mathrm{d}x_m) Q^\circ_{T-s_m}(x_m, \mathrm{d}z)\\ &\qquad Q_{t_1-T}(z, \mathrm{d}y_1) Q_{t_2-t_1}(y_1, \mathrm{d}y_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, \mathrm{d}y_n),\end{aligned} \tag{6.2}$$

where $0 < s_1 < \cdots < s_m < T < t_1 < \cdots < t_n$ and $x_1, \ldots, x_m, z, y_1, \ldots, y_n \in (0, \infty)$. Then for any $T_1 \geq T_2 > 0$ the two measures $\mathbf{N}_0^{b,T_1}$ and $\mathbf{N}_0^{b,T_2}$ coincide on $\{w \in D_1[0,\infty) : \tau_0(w) > T_1\}$, so the increasing limit $\mathbf{N}_0 := \lim_{T\to 0} \mathbf{N}_0^{b,T}$ exists and defines a σ-finite measure supported by $D_1[0,\infty)$. From (6.2) we get (6.1). The uniqueness of the measure $\mathbf{N}_0$ satisfying (6.1) follows by the measure extension theorem. □

The elements of $D_1[0,\infty)$ are called *excursions* and the measure $\mathbf{N}_0$ is referred to as the *excursion law* for the CB-process. In view of (3.17) and (6.1), we have formally, for $0 < t_1 < t_2 < \cdots < t_n$ and $x_1, x_2, \ldots, x_n \in (0, \infty)$,

$$\begin{aligned}
&\mathbf{N}_0(w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n) \\
&\quad = \lim_{x\to 0} x^{-1} Q^\circ_{t_1}(x, \mathrm{d}x_1) Q^\circ_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \\
&\quad = \lim_{x\to 0} x^{-1} \mathbf{Q}_x(w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n),
\end{aligned} \tag{6.3}$$

which explains why $\mathbf{N}_0$ is supported by $D_1[0,\infty)$.

From (6.1) we see that the excursion law is Markovian, namely, the path $\{w(t) : t > 0\}$ behaves under this law as a Markov process with transition semigroup $(Q^\circ_t)_{t\geq 0}$. Based on the excursion law, a reconstruction of the CB-process is given in the following theorem:

Theorem 6.2 *Suppose that $\phi'(\infty) = \infty$. Let $z \geq 0$ and let $N_z = \sum_{i=1}^\infty \delta_{w_i}$ be a Poisson random measure on $D[0,\infty)$ with intensity $z\mathbf{N}_0(\mathrm{d}w)$. Let $X_0 = z$ and for $t > 0$ let*

$$X^z_t = \int_{D[0,\infty)} w(t) N_z(\mathrm{d}w) = \sum_{i=1}^\infty w_i(t). \tag{6.4}$$

For $t \geq 0$ let $\mathscr{G}^z_t$ be the σ-algebra generated by the collection of random variables $\{N_z(A) : A \in \mathscr{A}_t\}$. Then $\{(X^z_t, \mathscr{G}^z_t) : t \geq 0\}$ is a CB-process with branching mechanism ϕ.

Proof. It is easy to see that $\{X^z_t : t \geq 0\}$ is adapted relative to the filtration $\{\mathscr{G}^z_t : t \geq 0\}$. We claim that the random variable X^z_t has distribution $Q_t(z, \cdot)$ on $[0,\infty)$. Indeed, for $t = 0$ this is immediate. For any $t > 0$ and $\lambda \geq 0$ we have

$$\begin{aligned}
\mathbf{P}\big[\exp\{-\lambda X^z_t\}\big] &= \exp\bigg\{-z\int_{D[0,\infty)} (1 - \mathrm{e}^{-\lambda w(t)}) \mathbf{N}_0(\mathrm{d}w)\bigg\} \\
&= \exp\bigg\{-z\int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u}) l_t(\mathrm{d}u)\bigg\} = \exp\{-z v_t(\lambda)\}.
\end{aligned}$$

By the Markov property (6.1), for any $t \geq r > 0$ and any bounded $\mathscr{A}_r$-measurable function H on $D[0,\infty)$ we have

$$\begin{aligned}
&\int_{D[0,\infty)} H(w)(1 - \mathrm{e}^{-\lambda w(t)}) \mathbf{N}_0(\mathrm{d}w) \\
&\qquad = \int_{D[0,\infty)} H(w)(1 - \mathrm{e}^{-v_{t-r}(\lambda) w(r)}) \mathbf{N}_0(\mathrm{d}w).
\end{aligned}$$

It follows that, for any bounded positive $\mathscr{A}_r$-measurable function F on $D[0,\infty)$,

$$\begin{aligned}
&\mathbf{P}\Big[\exp\Big\{-\int_{D[0,\infty)} F(w)N_z(\mathrm{d}w)\Big\}\cdot\exp\Big\{-\lambda X_t^z\Big\}\Big] \\
&\quad= \mathbf{P}\Big[\exp\Big\{-\int_{D[0,\infty)} \big[F(w)+\lambda w(t)\big]N_z(\mathrm{d}w)\Big\}\Big] \\
&\quad= \exp\Big\{-z\int_{D[0,\infty)} \big(1-\mathrm{e}^{-F(w)-\lambda w(t)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\quad= \exp\Big\{-z\int_{D[0,\infty)} \big(1-\mathrm{e}^{-F(w)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\qquad\qquad\cdot\exp\Big\{-z\int_{D[0,\infty)} \mathrm{e}^{-F(w)}\big(1-\mathrm{e}^{-\lambda w(t)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\quad= \exp\Big\{-z\int_{D[0,\infty)} \big(1-\mathrm{e}^{-F(w)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\qquad\qquad\cdot\exp\Big\{-z\int_{D[0,\infty)} \mathrm{e}^{-F(w)}\big(1-\mathrm{e}^{-w(r)v_{t-r}(\lambda)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\quad= \exp\Big\{-z\int_{D[0,\infty)} \big(1-\mathrm{e}^{-F(w)}\,\mathrm{e}^{-w(r)v_{t-r}(\lambda)}\big)\mathbf{N}_0(\mathrm{d}w)\Big\} \\
&\quad= \mathbf{P}\Big[\exp\Big\{-\int_{D[0,\infty)} \big[F(w)+v_{t-r}(\lambda)w(r)\big]N_z(\mathrm{d}w)\Big\}\Big] \\
&\quad= \mathbf{P}\Big[\exp\Big\{-\int_{D[0,\infty)} F(w)N_z(\mathrm{d}w)\Big\}\cdot\exp\Big\{-v_{t-r}(\lambda)X_r^z\Big\}\Big].
\end{aligned}$$

Clearly, the σ-algebra $\mathscr{G}_r^z$ is generated by the collection of random variables

$$\exp\Big\{-\int_{D[0,\infty)} F(w)N_z(\mathrm{d}w)\Big\},$$

where F runs over all bounded positive $\mathscr{A}_r$-measurable functions on $D[0,\infty)$. Then $\{(X_t^z,\mathscr{G}_t^z): t\geq 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. $\square$

The above theorem gives a description of the structures of the population represented by the CB-process. From (6.4) we see that the population at any time $t>0$ consists of at most countably many families, which evolve as the excursions $\{w_i: i=1,2,\cdots\}$ selected by the Poisson random measure $N_z(\mathrm{d}w)$. Unfortunately, this reconstruction is only available under the condition $\phi'(\infty)=\infty$. To give reconstructions of the CB- and CBI-processes when this condition is not necessarily satisfied, we need to consider some inhomogeneous immigration structures and more general Markovian measures on the path space. As a consequence of Theorem 6.1 we obtain the following:

Proposition 6.3 *Let $\beta\geq 0$ be a constant and ν a σ-finite measure on $(0,\infty)$ such that $\int_{(0,\infty)} u\nu(\mathrm{d}u)<\infty$. If $\beta>0$, assume in addition $\phi'(\infty)=\infty$. Then we can define a σ-finite measure $\mathbf{N}$ on $(D[0,\infty),\mathscr{A})$ by*

$$\mathbf{N}(\mathrm{d}w)=\beta\mathbf{N}_0(\mathrm{d}w)+\int_{(0,\infty)}\nu(\mathrm{d}x)\mathbf{Q}_x(\mathrm{d}w),\qquad w\in D[0,\infty). \tag{6.5}$$

Moreover, for $0 < t_1 < t_2 < \cdots < t_n$ *and* $x_1, x_2, \ldots, x_n \in (0, \infty)$*, we have*

$$\begin{aligned}\mathbf{N}(w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n)& \\ = H_{t_1}(\mathrm{d}x_1) Q^\circ_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n),&\end{aligned} \tag{6.6}$$

where

$$H_t(\mathrm{d}y) = \beta l_t(\mathrm{d}y) + \int_{(0,\infty)} \nu(\mathrm{d}x) Q^\circ_t(x, \mathrm{d}y), \qquad y > 0.$$

Clearly, the measure $\mathbf{N}$ defined by (6.5) is actually supported by $D_0[0, \infty)$. Under this law, the path $\{w(t) : t > 0\}$ behaves as a Markov process with transition semigroup $(Q^\circ_t)_{t\geq 0}$ and one-dimensional distributions $(H_t)_{t>0}$.

Theorem 6.4 *Suppose that the conditions of Proposition 6.3 are satisfied and let* $\mathbf{N}$ *be defined by* (6.5)*. Let* ρ *be a Borel measure on* $(0, \infty)$ *such that* $\rho(0, t] < \infty$ *for each* $0 < t < \infty$*. Suppose that* $N = \sum_{i=1}^\infty \delta_{(s_i, w_i)}$ *is a Poisson random measure on* $(0, \infty) \times D[0, \infty)$ *with intensity* $\rho(\mathrm{d}s)\mathbf{N}(\mathrm{d}w)$*. For* $t \geq 0$ *let*

$$Y_t = \int_{(0,t]} \int_{D[0,\infty)} w(t - s) N(\mathrm{d}s, \mathrm{d}w) = \sum_{0<s_i\leq t} w_i(t - s_i) \tag{6.7}$$

and let $\mathscr{G}_t$ *be the* σ*-algebra generated by the random variables* $\{N((0, u] \times A) : A \in \mathscr{A}_{t-u}, 0 \leq u \leq t\}$*. Then* $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ *is a Markov process with inhomogeneous transition semigroup* $(P_{r,t})_{t\geq r\geq 0}$ *given by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} P_{r,t}(x, \mathrm{d}y) = \exp\bigg\{ - x v_{t-r}(\lambda) - \int_{(r,t]} \psi(v_{t-s}(\lambda)) \rho(\mathrm{d}s) \bigg\}, \tag{6.8}$$

where the function ψ *is given by* (5.9)*.*

Proof. From (6.7) we see that $\{Y_t : t \geq 0\}$ is adapted to the filtration $\{\mathscr{G}_t : t \geq 0\}$. Let $t \geq r \geq u \geq 0$ and let F be a bounded positive function on $D[0, \infty)$ measurable relative to $\mathscr{A}_{r-u}$. For $\lambda \geq 0$, we can use the Markov property (6.6) to see

$$\begin{aligned}
&\mathbf{P}\bigg[\exp\bigg\{ - \int_{(0,u]} \int_{D[0,\infty)} F(w) N(\mathrm{d}s, \mathrm{d}w) - \lambda Y_t \bigg\}\bigg] \\
&\quad = \mathbf{P}\bigg[\exp\bigg\{ - \int_{(0,t]} \int_{D[0,\infty)} \big[F(w)1_{\{s\leq u\}} + \lambda w(t - s)\big] N(\mathrm{d}s, \mathrm{d}w) \bigg\}\bigg] \\
&\quad = \exp\bigg\{ - \int_{(0,t]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-F(w)1_{\{s\leq u\}}} \mathrm{e}^{-\lambda w(t-s)}\big) \mathbf{N}(\mathrm{d}w) \bigg\} \\
&\quad = \exp\bigg\{ - \int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-F(w)1_{\{s\leq u\}}} \mathrm{e}^{-w(t-s)}\big) \mathbf{N}(\mathrm{d}w) \bigg\} \\
&\qquad \cdot \exp\bigg\{ - \int_{(r,t]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-\lambda w(t-s)}\big) \mathbf{N}(\mathrm{d}w) \bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Big\{ -\int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-F(w)1_{\{s\le u\}}}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&\quad\cdot \exp\Big\{ -\int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \mathrm{e}^{-F(w)1_{\{s\le u\}}}\big(1 - \mathrm{e}^{-w(t-s)}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&\quad\cdot \exp\Big\{ -\int_{(r,t]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-\lambda w(t-s)}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&= \exp\Big\{ -\int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-F(w)1_{\{s\le u\}}}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&\quad\cdot \exp\Big\{ -\int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \mathrm{e}^{-F(w)1_{\{s\le u\}}}\big(1 - \mathrm{e}^{-v_{t-r}(\lambda)w(r-s)}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&\quad\cdot \exp\Big\{ -\int_{(r,t]} \rho(\mathrm{d}s) \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\lambda y}\big)H_{t-s}(\mathrm{d}w)\Big\} \\
&= \exp\Big\{ -\int_{(0,r]} \rho(\mathrm{d}s) \int_{D[0,\infty)} \big(1 - \mathrm{e}^{-F(w)1_{\{s\le u\}}}\,\mathrm{e}^{-v_{t-r}(\lambda)w(r-s)}\big)\mathbf{N}(\mathrm{d}w)\Big\} \\
&\quad\cdot \exp\Big\{ -\beta\int_{(r,t]} \rho(\mathrm{d}s) \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda y})l_{t-s}(\mathrm{d}y) \\
&\qquad\qquad - \int_{(r,t]} \rho(\mathrm{d}s) \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda y})\nu Q^\circ_{t-s}(\mathrm{d}y)\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{ -\int_{(0,r]} \int_{D[0,\infty)} \big[F(w)1_{\{s\le u\}} + v_{t-r}(\lambda)w(r-s)\big]N(\mathrm{d}s,\mathrm{d}w)\Big\}\Big] \\
&\quad\cdot \exp\Big\{ -\int_{(r,t]} \Big[\beta v_{t-s}(\lambda) + \int_{(0,\infty)} (1 - \mathrm{e}^{-yv_{t-s}(\lambda)})\nu(\mathrm{d}y)\Big]\rho(\mathrm{d}s)\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{ -\int_{(0,u]} \int_{D[0,\infty)} F(w)N(\mathrm{d}s,\mathrm{d}w)\Big\} \\
&\qquad\qquad \cdot \exp\Big\{ -v_{t-r}(\lambda)Y_r - \int_{(r,t]} \psi(v_{t-s}(\lambda))\rho(\mathrm{d}s)\Big\}\Big].
\end{aligned}
$$

Then $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ is a Markov process in $[0,\infty)$ with inhomogeneous transition semigroup $(P_{r,t})_{t\geq r\geq 0}$ given by (6.8). □

Corollary 6.5 *Suppose that $\phi'(\infty) = \infty$. Let $\beta > 0$ and let $N_\beta = \sum_{i=1}^\infty \delta_{(s_i,w_i)}$ be a Poisson random measure on $(0,\infty)\times D[0,\infty)$ with intensity $\beta\mathrm{d}s\mathbf{N}_0(\mathrm{d}w)$. For $t\geq 0$ let*

$$
Y_t^\beta = \int_{(0,t]} \int_{D[0,\infty)} w(t-s)N_\beta(\mathrm{d}s,\mathrm{d}w) = \sum_{0<s_i\le t} w_i(t-s_i)
$$

and let $\mathscr{G}_t^\beta$ be the σ-algebra generated by the random variables $\{N_\beta((0,u]\times A) : A\in \mathscr{A}_{t-u}, 0\le u\le t\}$. Then $\{(Y_t^\beta, \mathscr{G}_t^\beta) : t\geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanism ψ_β defined by $\psi_\beta(\lambda) = \beta\lambda$, $\lambda\geq 0$.

Corollary 6.6 *Let ν be a σ-finite measure on $(0,\infty)$ such that $\int_{(0,\infty)} u\nu(\mathrm{d}u) < \infty$. Let $N_\nu = \sum_{i=1}^\infty \delta_{(s_i,w_i)}$ be a Poisson random measure on $(0,\infty)\times D[0,\infty)$ with intensity*

$\mathrm{d}s\mathbf{N}_\nu(\mathrm{d}w)$, *where*

$$\mathbf{N}_\nu(\mathrm{d}w) = \int_{(0,\infty)} \nu(\mathrm{d}x)\mathbf{Q}_x(\mathrm{d}w). \tag{6.9}$$

For $t \geq 0$ *let*

$$Y_t^\nu = \int_{(0,t]}\int_{D[0,\infty)} w(t-s)N_\nu(\mathrm{d}s,\mathrm{d}w) = \sum_{0<s_i\leq t} w_i(t-s_i)$$

and let $\mathscr{G}_t^\nu$ *be the* σ*-algebra generated by the random variables* $\{N_\nu((0,u]\times A) : A \in \mathscr{A}_{t-u}, 0 \leq u \leq t\}$. *Then* $\{(Y_t^\nu, \mathscr{G}_t^\nu) : t \geq 0\}$ *is a CBI-process with branching mechanism* ϕ *and immigration mechanism* ψ_ν *defined by*

$$\psi_\nu(\lambda) = \int_{(0,\infty)} (1-\mathrm{e}^{-u\lambda})\nu(\mathrm{d}u), \qquad \lambda \geq 0. \tag{6.10}$$

The transition semigroup $(P_{r,t})_{t\geq r\geq 0}$ defined by (6.8) is a generalization of the one given by (5.10); see also Li (1996, 2003) and Li (2011, p.224). A Markov process with transition semigroup $(P_{r,t})_{t\geq r\geq 0}$ is naturally called a CBI-process with *inhomogeneous immigration rate* ρ. In view of (6.7), the population $\{Y_t : t \geq 0\}$ consists of a countable families of immigrants, whose immigration times $\{s_i : i = 1, 2, \cdots\}$ and evolution trajectories $\{w_i : i = 1, 2, \cdots\}$ are both selected by the Poisson random measure $N(\mathrm{d}s, \mathrm{d}w)$. The processes constructed in Corollaries 6.5 and 6.6 can be interpreted similarly.

Theorem 6.7 *Suppose that* $\delta := \phi'(\infty) < \infty$. *Let* $z > 0$ *and let* $N_z = \sum_{i=1}^\infty \delta_{(s_i,w_i)}$ *be a Poisson random measure on* $(0,\infty)\times D[0,\infty)$ *with intensity* $z\,\mathrm{e}^{-\delta s}\,\mathrm{d}s\mathbf{N}_m(\mathrm{d}w)$, *where* $\mathbf{N}_m$ *is defined by* (6.9) *with* $\nu = m$. *For* $t \geq 0$ *let*

$$X_t^z = z\,\mathrm{e}^{-\delta t} + \int_{(0,t]}\int_{D[0,\infty)} w(t-s)N_z(\mathrm{d}s,\mathrm{d}w) \tag{6.11}$$

and let $\mathscr{G}_t^z$ *be the* σ*-algebra generated by the random variables* $\{N_z((0,u]\times A) : A \in \mathscr{A}_{t-u}, 0 \leq u \leq t\}$. *Then* $\{(X_t^z, \mathscr{G}_t^z) : t \geq 0\}$ *is a CB-process with branching mechanism* ϕ.

Proof. Let $Z_t = X_t^z - z\,\mathrm{e}^{-\delta t}$ denote the second term on the right-hand side of (6.11). By Theorem 6.4 we infer that $\{(Z_t, \mathscr{G}_t^z) : t \geq 0\}$ is a Markov process with inhomogeneous transition semigroup $(P_{r,t}^z)_{t\geq r\geq 0}$ given by

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\,P_{r,t}^z(x,\mathrm{d}y) = \exp\bigg\{-xv_{t-r}(\lambda) - z\int_r^t \psi_m(v_{t-s}(\lambda))\,\mathrm{e}^{-\delta s}\,\mathrm{d}s\bigg\},$$

where ψ_m is defined by (6.10) with $\nu = m$. Let $t \geq r \geq u \geq 0$ and let F be a bounded positive $\mathscr{A}_{r-u}$-measurable function on $D[0,\infty)$. For $\lambda \geq 0$ we have

$$\mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - \lambda X_t^z\bigg\}\bigg]$$

$$\begin{aligned}
&= \mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - \lambda z\,\mathrm{e}^{-\delta t} - \lambda Z_t\bigg\}\bigg] \\
&= \mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - \lambda z\,\mathrm{e}^{-\delta t}\bigg\}\bigg] \\
&\qquad\cdot\exp\bigg\{-v_{t-r}(\lambda)Z_r - z\int_r^t \psi_m(v_{t-s}(\lambda))\,\mathrm{e}^{-\delta s}\,\mathrm{d}s\bigg\} \\
&= \mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - \lambda z\,\mathrm{e}^{-\delta t}\bigg\}\bigg] \\
&\qquad\cdot\exp\bigg\{-v_{t-r}(\lambda)Z_r - \mathrm{e}^{-\delta r} z\int_0^{t-r} \psi_m(v_{t-r-s}(\lambda))\,\mathrm{e}^{-\delta s}\,\mathrm{d}s\bigg\} \\
&= \mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - v_{t-r}(\lambda)Z_r - z\,\mathrm{e}^{-\delta r}\,v_{t-r}(\lambda)\bigg\}\bigg] \\
&= \mathbf{P}\bigg[\exp\bigg\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_z(\mathrm{d}s,\mathrm{d}w) - v_{t-r}(\lambda)X_r^z\bigg\}\bigg],
\end{aligned}$$

where we have used (3.20). Then $\{(X_t^z, \mathscr{G}_t^z) : t \geq 0\}$ is a CB-process with transition semigroup $(Q_t)_{t\geq 0}$ defined by (2.19). □

Theorem 6.8 *Suppose that $\delta := \phi'(\infty) < \infty$. Let $\beta > 0$ and let $N_\beta = \sum_{i=1}^\infty \delta_{(s_i,w_i)}$ be a Poisson random measure on $(0,\infty)\times D[0,\infty)$ with intensity $\beta\delta^{-1}(1-\mathrm{e}^{-\delta s})\,\mathrm{d}s\mathbf{N}_m(\mathrm{d}w)$, where $\mathbf{N}_m$ is defined by (6.9) with $\nu = m$. For $t \geq 0$ let*

$$Y_t^\beta = \beta\delta^{-1}(1-\mathrm{e}^{-\delta t}) + \int_{(0,t]}\int_{D[0,\infty)} w(t-s)N_\beta(\mathrm{d}s,\mathrm{d}w) \tag{6.12}$$

and let $\mathscr{G}_t^\beta$ be the σ-algebra generated by the random variables $\{N_\beta((0,u]\times A) : A \in \mathscr{A}_{t-u}, 0 \leq u \leq t\}$. Then $\{(X_t^\beta, \mathscr{G}_t^\beta) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ and immigration mechanism ψ_β defined by $\psi_\beta(\lambda) = \beta\lambda$, $\lambda \geq 0$.

Proof. Let Z_t denote the second term on the right-hand side of (6.12). By Theorem 6.4, the process $\{(Z_t, \mathscr{G}_t^\beta) : t \geq 0\}$ is a Markov process with inhomogeneous transition semigroup $(P_{r,t}^\beta)_{t\geq r\geq 0}$ given by

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\,P_{r,t}^\beta(x,\mathrm{d}y) = \exp\bigg\{-xv_{t-r}(\lambda) - \beta\delta^{-1}\int_r^t \psi_m(v_{t-s}(\lambda))(1-\mathrm{e}^{-\delta s})\mathrm{d}s\bigg\},$$

where ψ_m is defined by (6.10) with $\nu = m$. For $t \geq 0$ and $\lambda \geq 0$, we can use Theorem 3.15 to see

$$\begin{aligned}
\int_0^t v_s(\lambda)\mathrm{d}s &= \lambda\int_0^t \mathrm{e}^{-\delta s}\,\mathrm{d}s + \int_0^t \mathrm{d}s\int_0^s \mathrm{e}^{-\delta u}\,\psi_m(v_{s-u}(\lambda))\mathrm{d}u \\
&= \lambda\int_0^t \mathrm{e}^{-\delta s}\,\mathrm{d}s + \int_0^t \mathrm{d}s\int_0^{t-s} \mathrm{e}^{-\delta u}\,\psi_m(v_{t-s-u}(\lambda))\mathrm{d}u \\
&= \lambda\delta^{-1}(1-\mathrm{e}^{-\delta t}) + \int_0^t \mathrm{d}s\int_s^t \mathrm{e}^{-\delta(u-s)}\,\psi_m(v_{t-u}(\lambda))\mathrm{d}u
\end{aligned}$$

$$= \lambda\delta^{-1}(1-\mathrm{e}^{-\delta t}) + \int_0^t \mathrm{d}u \int_0^u \mathrm{e}^{-\delta(u-s)}\,\psi_m(v_{t-u}(\lambda))\mathrm{d}s$$
$$= \lambda\delta^{-1}(1-\mathrm{e}^{-\delta t}) + \delta^{-1}\int_0^t (1-\mathrm{e}^{-\delta u})\psi_m(v_{t-u}(\lambda))\mathrm{d}u. \tag{6.13}$$

Let $t \geq r \geq u \geq 0$ and let F be a bounded positive $\mathscr{A}_{r-u}$-measurable function on $D[0,\infty)$. Then

$$\begin{aligned}
&\mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda Y_t^\beta\Big\}\Big] \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda\beta\delta^{-1}(1-\mathrm{e}^{-\delta t}) - \lambda Z_t\Big\}\Big] \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda\beta\delta^{-1}(1-\mathrm{e}^{-\delta t})\Big\}\Big] \\
&\qquad\cdot\exp\Big\{-v_{t-r}(\lambda)Z_r - \beta\delta^{-1}\int_r^t \psi_m(v_{t-s}(\lambda))(1-\mathrm{e}^{-\delta s})\mathrm{d}s\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda\beta\delta^{-1}(1-\mathrm{e}^{-\delta t})\Big\}\Big] \\
&\qquad\cdot\exp\Big\{-v_{t-r}(\lambda)Y_r^\beta + v_{t-r}(\lambda)\beta\delta^{-1}(1-\mathrm{e}^{-\delta r})\Big\} \\
&\qquad\cdot\exp\Big\{-\beta\delta^{-1}\int_0^{t-r} \psi_m(v_{t-r-s}(\lambda))(1-\mathrm{e}^{-\delta(r+s)})\mathrm{d}s\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda\beta\delta^{-1}(1-\mathrm{e}^{-\delta t})\Big\}\Big] \\
&\qquad\cdot\exp\Big\{-v_{t-r}(\lambda)Y_r^\beta + \lambda\beta\delta^{-1}\,\mathrm{e}^{-\delta(t-r)}(1-\mathrm{e}^{-\delta r})\Big\} \\
&\qquad\cdot\exp\Big\{\beta\delta^{-1}(1-\mathrm{e}^{-\delta r})\int_0^{t-r} \mathrm{e}^{-\delta s}\,\psi_m(v_{t-r-s}(\lambda))\mathrm{d}s\Big\} \\
&\qquad\cdot\exp\Big\{-\beta\delta^{-1}\int_0^{t-r} \psi_m(v_{t-r-s}(\lambda))(1-\mathrm{e}^{-\delta(r+s)})\mathrm{d}s\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) - \lambda\beta\delta^{-1}(1-\mathrm{e}^{-\delta(t-r)})\Big\}\Big] \\
&\qquad\cdot\exp\Big\{-v_{t-r}(\lambda)Y_r^\beta - \beta\delta^{-1}\int_0^{t-r} \psi_m(v_{t-r-s}(\lambda))(1-\mathrm{e}^{-\delta s})\mathrm{d}s\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{-\int_{(0,u]}\int_{D[0,\infty)} F(w)N_\beta(\mathrm{d}s,\mathrm{d}w) \\
&\qquad\qquad - v_{t-r}(\lambda)Y_r^\beta - \beta\int_0^{t-r} v_s(\lambda))\mathrm{d}s\Big\}\Big],
\end{aligned}$$

where we used (6.13) for the last equality. That gives the desired result. □

Since the CB- and CBI-processes have Feller transition semigroups, they have càdlàg realizations. By Proposition A.7 of Li (2011), any realizations of the processes has a càdlàg modification. In the case of $\phi'(\infty) = \infty$, let $(X_t^z, \mathscr{G}_t^z)$, $(Y_t^\beta, \mathscr{G}_t^\beta)$ and $(Y_t^\nu, \mathscr{G}_t^\nu)$ be defined as in Theorem 6.2, Corollary 6.5 and Corollary 6.6, respectively. In the case of

$\phi'(\infty) < \infty$, we define those processes as in Theorem 6.7, Theorem 6.8 and Corollary 6.6, respectively. In both cases, let $Y_t = X_t^z + Y_t^\beta + Y_t^\nu$ and $\mathscr{G}_t = \sigma(\mathscr{G}_t^z \cup \mathscr{G}_t^\beta \cup \mathscr{G}_t^\nu)$. It is not hard to show that $\{(Y_t, \mathscr{G}_t) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ given by (2.13) and immigration mechanism ψ given by (5.9).

The existence of the excursion law for the branching mechanism $\phi(z) = bz + cz^2$ was first proved by Pitman and Yor (1982). As a special case of the so-called *Kuznetsov measure*, the existence of the law for measure-valued branching processes was derived from a general result on Markov processes in Li (2003, 2011), where it was also shown that the law only charges sample paths starting with zero. In the setting of measure-valued processes, El Karoui and Roelly (1991) used (6.3) to construct the excursion law; see also Duquesne and Labbé (2014). The construction of the CB- or CBI-process based on a excursion law was first given by Pitman and Yor (1982). This type of constructions have also been used in the measure-valued setting by a number of authors; see, e.g., Dawson and Li (2003), El Karoui and Roelly (1991), Li (1996, 2003, 2011) and Li and Shiga (1995).

7 Martingale problem formulations

In this section we give several formulations of the CBI-process in terms of martingale problems. Let (ϕ, ψ) be given by (2.13) and (5.9), respectively. We assume (5.11) is satisfied and define $\psi'(0)$ by (5.13). Let $C^2[0,\infty)$ denote the set of bounded continuous real functions on $[0,\infty)$ with bounded continuous derivatives up to the second order. For $f \in C^2[0,\infty)$ define

$$\begin{aligned} Lf(x) &= cxf''(x) + x\int_{(0,\infty)} \big[f(x+z) - f(x) - zf'(x)\big]m(\mathrm{d}z) \\ &\quad + (\beta - bx)f'(x) + \int_{(0,\infty)} \big[f(x+z) - f(x)\big]\nu(\mathrm{d}z). \end{aligned} \tag{7.1}$$

We shall identify the operator L as the generator of the CBI-process.

Proposition 7.1 *Let* $(P_t)_{t\geq 0}$ *be the transition semigroup defined by* (2.19) *and* (5.10). *Then for any* $t \geq 0$ *and* $\lambda \geq 0$ *we have*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) = \mathrm{e}^{-x\lambda} + \int_0^t \mathrm{d}s \int_{[0,\infty)} [y\phi(\lambda) - \psi(\lambda)]\,\mathrm{e}^{-y\lambda} P_s(x, \mathrm{d}y). \tag{7.2}$$

Proof. Recall that $v_t'(\lambda) = (\partial/\partial\lambda)v_t(\lambda)$. By differentiating both sides of (5.10) we get

$$\int_{[0,\infty)} y\,\mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) = \int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y)\Big[xv_t'(\lambda) + \int_0^t \psi'(v_s(\lambda))v_s'(\lambda)\mathrm{d}s\Big].$$

From this and (3.7) it follows that

$$\begin{aligned} \frac{\partial}{\partial t}\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) &= -\Big[x\frac{\partial}{\partial t}v_t(\lambda) + \psi(v_t(\lambda))\Big]\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) \\ &= \Big[x\phi(\lambda)v_t'(\lambda) - \psi(\lambda)\Big]\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) \\ &\quad - \int_0^t \psi'(v_s(\lambda))\frac{\partial}{\partial s}v_s(\lambda)\mathrm{d}s\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) \\ &= \Big[x\phi(\lambda)v_t'(\lambda) - \psi(\lambda)\Big]\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) \\ &\quad + \phi(\lambda)\int_0^t \psi'(v_s(\lambda))v_s'(\lambda)\mathrm{d}s\int_{[0,\infty)} \mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y) \\ &= \int_{[0,\infty)} [y\phi(\lambda) - \psi(\lambda)]\,\mathrm{e}^{-y\lambda} P_t(x, \mathrm{d}y). \end{aligned}$$

That gives (7.2). □

Suppose that $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses and $\{y(t) : t \geq 0\}$ is a càdlàg process in $[0,\infty)$ that is adapted to $(\mathscr{G}_t)_{t\geq 0}$ and satisfies $\mathbf{P}[y(0)] < \infty$. Let $C^{1,2}([0,\infty)^2)$ be the set of bounded continuous real functions $(t, x) \mapsto G(t, x)$ on $[0,\infty)^2$ with bounded continuous derivatives up to the first order relative to $t \geq 0$ and up to the second order relative to $x \geq 0$. Let us consider the following properties:

(1) For every $T \geq 0$ and $\lambda \geq 0$,

$$\exp\bigg\{ - v_{T-t}(\lambda)y(t) - \int_0^{T-t} \psi(v_s(\lambda))\mathrm{d}s\bigg\}, \qquad 0 \leq t \leq T,$$

is a martingale.

(2) For every $\lambda \geq 0$,

$$H_t(\lambda) := \exp\bigg\{ - \lambda y(t) + \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{d}s\bigg\}, \quad t \geq 0,$$

is a local martingale.

(3) The process $\{y(t) : t \geq 0\}$ has no negative jumps and the optional random measure

$$N_0(\mathrm{d}s, \mathrm{d}z) := \sum_{s>0} 1_{\{\Delta y(s)\neq 0\}}\delta_{(s,\Delta y(s))}(\mathrm{d}s, \mathrm{d}z),$$

where $\Delta y(s) = y(s) - y(s-)$, has predictable compensator $\hat{N}_0(\mathrm{d}s, \mathrm{d}z) = \mathrm{d}s\nu(\mathrm{d}z) + y(s-)\mathrm{d}sm(\mathrm{d}z)$. Let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) = N_0(\mathrm{d}s, \mathrm{d}z) - \hat{N}_0(\mathrm{d}s, \mathrm{d}z)$. We have

$$y(t) = y(0) + M^c(t) + M^d(t) - b\int_0^t y(s-)\mathrm{d}s + \psi'(0)t,$$

where $\{M^c(t) : t \geq 0\}$ is a continuous local martingale with quadratic variation $2cy(t-)\mathrm{d}t$ and

$$M^d(t) = \int_0^t \int_{(0,\infty)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0,$$

is a purely discontinuous local martingale.

(4) For every $f \in C^2[0, \infty)$ we have

$$f(y(t)) = f(y(0)) + \int_0^t Lf(y(s))\mathrm{d}s + \text{local mart.} \tag{7.3}$$

(5) For any $G \in C^{1,2}([0, \infty)^2)$ we have

$$G(t, y(t)) = G(0, y(0)) + \int_0^t \big[G_t'(s, y(s)) + LG(s, y(s))\big]\mathrm{d}s + \text{local mart.} \tag{7.4}$$

where L acts on the function $x \mapsto G(s, x)$.

Theorem 7.2 *The above properties* (1), (2), (3), (4) *and* (5) *are equivalent to each other. Those properties hold if and only if* $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ *is a CBI-process with branching mechanism* ϕ *and immigration mechanism* ψ.

Proof. Clearly, (1) holds if and only if $\{y(t) : t \geq 0\}$ is a Markov process relative to $(\mathscr{G}_t)_{t\geq 0}$ with transition semigroup $(P_t)_{t\geq 0}$ defined by (5.10). Then we only need to prove the equivalence of the five properties.

(1)⇒(2): Suppose that (1) holds. Then $\{y(t) : t \geq 0\}$ is a CBI-process with transition semigroup $(P_t)_{t\geq 0}$ given by (5.10). By (7.2) and the Markov property it is easy to see that

$$Y_t(\lambda) := \mathrm{e}^{-\lambda y(t)} + \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\, \mathrm{e}^{-\lambda y(s)}\, \mathrm{d}s \tag{7.5}$$

is a martingale. By integration by parts applied to

$$Z_t(\lambda) := \mathrm{e}^{-\lambda y(t)} \text{ and } W_t(\lambda) := \exp\bigg\{ \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{d}s \bigg\} \tag{7.6}$$

we obtain

$$\mathrm{d}H_t(\lambda) = \mathrm{e}^{-\lambda y(t-)}\, \mathrm{d}W_t(\lambda) + W_t(\lambda)\mathrm{d}\, \mathrm{e}^{-\lambda y(t)} = W_t(\lambda)\mathrm{d}Y_t(\lambda).$$

Then $\{H_t(\lambda)\}$ is a local martingale.

(2)⇒(3): For any $\lambda \geq 0$ let $Z_t(\lambda)$ and $W_t(\lambda)$ be defined by (7.6). We have $Z_t(\lambda) = H_t(\lambda)W_t(\lambda)^{-1}$ and so

$$\mathrm{d}Z_t(\lambda) = W_t(\lambda)^{-1}\mathrm{d}H_t(\lambda) - Z_{t-}(\lambda)[\psi(\lambda) - \phi(\lambda)y(t-)]\mathrm{d}t \tag{7.7}$$

by integration by parts. Then the strictly positive process $t \mapsto Z_t(\lambda)$ is a special semi-martingale; see, e.g., Dellacherie and Meyer (1982, p.213). By Itô's formula we find $t \mapsto y(t)$ is a semi-martingale. Now define the optional random measure $N_0(\mathrm{d}s, \mathrm{d}z)$ on $[0, \infty) \times \mathbb{R}$ by

$$N_0(\mathrm{d}s, \mathrm{d}z) = \sum_{s>0} 1_{\{\Delta y(s)\neq 0\}}\delta_{(s,\Delta y(s))}(\mathrm{d}s, \mathrm{d}z),$$

where $\Delta y(s) = y(s) - y(s-)$. Let $\hat{N}_0(\mathrm{d}s, \mathrm{d}z)$ denote the predictable compensator of $N_0(\mathrm{d}s, \mathrm{d}z)$ and let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z)$ denote the compensated random measure; see, e.g., Dellacherie and Meyer (1982, p.375). It follows that

$$y(t) = y(0) + U(t) + M^c(t) + M^d(t), \tag{7.8}$$

where $t \mapsto U(t)$ is a right-continuous adapted process with locally bounded variations, $t \mapsto M^c(t)$ is a continuous local martingale and

$$t \mapsto M^d(t) := \int_0^t \int_{\mathbb{R}} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z)$$

is a purely discontinuous local martingale; see, e.g., Dellacherie and Meyer (1982, p.353 and p.376) or Jacod and Shiryaev (2003, pp.84–85). Let $\{C(t)\}$ denote the quadratic variation process of $\{M^c(t)\}$. By Itô's formula,

$$Z_t(\lambda) = Z_0(\lambda) - \lambda \int_0^t Z_{s-}(\lambda)\mathrm{d}y(s) + \frac{1}{2}\lambda^2 \int_0^t Z_{s-}(\lambda)\mathrm{d}C(s)$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}} Z_{s-}(\lambda)\big(\mathrm{e}^{-z\lambda}-1+z\lambda\big)N_0(\mathrm{d}s,\mathrm{d}z)\\
&= Z_0(\lambda)-\lambda\int_0^t Z_{s-}(\lambda)\mathrm{d}U(s)+\frac{1}{2}\lambda^2\int_0^t Z_{s-}(\lambda)\mathrm{d}C(s)\\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}(\lambda)\big(\mathrm{e}^{-z\lambda}-1+z\lambda\big)\hat{N}_0(\mathrm{d}s,\mathrm{d}z)+\text{local mart.} \qquad (7.9)
\end{aligned}
$$

In view of (7.7) and (7.9) we get

$$
[y(s-)\phi(\lambda)-\psi(\lambda)]\mathrm{d}s=-\lambda\mathrm{d}U(s)+\frac{1}{2}\lambda^2\mathrm{d}C(s)+\int_{\mathbb{R}}\big(\mathrm{e}^{-z\lambda}-1+z\lambda\big)\hat{N}_0(\mathrm{d}s,\mathrm{d}z)
$$

by the uniqueness of canonical decompositions of special semi-martingales; see, e.g., Dellacherie and Meyer (1982, p.213). By substituting the representations (2.13) and (5.9) for ϕ and ψ into the above equation and comparing both sides we find

$$
\mathrm{d}C(s)=2cy(s-)\mathrm{d}s,\ \mathrm{d}U(s)=[\psi'(0)-by(s-)]\mathrm{d}s
$$

and

$$
\hat{N}_0(\mathrm{d}s,\mathrm{d}z)=\mathrm{d}s\nu(\mathrm{d}z)+y(s-)\mathrm{d}s m(\mathrm{d}z).
$$

Then the process $t\mapsto y(t)$ has no negative jumps.

(3)$\Rightarrow$(4): This follows by Itô's formula.

(4)$\Rightarrow$(5): For $t\geq 0$ and $k\geq 1$ we have

$$
\begin{aligned}
G(t,y(t)) &= G(0,y(0))+\sum_{j=0}^{\infty}\big[G(t\wedge j/k,y(t\wedge(j+1)/k))\\
&\qquad -G(t\wedge j/k,y(t\wedge j/k))\big]\\
&\quad+\sum_{j=0}^{\infty}\big[G(t\wedge(j+1)/k,y(t\wedge(j+1)/k))\\
&\qquad -G(t\wedge j/k,y(t\wedge(j+1)/k))\big],
\end{aligned}
$$

where the summations only consist of finitely many nontrivial terms. By applying (4) term by term we obtain

$$
\begin{aligned}
G(t,y(t)) &= G(0,y(0))+\sum_{j=0}^{\infty}\int_{t\wedge j/k}^{t\wedge(j+1)/k}\Big\{[\beta-by(s)]G'_x(t\wedge j/k,y(s))\\
&\quad+cy(s)G''_{xx}(t\wedge j/k,y(s))+y(s)\int_{(0,\infty)}\Big[G(t\wedge j/k,y(s)+z)\\
&\quad-G(t\wedge j/k,y(s))-zG'_x(t\wedge j/k,y(s))\Big]m(\mathrm{d}z)\\
&\quad+\int_{(0,\infty)}\Big[G(t\wedge j/k,y(s)+z)-G(t\wedge j/k,y(s))\Big]\nu(\mathrm{d}z)\Big\}\mathrm{d}s\\
&\quad+\sum_{j=0}^{\infty}\int_{t\wedge j/k}^{t\wedge(j+1)/k}G'_t(s,y(t\wedge(j+1)/k))\mathrm{d}s+M_k(t),
\end{aligned}
$$

where $\{M_k(t)\}$ is a local martingale. Since $\{y(t)\}$ is a càdlàg process, letting $k \to \infty$ in the equation above gives

$$\begin{aligned} G(t, y(t)) &= G(0, y(0)) + \int_0^t \Big\{ G_t'(s, y(s)) + [\beta - by(s)]G_x'(s, y(s)) \\ &\quad + cy(s)G_{xx}''(s, y(s)) + y(s) \int_{(0,\infty)} \Big[G(s, y(s) + z) \\ &\quad - G(s, y(s)) - zG_x'(s, y(s)) \Big] m(\mathrm{d}z) \\ &\quad + \int_{(0,\infty)} \Big[G(s, y(s) + z) - G(s, y(s)) \Big] \nu(\mathrm{d}z) \Big\} \mathrm{d}s + M(t), \end{aligned}$$

where $\{M(t)\}$ is a local martingale. Then we have (7.4).

(5)⇒(1): For fixed $T \geq 0$ and $\lambda \geq 0$ we define the function

$$G_T(t, x) = \exp\Big\{ - v_{T-t}(\lambda)x - \int_0^{T-t} \psi(v_s(\lambda))\mathrm{d}s \Big\}, \quad 0 \leq t \leq T, x \geq 0,$$

which can be extended to a function in $C^{1,2}([0, \infty)^2)$. Using (3.6) we see

$$\frac{\mathrm{d}}{\mathrm{d}t} G_T(t, x) + LG_T(t, x) = 0, \quad 0 \leq t \leq T, x \geq 0,$$

Then (7.4) implies that $t \mapsto G(t \wedge T, y(t \wedge T))$ is a local martingale, and hence a martingale by the boundedness. □

Corollary 7.3 *Let $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ be a càdlàg realization of the CBI-process satisfying $\mathbf{P}[y(0)] < \infty$. Then for every $T \geq 0$ there is a constant $C_T \geq 0$ such that*

$$\mathbf{P}\Big[\sup_{0 \leq t \leq T} y(t) \Big] \leq C_T \big\{ \mathbf{P}[y(0)] + \psi'(0) + \sqrt{\mathbf{P}[y(0)]} + \sqrt{\psi'(0)} \big\}.$$

Proof. By the above property (3) and Doob's martingale inequality we have

$$\begin{aligned} &\mathbf{P}\Big[\sup_{0 \leq t \leq T} |y(t) - y(0)| \Big] \\ &\quad \leq T\psi'(0) + \mathbf{P}\Big[|b| \int_0^T y(s)\mathrm{d}s \Big] + \mathbf{P}\Big[\sup_{0 \leq t \leq T} |M_t^c| \Big] \\ &\qquad + \mathbf{P}\Big[\int_0^T \int_{(1,\infty)} zN_0(\mathrm{d}s, \mathrm{d}z) \Big] + \mathbf{P}\Big[\int_0^T \int_{(1,\infty)} z\hat{N}_0(\mathrm{d}s, \mathrm{d}z) \Big] \\ &\qquad + \mathbf{P}\Big[\sup_{0 \leq t \leq T} \Big| \int_{(0,1]} \int_0^1 z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) \Big| \Big] \\ &\quad \leq T\psi'(0) + \mathbf{P}\Big[|b| \int_0^T y(s)\mathrm{d}s \Big] + 2\Big\{ \mathbf{P}\Big[c \int_0^T y(s)\mathrm{d}s \Big] \Big\}^{1/2} \\ &\qquad + 2T \int_{(1,\infty)} z\nu(\mathrm{d}z) + 2\mathbf{P}\Big[\int_0^T y(s)\mathrm{d}s \int_{(1,\infty)} zm(\mathrm{d}z) \Big] \end{aligned}$$

$$+2\bigg\{\mathbf{P}\bigg[T\int_{(0,1]} z^2\nu(\mathrm{d}z) + \int_0^T y(s)\mathrm{d}s\int_{(0,1]} z^2 m(\mathrm{d}z)\bigg]\bigg\}^{1/2}.$$

Then the desired inequality follows by simple estimates based on (5.12). □

Corollary 7.4 *Let* $\{(y(t),\mathscr{G}_t) : t \geq 0\}$ *be a càdlàg realization of the CBI-process satisfying* $\mathbf{P}[y(0)] < \infty$*. Then the above properties* (3)*,* (4) *and* (5) *hold with the local martingales being martingales.*

Proof. Since the arguments are similar, we only give those for (4). Let $f \in C^2[0,\infty)$ and let

$$M(t) = f(y(t)) - f(y(0)) - \int_0^t Lf(y(s))\mathrm{d}s, \qquad t \geq 0.$$

By property (4) we know $\{M(t)\}$ is a local martingale. Let $\{\tau_n\}$ be a localization sequence of stopping times for $\{M(t)\}$. For any $t \geq r \geq 0$ and any bounded $\mathscr{G}_r$-measurable random variable F, we have

$$\begin{aligned}\mathbf{P}\bigg\{\bigg[f(y(t\wedge\tau_n)) - f(y(0)) - \int_0^{t\wedge\tau_n} Lf(y(s))\mathrm{d}s\bigg]F\bigg\} \\ = \mathbf{P}\bigg\{\bigg[f(y(r\wedge\tau_n)) - f(y(0)) - \int_0^{r\wedge\tau_n} Lf(y(s))\mathrm{d}s\bigg]F\bigg\}.\end{aligned}$$

In view of (7.1), there is a constant $C \geq 0$ so that $|Lf(x)| \leq C(1+x)$. By Corollary 7.3 we can let $n \to \infty$ and use dominated convergence in the above equality to see $\{M(t)\}$ is a martingale. □

Note that property (4) implies that the generator of the CBI-process is the closure of the operator L in the sense of Ethier and Kurtz (1986). This explicit form of the generator was first given in Kawazu and Watanabe (1971). The results of Theorem 7.2 were presented for measure-valued processes in El Karoui and Roelly (1991) and Li (2011).

8 Stochastic equations for CBI-processes

In this section we establish some stochastic equations for the CBI-processes. Suppose that (ϕ, ψ) are branching and immigration mechanisms given respectively by (2.13) and (5.9) with $\nu(\mathrm{d}u)$ satisfying condition (5.11). Let $(P_t)_{t\geq 0}$ be the transition semigroup defined by (2.19) and (5.10). In this and the next section, for any $b \geq a \geq 0$ we understand

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}.$$

Let $\{B(t)\}$ be a standard Brownian motion and $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ a Poisson time-space random measure on $(0,\infty)^3$ with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$. Let $\{\eta(t)\}$ be an increasing Lévy process with $\eta(0) = 0$ and with Laplace exponent $\psi(z) = -\log \mathbf{P}\exp\{-z\eta(1)\}$. We assume that $\{B(t)\}$, $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\eta(t)\}$ are defined on a complete probability space and are independent of each other. Consider the stochastic integral equation

$$\begin{aligned} y(t) \;=\; & y(0) + \int_0^t \sqrt{2cy(s-)}\mathrm{d}B(s) - b\int_0^t y(s-)\mathrm{d}s \\ & + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \eta(t), \end{aligned} \tag{8.1}$$

where $\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ denotes the compensated measure. We understand the forth term on the right-hand side of (8.1) as an integral over the set $\{(s,z,u) : 0 < s \leq t, 0 < z < \infty, 0 < u \leq y(s-)\}$ and give similar interpretations for other stochastic integrals in this section. The reader is referred to Ikeda and Watanabe (1989) and Situ (2005) for the basic theory of stochastic equations.

Theorem 8.1 *A positive càdlàg process* $\{y(t) : t \geq 0\}$ *is a CBI-process with branching and immigration mechanisms* (ϕ, ψ) *given respectively by* (2.13) *and* (5.9) *if and only if it is a weak solution to* (8.1).

Proof. Suppose that the positive càdlàg process $\{y(t)\}$ is a weak solution to (8.1). By Itô's formula one can see $\{y(t)\}$ solves the martingale problem (7.3). By Theorem 7.2 we infer that $\{y(t)\}$ is a CBI-process with branching and immigration mechanisms given respectively by (2.13) and (5.9). Conversely, suppose that $\{y(t)\}$ is a càdlàg realization of the CBI-process with branching and immigration mechanisms given respectively by (2.13) and (5.9). By Theorem 7.2 the process has no negative jumps and the random measure

$$N_0(\mathrm{d}s, \mathrm{d}z) := \sum_{s>0} 1_{\{\Delta y(s)>0\}}\delta_{(s,\Delta y(s))}(\mathrm{d}s, \mathrm{d}z)$$

has predictable compensator

$$\hat{N}_0(\mathrm{d}s, \mathrm{d}z) = y(s-)\mathrm{d}sm(\mathrm{d}z) + \mathrm{d}s\nu(\mathrm{d}z).$$

Moreover, we have

$$\begin{aligned} y(t) \;=\; & y(0) + t\Big[\beta + \int_0^\infty u\nu(\mathrm{d}u)\Big] - \int_0^t by(s-)\mathrm{d}s \\ & + M^c(t) + \int_0^t \int_0^\infty z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

where $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) = N_0(\mathrm{d}s, \mathrm{d}z) - \hat{N}_0(\mathrm{d}s, \mathrm{d}z)$ and $t \mapsto M^c(t)$ is a continuous local martingale with quadratic variation $2cy(t-)\mathrm{d}t$. By Theorem III.7.1$'$ in Ikeda and Watanabe (1989, p.90), on an extension of the original probability space there is a standard Brownian motion $\{B(t)\}$ so that

$$M^c(t) = \int_0^t \sqrt{2cy(s-)}\mathrm{d}B(s).$$

By Theorem III.7.4 in Ikeda and Watanabe (1989, p.93), on a further extension of the probability space we can define independent Poisson random measures $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N(\mathrm{d}s, \mathrm{d}z)$ with intensities $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ and $\mathrm{d}s\nu(\mathrm{d}z)$, respectively, so that

$$\int_0^t \int_0^\infty z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) = \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Then $\{y(t)\}$ is a weak solution to (8.1). □

Theorem 8.2 *For any initial value $y(0) = x \geq 0$, there is a pathwise unique positive strong solution to* (8.1).

Proof. By Theorem 8.1 there is a weak solution to (8.1). Then we only need to prove the pathwise uniqueness of the solution; see, e.g., Situ (2005, p.76 and p.104). Suppose that $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ are two positive solutions of (8.1) with deterministic initial states. By Theorem 8.1, both of them are CBI-processes. We may assume $x(0)$ and $y(0)$ are deterministic upon taking a conditional probability. In view of (5.12), the processes have locally bounded first moments. Let $\zeta(t) = x(t) - y(t)$ for $t \geq 0$. For each integer $n \geq 0$ define $a_n = \exp\{-n(n+1)/2\}$. Then $a_n \to 0$ decreasingly as $n \to \infty$ and

$$\int_{a_n}^{a_{n-1}} z^{-1}\mathrm{d}z = n, \qquad n \geq 1.$$

Let $x \mapsto g_n(x)$ be a positive continuous function supported by (a_n, a_{n-1}) so that

$$\int_{a_n}^{a_{n-1}} g_n(x)\mathrm{d}x = 1$$

and $g_n(x) \leq 2(nx)^{-1}$ for every $x > 0$. For $n \geq 0$ and $z \in \mathbb{R}$ let

$$f_n(z) = \int_0^{|z|} \mathrm{d}y \int_0^y g_n(x)\mathrm{d}x.$$

Then $f_n(z) \to |z|$ increasingly as $n \to \infty$. Moreover, we have $|f_n'(z)| \leq 1$ and

$$0 \leq |z| f_n''(z) = |z| g_n(|z|) \leq 2/n. \tag{8.2}$$

For $z, \zeta \in \mathbb{R}$ it is easy to see that

$$|f_n(\zeta + z) - f_n(\zeta) - z f_n'(\zeta)| \leq |f_n(\zeta + z) - f_n(\zeta)| + |z f_n'(\zeta)| \leq 2|z|.$$

By Taylors expansion, when $z\zeta \geq 0$, there is η between ζ and $\zeta + z$ so that

$$|\zeta||f_n(\zeta + z) - f_n(\zeta) - z f_n'(\zeta)| \leq |\zeta||f_n''(\eta)|z^2/2 \leq |\eta||f_n''(\eta)|z^2/2 \leq z^2/n.$$

where we used (8.2) for the last inequality. It follows that, when $z\zeta \geq 0$,

$$|\zeta||f_n(\zeta + z) - f_n(\zeta) - z f_n'(\zeta)| \leq (2|z\zeta|) \wedge (z^2/n) \leq (1 + 2|\zeta|)[|z| \wedge (z^2/n)]. \tag{8.3}$$

From (8.1) we have

$$\begin{aligned}
\zeta(t) \ =\ & \zeta(0) - b\int_0^t \zeta(s-)\mathrm{d}s + \sqrt{2c}\int_0^t \big(\sqrt{x(s)} - \sqrt{y(s)}\big)\mathrm{d}B(s) \\
& + \int_0^t \int_0^\infty \int_{y(s-)}^{x(s-)} z 1_{\{\zeta(s-)>0\}} \tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\
& - \int_0^t \int_0^\infty \int_{x(s-)}^{y(s-)} z 1_{\{\zeta(s-)<0\}} \tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).
\end{aligned}$$

By this and Itô's formula,

$$\begin{aligned}
f_n(\zeta(t)) \ =\ & f_n(\zeta(0)) - b\int_0^t f_n'(\zeta(s))\zeta(s)\mathrm{d}s + c\int_0^t f_n''(\zeta(s))\big[\sqrt{x(s)} - \sqrt{y(s)}\,\big]^2\mathrm{d}s \\
& + \int_0^t \zeta(s)1_{\{\zeta(s)>0\}}\mathrm{d}s \int_0^\infty [f_n(\zeta(s) + z) - f_n(\zeta(s)) - z f_n'(\zeta(s))] m(\mathrm{d}z) \\
& - \int_0^t \zeta(s)1_{\{\zeta(s)<0\}}\mathrm{d}s \int_0^\infty [f_n(\zeta(s) - z) - f_n(\zeta(s)) + z f_n'(\zeta(s))] m(\mathrm{d}z) \\
& + \text{mart.}
\end{aligned}$$

Taking the expectation in both sides of the above and using (8.2) and (8.3) we see

$$\mathbf{P}[f_n(\zeta(t))] \leq f_n(\zeta(0)) + |b|\int_0^t \mathbf{P}[|\zeta(s)|]\mathrm{d}s + \varepsilon_n(t), \tag{8.4}$$

where

$$\varepsilon_n(t) = 2cn^{-1}t + \int_0^t (1 + 2\mathbf{P}[|\zeta(s)|])\mathrm{d}s \int_0^\infty [z \wedge (n^{-1}z^2)] m(\mathrm{d}z).$$

Clearly, we have $\lim_{n\to\infty} \varepsilon_n(t) = 0$. Then letting $n \to \infty$ in (8.4) we get

$$\mathbf{P}[|x(t) - y(t)|] \leq |x(0) - y(0)| + |b|\int_0^t \mathbf{P}[|x(s) - y(s)|]\mathrm{d}s.$$

If $x(0) = y(0)$, we have $\mathbf{P}[|x(t) - y(t)|] = 0$ by Gronwall's inequality, and so $\mathbf{P}\{x(t) = y(t)\} = 1$ for $t \geq 0$. Then $\mathbf{P}\{x(t) = y(t)$ for $t \geq 0\} = 1$ by the right continuity of the processes. That gives the pathwise uniqueness for (8.1). $\square$

We can give a formulation of the CBI-process in terms another stochastic integral equation weakly equivalent to (8.1). Let $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\eta(s)\}$ be as in (8.1). Let $\{W(\mathrm{d}s, \mathrm{d}u)\}$ be a Gaussian time-space white noise on $(0, \infty)^2$ with intensity $\mathrm{d}s\mathrm{d}u$. We assume $\{W(\mathrm{d}s, \mathrm{d}u)\}$, $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\eta(s)\}$ are defined on a complete probability space and are independent of each other. Consider the stochastic integral equation

$$\begin{aligned} y(t) \;=\; & y(0) + \sqrt{2c}\int_0^t \int_0^{y(s-)} W(\mathrm{d}s, \mathrm{d}u) - b\int_0^t y(s-)\mathrm{d}s \\ & + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \eta(t). \end{aligned} \tag{8.5}$$

The reader may refer to Li (2011, Section 7.3) and Walsh (1986, Chapter 2) for discussions of stochastic integration with respect to Gaussian time-space white noises.

Theorem 8.3 *A positive càdlàg process $\{y(t) : t \geq 0\}$ is a CBI-process with branching and immigration mechanisms (ϕ, ψ) given respectively by* (2.13) *and* (5.9) *if and only if it is a weak solution to* (8.5).

Proof. Suppose that $\{y(t)\}$ is a CBI-process with branching and immigration mechanisms given respectively by (2.13) and (5.9). By Theorem 8.1, the process is a weak solution to (8.1). By El Karoui and Méléard (1990, Theorem III.6), on an extension of the probability space we can define a Gaussian time-space white noise $W(\mathrm{d}s, \mathrm{d}u)$ with intensity $\mathrm{d}s\mathrm{d}u$ so that

$$\int_0^t \sqrt{y(s-)}\mathrm{d}B(s) = \int_0^t \int_0^{y(s-)} W(\mathrm{d}s, \mathrm{d}u).$$

Then $\{y(t)\}$ is a weak solution to (8.5). Conversely, suppose that $\{y(t)\}$ is a weak solution to (8.5). By Itô's formula one can see $\{y(t)\}$ solves the martingale problem (7.3). By Theorem 7.2 we infer that $\{y(t)\}$ is a CBI-process with branching and immigration mechanisms given respectively by (2.13) and (5.9). $\square$

Theorem 8.4 *Suppose that $\{y_1(t) : t \geq 0\}$ and $\{y_2(t) : t \geq 0\}$ are two positive solutions to* (8.5) *with $\mathbf{P}\{y_1(0) \leq y_2(0)\} = 1$. Then we have $\mathbf{P}\{y_1(t) \leq y_2(t)$ for all $t \geq 0\} = 1$.*

Proof. Let $\zeta(t) = y_1(t) - y_2(t)$ for $t \geq 0$. For $n \geq 0$ let f_n be the function defined as in the proof of Theorem 8.2. Let $h_n(z) = f_n(z \vee 0)$ for $z \in \mathbb{R}$. Then $h_n(z) \to z_+ := z \vee 0$ increasingly as $n \to \infty$. From (8.5) it follows that

$$\zeta(t) \;=\; \zeta(0) - b\int_0^t \zeta(s-)\mathrm{d}s + \sqrt{2c}\int_0^t \int_{y_2(s-)}^{y_1(s-)} 1_{\{\zeta(s-)>0\}} W(\mathrm{d}s, \mathrm{d}u)$$

$$
\begin{aligned}
&-\sqrt{2c}\int_0^t\int_{y_1(s-)}^{y_2(s-)} 1_{\{\zeta(s-)<0\}}W(\mathrm{d}s,\mathrm{d}u)\\
&+\int_0^t\int_0^\infty\int_{y_2(s-)}^{y_1(s-)} 1_{\{\zeta(s-)>0\}}z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\
&-\int_0^t\int_0^\infty\int_{y_1(s-)}^{y_2(s-)} 1_{\{\zeta(s-)<0\}}z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}
$$

Since $h_n(z)=0$ for $z\le 0$, by Itô's formula we have

$$
\begin{aligned}
h_n(\zeta(t)) &= -b\int_0^t h_n'(\zeta(s-))\zeta(s-)\mathrm{d}s + c\int_0^t h_n''(\zeta(s-))|\zeta(s-)|\mathrm{d}s\\
&\quad+\int_0^t \zeta(s-)1_{\{\zeta(s-)>0\}}\mathrm{d}s\int_0^\infty\Big[h_n(\zeta(s-)+z)-h_n(\zeta(s-))\\
&\quad-zh_n'(\zeta(s-))\Big]m(\mathrm{d}z)-\int_0^t\zeta(s-)1_{\{\zeta(s-)<0\}}\mathrm{d}s\int_0^\infty\Big[h_n(\zeta(s-)-z)\\
&\quad-h_n(\zeta(s-))+zh_n'(\zeta(s-))\Big]m(\mathrm{d}z)+\text{local mart.}\\
&= -b\int_0^t h_n'(\zeta(s-))\zeta(s-)_+\mathrm{d}s + c\int_0^t h_n''(\zeta(s-))\zeta(s-)_+\mathrm{d}s\\
&\quad+\int_0^t\zeta(s-)_+\mathrm{d}s\int_0^\infty\Big[h_n(\zeta(s-)+z)-h_n(\zeta(s-))\\
&\quad-zh_n'(\zeta(s-))\Big]m(\mathrm{d}z)+\text{local mart.}
\end{aligned}
$$

For any $k\ge 1$ define $\tau_k=\inf\{t\ge 0:\zeta(t)_+\ge k\}$. Taking the expectation in the above equality at time $t\wedge\tau_k$ and using (8.2) and (8.3) we have

$$
\mathbf{P}[h_n(\zeta(t\wedge\tau_k))]\le|b|\mathbf{P}\bigg[\int_0^{t\wedge\tau_k}\zeta(s-)_+\mathrm{d}s\bigg]+\varepsilon_n(t),
$$

where

$$
\varepsilon_n(t)=2cn^{-1}t+\mathbf{P}\bigg[\int_0^{t\wedge\tau_k}(1+2\zeta(s-)_+)\mathrm{d}s\bigg]\int_0^\infty(z\wedge n^{-1}z^2)m(\mathrm{d}z).
$$

Then we let $n\to\infty$ to obtain

$$
\mathbf{P}[\zeta(t\wedge\tau_k)_+]\le|b|\mathbf{P}\bigg[\int_0^{t\wedge\tau_k}\zeta(s-)_+\mathrm{d}s\bigg]\le|b|\int_0^t\mathbf{P}[\zeta(s\wedge\tau_k)_+]\mathrm{d}s.
$$

By Gronwall's inequality, for each $t\ge 0$ we have

$$
\mathbf{P}[(y_1(t\wedge\tau_k)-y_2(t\wedge\tau_k))_+]=\mathbf{P}[\zeta(t\wedge\tau_k)_+]=0.
$$

By letting $k\to\infty$ and using Fatou's lemma we see $\mathbf{P}[(y_1(t)-y_2(t))_+]=0$ for $t\ge 0$, and so $\mathbf{P}\{y_1(t)\le y_2(t)\text{ for all }t\ge 0\}=1$ by the right continuity of the processes. □

Theorem 8.5 *For any initial value $y(0)=x\ge 0$, there is a pathwise unique positive strong solution to* (8.5).

Proof. By Theorem 8.3 there is a weak solution $\{y(t)\}$ to (8.5). The pathwise uniqueness of the solution follows from Theorem 8.4. Then $\{y(t)\}$ is a strong solution to (8.5). See, e.g., Situ (2005, p.76 and p.104). $\square$

From (8.1) or (8.5) we see that the immigration of the CBI-process $\{y(t)\}$ is represented by the increasing Lévy process $\{\eta(t)\}$. By the Lévy–Itô decomposition, there is a Poisson time-space random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)$ such that

$$\eta(t) = \beta t + \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0.$$

Then the immigration of $\{y(t)\}$ involves two parts: the *continuous part* determined by the drift coefficient β and the *discontinuous part* given by the Poisson random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$.

Now let us consider a special CBI-process. Let $c, q \geq 0$, $b \in \mathbb{R}$ and $1 < \alpha < 2$ be given constants. Let $\{B(t)\}$ be a standard Brownian motion. Let $\{z(t)\}$ be a spectrally positive α-stable Lévy process with Lévy measure

$$\gamma(\mathrm{d}z) := (\alpha - 1)\Gamma(2 - \alpha)^{-1} z^{-1-\alpha}\mathrm{d}z, \qquad z > 0$$

and $\{\eta(t)\}$ an increasing Lévy process with $\eta(0) = 0$ and with Laplace exponent ψ. We assume that $\{B(t)\}$, $\{z(t)\}$ and $\{\eta(t)\}$ are defined on a complete probability space and are independent of each other. Consider the stochastic differential equation

$$\mathrm{d}y(t) = \sqrt{2cy(t-)}\mathrm{d}B(t) + \sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}z(t) - by(t-)\mathrm{d}t + \mathrm{d}\eta(t), \tag{8.6}$$

Theorem 8.6 *A positive càdlàg process $\{y(t) : t \geq 0\}$ is a CBI-process with branching mechanism $\phi(z) = bz + cz^2 + qz^\alpha$ and immigration mechanism ψ given by* (5.9) *if and only if it is a weak solution to* (8.6).

Proof. Suppose that $\{y(t)\}$ is a weak solution to (8.6). By Itô's formula one can see that $\{y(t)\}$ solves the martingale problem (7.3) associated with the generator L defined by (7.1) with $m(\mathrm{d}z) = \alpha q\gamma(\mathrm{d}z)$. Then $\{y(t)\}$ is a CBI-process with branching mechanism $\phi(z) = bz + cz^2 + qz^\alpha$ and immigration mechanism ψ given by (5.9). Conversely, suppose that $\{y(t)\}$ is a CBI-process with branching mechanism $\phi(z) = bz + cz^2 + qz^\alpha$ and immigration mechanism ψ given by (5.9). Then $\{y(t)\}$ is a weak solution to (8.1) with $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ being a Poisson random measure on $(0, \infty)^3$ with intensity $\alpha q\mathrm{d}s\gamma(\mathrm{d}z)\mathrm{d}u$. Let us assume $q > 0$, for otherwise the proof is easier. Define the random measure $\{N_0(\mathrm{d}s, \mathrm{d}z)\}$ on $(0, \infty)^2$ by

$$\begin{aligned} N_0((0,t] \times B) \;=\; & \int_0^t \int_0^\infty \int_0^{y(s-)} 1_{\{y(s-)>0\}} 1_B\Big(\frac{z}{\sqrt[\alpha]{\alpha q y(s-)}}\Big) M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ & + \int_0^t \int_0^\infty \int_0^{1/\alpha q} 1_{\{y(s-)=0\}} 1_B(z) M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

It is easy to compute that $\{N_0(\mathrm{d}s, \mathrm{d}z)\}$ has predictable compensator

$$\hat{N}_0((0,t] \times B) \;=\; \int_0^t \int_0^\infty 1_{\{y(s-)>0\}} 1_B\Big(\frac{z}{\sqrt[\alpha]{\alpha q y(s-)}}\Big) \frac{\alpha q y(s-)(\alpha - 1)\mathrm{d}s\mathrm{d}z}{\Gamma(2 - \alpha) z^{1+\alpha}}$$

$$
\begin{aligned}
&+\int_0^t\int_0^\infty 1_{\{y(s-)=0\}}1_B(z)\frac{(\alpha-1)\mathrm{d}s\mathrm{d}z}{\Gamma(2-\alpha)z^{1+\alpha}}\\
&=\int_0^t\int_0^\infty 1_B(z)\frac{(\alpha-1)\mathrm{d}s\mathrm{d}z}{\Gamma(2-\alpha)z^{1+\alpha}}.
\end{aligned}
$$

Thus $\{N_0(\mathrm{d}s,\mathrm{d}z)\}$ is a Poisson random measure with intensity $\mathrm{d}s\gamma(\mathrm{d}z)$; see, e.g., Theorem III.7.4 in Ikeda and Watanabe (1989, p.93). Now define the Lévy processes

$$
z(t)=\int_0^t\int_0^\infty z\tilde{N}_0(\mathrm{d}s,\mathrm{d}z)\ \text{ and }\ \eta(t)=\beta t+\int_0^t\int_0^\infty zN(\mathrm{d}s,\mathrm{d}z),
$$

where $\tilde{N}_0(\mathrm{d}s,\mathrm{d}z)=N_0(\mathrm{d}s,\mathrm{d}z)-\hat{N}_0(\mathrm{d}s,\mathrm{d}z)$. It is easy to see that

$$
\begin{aligned}
\int_0^t \sqrt[\alpha]{\alpha q y(s-)}\mathrm{d}z(s) &= \int_0^t\int_0^\infty \sqrt[\alpha]{\alpha q y(s-)}\, z\tilde{N}_0(\mathrm{d}s,\mathrm{d}z)\\
&= \int_0^t\int_0^\infty\int_0^{y(s-)} z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}
$$

Then $\{y(t)\}$ is a weak solution to (8.6). $\square$

Theorem 8.7 *For any initial value $y(0)=x\geq 0$, there is a pathwise unique positive strong solution to* (8.6).

Proof. By Theorem 8.6 there is a weak solution $\{y(t)\}$ to (8.6), so it suffices to prove the pathwise uniqueness of the solution. We first recall that the one-sided α-stable process $\{z(t)\}$ can be represented as

$$
z(t)=\int_0^t\int_0^\infty z\tilde{M}(\mathrm{d}s,\mathrm{d}z),
$$

where $M(\mathrm{d}s,\mathrm{d}z)$ is a Poisson random measure on $(0,\infty)^2$ with intensity $\mathrm{d}s\gamma(\mathrm{d}z)$. Let

$$
z_1(t)=\int_0^t\int_0^1 z\tilde{M}(\mathrm{d}s,\mathrm{d}z) \quad\text{and}\quad z_2(t)=\int_0^t\int_1^\infty zM(\mathrm{d}s,\mathrm{d}z).
$$

Since $t\mapsto z_2(t)$ has at most finitely many jumps in each bounded interval, we only need to prove the pathwise uniqueness of

$$
\begin{aligned}
\mathrm{d}y(t) &= \sqrt{2cy(t-)}\mathrm{d}B(t)+\sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}z_1(t)-by(t-)\mathrm{d}t\\
&\quad -\alpha^{-1}(\alpha-1)\Gamma(2-\alpha)^{-1}\sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}t+\mathrm{d}\eta(t),
\end{aligned}
\tag{8.7}
$$

Suppose that $\{x(t)\}$ and $\{y(t)\}$ are two positive solutions to (8.7) with deterministic initial values. Let $\zeta_\theta(t)=\sqrt[\theta]{x(t)}-\sqrt[\theta]{y(t)}$ for $0<\theta\leq 2$ and $t\geq 0$. Then we have

$$
\begin{aligned}
\mathrm{d}\zeta_1(t) &= \sqrt{2c}\zeta_2(t-)\mathrm{d}B(t)+\sqrt[\alpha]{\alpha q}\zeta_\alpha(t-)\mathrm{d}z_1(t)-b\zeta_1(t-)\mathrm{d}t\\
&\quad -\alpha^{-1}(\alpha-1)\Gamma(2-\alpha)^{-1}\sqrt[\alpha]{\alpha q}\zeta_\alpha(t-)\mathrm{d}t.
\end{aligned}
$$

For $n \geq 0$ let f_n be the function defined as in the proof of Theorem 8.2. By Itô's formula,

$$\begin{aligned} f_n(\zeta_1(t)) \;=\; & f_n(\zeta_1(0)) + c\int_0^t f_n''(\zeta_1(s-))\zeta_1(s-)^2 \mathrm{d}s - b\int_0^t f_n'(\zeta_1(s-))\zeta_1(s-)\mathrm{d}s \\ & - \alpha^{-1}(\alpha-1)\Gamma(2-\alpha)^{-1}\sqrt[\alpha]{\alpha q}\int_0^t f_n'(\zeta_1(s-))\zeta_\alpha(s-)\mathrm{d}s \\ & + \int_0^t ds\int_0^1 \Big[f_n(\zeta_1(s-) + \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)z) - f_n(\zeta_1(s-)) \\ & - \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)z f_n'(\zeta_1(s-))\Big]\gamma(\mathrm{d}z) + \text{local mart.} \end{aligned} \tag{8.8}$$

For any $k \geq 1 + x(0) \vee y(0)$ let $\tau_k = \inf\{s \geq 0 : x(s) \geq k$ or $y(s) \geq k\}$. For $0 \leq t \leq \tau_k$ we have $|\zeta_1(t-)| \leq k$, $|\zeta_\alpha(t-)| \leq \sqrt[\alpha]{k}$ and

$$|\zeta_1(t)| \leq |\zeta_1(t-)| + |\zeta_1(t) - \zeta_1(t-)| \leq k + \sqrt[\alpha]{\alpha q k}.$$

By Taylor's expansion, there exists $0 < \xi < z$ so that

$$\begin{aligned} & [f_n(\zeta_1(s-) + \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)z) - f_n(\zeta_1(s-)) - \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)z f_n'(\zeta_1(s-))] \\ & \quad = 2^{-1}(\alpha q)^{2/\alpha} f_n''(\zeta_1(s-) + \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)\xi)\zeta_\alpha(s-)^2 z^2 \\ & \quad \leq 2^{-1}(\alpha q)^{2/\alpha}\sqrt[\alpha]{k} f_n''(\zeta_1(s-) + \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)\xi)|\zeta_1(s-) + \sqrt[\alpha]{\alpha q}\zeta_\alpha(s-)\xi| z^2 \\ & \quad \leq n^{-1}(\alpha q)^{2/\alpha}\sqrt[\alpha]{k} z^2, \end{aligned}$$

where we have used (8.2) and the fact $\zeta_1(s-)\zeta_\alpha(s-) \geq 0$. Taking the expectation in both sides of (8.8) gives

$$\begin{aligned} \mathbf{P}[f_n(\zeta_1(t\wedge\tau_k))] \;\leq\; & \mathbf{P}[f_n(\zeta_1(0))] + |b|\int_0^t \mathbf{P}[|\zeta_1(s\wedge\tau_k)|]\mathrm{d}s + 2cn^{-1}kt \\ & + n^{-1}(\alpha q)^{2/\alpha}\sqrt[\alpha]{k}\int_0^t \mathrm{d}s\int_0^1 z^2\gamma(\mathrm{d}z). \end{aligned}$$

Now, if $x(0) = y(0)$, we can let $n \to \infty$ in the inequality above to get

$$\mathbf{P}[|x(t\wedge\tau_k) - y(t\wedge\tau_k)|] \leq |b|\int_0^t \mathbf{P}[|x(s\wedge\tau_k) - y(s\wedge\tau_k)|]\mathrm{d}s.$$

Then $\mathbf{P}[|x(t\wedge\tau_k) - y(t\wedge\tau_k)|] = 0$ for $t \geq 0$ by Gronwall's inequality. By letting $k \to \infty$ and using Fatou's lemma we obtain the pathwise uniqueness for (8.6). □

Example 8.1 The stochastic integral equation (8.5) can be thought as a continuous time-space counterpart of the definition (5.1) of the GWI-process. In fact, assuming $\mu = \mathbf{E}(\xi_{1,1}) < \infty$, from (5.1) we have

$$y(n) - y(n-1) = \sum_{i=1}^{y(n-1)} (\xi_{n,i} - \mu) - (1-\mu)y(n-1) + \eta_n. \tag{8.9}$$

It follows that

$$y(n) - y(0) = \sum_{k=1}^{n} \sum_{i=1}^{y(k-1)} (\xi_{k,i} - \mu) - (1-\mu) \sum_{k=1}^{n} y(k-1) + \sum_{k=1}^{n} \eta_k. \tag{8.10}$$

The exact continuous time-state counterpart of (8.10) would be the stochastic integral equation

$$y(t) = y(0) + \int_0^t \int_0^\infty \int_0^{y(s-)} \xi \tilde{M}(\mathrm{d}s, \mathrm{d}\xi, \mathrm{d}u) - \int_0^t by(s)\mathrm{d}s + \eta(t), \tag{8.11}$$

which is a typical special form of (8.5); see Bertoin and Le Gall (2006) and Dawson and Li (2006). Here the ξ's selected by the Poisson random measure $M(\mathrm{d}s, \mathrm{d}\xi, \mathrm{d}u)$ are distributed in a i.i.d. fashion and the compensation of the measure corresponds to the centralization in (8.10). The increasing Lévy process $t \mapsto \eta(t)$ in (8.11) corresponds to the increasing random walk $n \mapsto \sum_{k=1}^n \eta_k$ in (8.10). The additional term in (8.5) involving the stochastic integral with respect to the Gaussian white noise is just a continuous time-space parallel of that with respect to the compensated Poisson random measure.

Example 8.2 The stochastic differential equation (8.6) captures the structure of the CBI-process in a typical special case. Let $1 < \alpha \leq 2$. Under the condition $\mu := \mathbf{E}(\xi_{1,1}) < \infty$, from (8.9) we have

$$y(n) - y(n-1) = \sqrt[\alpha]{y(n-1)} \sum_{i=1}^{y(n-1)} \frac{\xi_{n,i} - \mu}{\sqrt[\alpha]{y(n-1)}} - (1-\mu)y(n-1) + \eta_n.$$

Observe that the partial sum on the right-hand side corresponds to a one-sided α-stable type central limit theorem. Then a continuous time-state counterpart of the above equation would be

$$\mathrm{d}y(t) = \sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}z(t) - by(t)\mathrm{d}t + \beta\mathrm{d}t, \qquad t \geq 0, \tag{8.12}$$

where $\{z(t) : t \geq 0\}$ is a standard Brownian motion if $\alpha = 2$ and a spectrally positive α-stable Lévy process with Lévy measure $(\alpha-1)\Gamma(2-\alpha)^{-1}z^{-1-\alpha}\mathrm{d}z$ if $1 < \alpha < 2$. This is a typical special form of (8.6).

Example 8.3 When $\alpha = 2$ and $\beta = 0$, the CB-process defined by (8.12) is a diffusion process, which is known as *Feller's branching diffusion*. This process was first studied by Feller (1951).

Example 8.4 In the special case of $\alpha = 2$, the CBI-process defined by (8.12) is known in mathematical finance as the *Cox–Ingersoll–Ross model* (CIR-model), which was used by Cox et al. (1985) to describe the evolution of interest rates. The asymptotic behavior of the estimators of the parameters in the CIR-model was studied by Overbeck and Rydén (1997). In the general case, the solution to (8.12) is called a *α-stable Cox–Ingersoll–Ross model* (α-stable CIR-model); see, e.g., Jiao et al. (2017) and Li and Ma (2015).

As a simple application of the stochastic equation (8.1) or (8.5), we can give a simple derivation of the joint Laplace transform of the CBI-process and its positive integral functional. The next theorem extends the results in Section 4.

Theorem 8.8 *Let $Y = (\Omega, \mathscr{F}, \mathscr{F}_t, y(t), \mathbf{P}_x)$ be a Hunt realization of the CBI-process. Then for $t, \lambda, \theta \geq 0$ we have*

$$\mathbf{P}_x \exp\bigg\{ -\lambda y(t) - \theta \int_0^t y(s)\mathrm{d}s \bigg\} = \exp\bigg\{ -xv(t) - \int_0^t \psi(v(s))\mathrm{d}s \bigg\},$$

where $t \mapsto v(t) = v(t, \lambda, \theta)$ is the unique positive solution to (4.14).

Proof. We can construct the process $\{y(t) : t \geq 0\}$ as the solution to (8.1) or (8.5) with $y(0) = x \geq 0$. Let

$$z(t) = \int_0^t y(s)\mathrm{d}s, \qquad t \geq 0.$$

Consider a function $G = G(t, y, z)$ on $[0, \infty)^3$ with bounded continuous derivatives up to the first order relative to $t \geq 0$ and $z \geq 0$ and up to the second order relative to $x \geq 0$. By Itô's formula,

$$\begin{aligned} G(t, y(t), z(t)) \;=\; & G(0, y(0), 0) + \text{local mart.} + \int_0^t \bigg\{ G_t'(s, y(s), z(s)) \\ & + y(s)G_z'(s, y(s), z(s)) + [\beta - by(s)]G_y'(s, y(s), z(s)) \\ & + cy(s)G_{yy}''(s, y(s), z(s)) + y(s) \int_0^\infty \Big[G(s, y(s) + z, z(s)) \\ & - G(s, y(s), z(s)) - zG_y'(s, y(s), z(s)) \Big] m(\mathrm{d}z) \bigg\} \mathrm{d}s \\ & + \int_0^t \mathrm{d}s \int_0^\infty \Big[G(s, y(s) + z, z(s)) - G(s, y(s), z(s)) \Big] \nu(\mathrm{d}z). \end{aligned}$$

We can apply the above formula to the function

$$G_T(t, y, z) = \exp\bigg\{ -v(T-t)x - \theta z - \int_0^{T-t} \psi(v(s))\mathrm{d}s \bigg\}.$$

Using (4.14) we see $t \mapsto G_T(t \wedge T, y(t \wedge T), z(t \wedge T))$ is a local martingale, and hence a martingale by the boundedness. From the relation $\mathbf{P}_x[G_T(t, y(t), z(t))] = G_T(0, x, 0)$ with $T = t$ we get the desired result. □

The existence and uniqueness of strong solution to (8.1) were first established in Dawson and Li (2006). The moment condition (5.11) was removed in Fu and Li (2010). A stochastic flow of discontinuous CB-processes with critical branching mechanism was constructed in Bertoin and Le Gall (2006) by using weak solutions of a special case of (8.1). The existence and uniqueness of strong solution to (8.6) were proved in Fu and Li (2010) and those for (8.5) were given in Dawson and Li (2012) and Li and Ma (2008). The results of Bertoin and Le Gall (2006) were extended to flows of CBI-processes in Dawson and Li (2012) and Li (2014) using strong solutions. Although the study of branching processes has a long history, the stochastic equations (8.1), (8.5) and (8.6) were not established until the works mentioned above.

A natural generalization of the CBI-process is the so-called *affine Markov process*; see Duffie et al. (2003) and the references therein. Those authors defined the regularity property of affine processes and gave a number of characterizations of those processes under the regularity assumption. By a result of Kawazu and Watanabe (1971), a stochastically continuous CBI-process is automatically regular. Under the first moment assumption, the regularity of affine processes was proved in Dawson and Li (2006). The regularity problem was settled in Keller-Ressel et al. (2011), where it was proved that any stochastically continuous affine process is regular. This problem is related to Hilbert's fifth problem; see Keller-Ressel et al. (2011) for details.

9 Local and global maximal jumps

In this section, we use stochastic equations of the CB- and CBI-processes to derive several characterizations of the distributions of their local and global maximal jumps. Let us consider a branching mechanism ϕ given by (2.13). Let $\{B(t)\}$ be a standard Brownian motion and $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ a Poisson time-space random measure on $(0, \infty)^3$ with intensity $\mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$. By Theorem 8.2, for any $x \geq 0$, there is a pathwise unique positive strong solution to

$$\begin{aligned} x(t) \;=\; & x + \int_0^t \sqrt{2cx(s-)}\mathrm{d}B(s) - b\int_0^t x(s-)\mathrm{d}s \\ & + \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{9.1}$$

By Theorem 8.1, the solution $\{x(t) : t \geq 0\}$ is a CB-process with branching mechanism ϕ. For $t \geq 0$ and $r > 0$ let

$$N_r(t) = \int_0^t \int_r^\infty \int_0^{x(s-)} M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

which denotes the number of jumps with sizes in (r, ∞) of the trajectory $t \mapsto x(t)$ on the interval $(0, t]$. By (3.5) we have

$$\mathbf{P}[N_r(t)] = m(r, \infty)\mathbf{P}\bigg[\int_0^t x(s)\mathrm{d}s\bigg] = xb^{-1}(1 - \mathrm{e}^{-bt})m(r, \infty),$$

where $b^{-1}(1 - \mathrm{e}^{-bt}) = t$ for $b = 0$ by convention. In particular, we have $\mathbf{P}\{N_r(t) < \infty\} = 1$. For $r > 0$ we can define another branching mechanism by

$$\phi_r(z) = b_r z + cz^2 + \int_0^r (\mathrm{e}^{-zu} - 1 + zu)m(\mathrm{d}u), \tag{9.2}$$

where

$$b_r = b + \int_r^\infty u m(\mathrm{d}u).$$

For $\theta \geq 0$ let $t \mapsto u(t, \theta)$ be the unique positive solution to (4.17). Let $t \mapsto u_r(t, \theta)$ be the unique positive solution to

$$\frac{\partial}{\partial t}u(t, \theta) = \theta - \phi_r(u(t, \theta)), \quad u(0, \theta) = 0. \tag{9.3}$$

The following theorem gives a characterization of the distribution of the local maximal jump of the CB-process:

Theorem 9.1 *Let $\Delta x(t) = x(t) - x(t-)$ for $t \geq 0$. Then for any $r > 0$ we have*

$$\mathbf{P}_x\Big\{\max_{0<s\leq t} \Delta x(s) \leq r\Big\} = \exp\{-xu_r(t)\},$$

where $u_r(t) = u_r(t, m(r, \infty))$.

Proof. Let $M_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $M^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote the restrictions of $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ to $(0, \infty) \times (0, r] \times (0, \infty)$ and $(0, \infty) \times (r, \infty) \times (0, \infty)$, respectively. We can rewrite (9.1) into

$$\begin{aligned} x(t) \;&=\; x + \int_0^t \sqrt{2cx(s-)}\mathrm{d}B(s) + \int_0^t \int_0^r \int_0^{x(s-)} z\tilde{M}_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &\quad - \int_0^t b_r x(s-)\mathrm{d}s + \int_0^t \int_r^\infty \int_0^{x(s-)} zM^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

where the last term collects the jumps with sizes in (r, ∞) of $\{x(t)\}$. Let $\{x_r(t)\}$ be the unique positive strong solution to

$$\begin{aligned} x_r(t) \;&=\; x - \int_0^t b_r x_r(s-)\mathrm{d}s + \int_0^t \sqrt{2cx_r(s-)}\mathrm{d}B(s) \\ &\quad + \int_0^t \int_0^r \int_0^{x_r(s-)} z\tilde{M}_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

Then $\{x_r(t)\}$ is a CB-process with branching mechanism ϕ_r. Let $\tau_r = \inf\{s \geq 0 : \Delta x(s) > r\}$. We have $x_r(s) = x(s)$ for $0 \leq s < \tau_r$ and

$$\begin{aligned} \Big\{\max_{0<s\leq t} \Delta x(s) \leq r\Big\} \;&=\; \Big\{\int_0^t \int_r^\infty \int_0^{x(s-)} M^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = 0\Big\} \\ &=\; \Big\{\int_0^t \int_r^\infty \int_0^{x_r(s-)} M^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = 0\Big\}. \end{aligned}$$

Since the strong solution $\{x_r(t)\}$ is progressively measurable with respect to the filtration generated by $\{B(t)\}$ and $\{M_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$, it is independent of $\{M^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$. Then $\{M^r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ is still a Poisson random measure conditionally upon $\{x_r(t)\}$. It follows that

$$\mathbf{P}_x\Big\{\max_{0<s\leq t} \Delta x(s) \leq r\Big\} = \mathbf{P}_x\bigg[\exp\bigg\{-m(r, \infty)\int_0^t x_r(s)\mathrm{d}s\bigg\}\bigg].$$

Then the desired result follows by Corollary 4.4. □

Corollary 9.2 *Suppose that the measure $m(\mathrm{d}u)$ has unbounded support. Then we have, as $r \to \infty$,*

$$\mathbf{P}_x\Big\{\max_{0<s\leq t} \Delta x(s) > r\Big\} \sim xb^{-1}(1 - \mathrm{e}^{-bt})m(r, \infty).$$

Proof. Recall that $t \mapsto u(t, \theta)$ is defined by (4.17) and $t \mapsto u_r(t, \theta)$ is defined by (9.3). It is easy to see that $u(t, 0) = u_r(t, 0) = 0$. Moreover, by (4.17) we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \theta}u(t, 0) = 1 - b\frac{\partial}{\partial \theta}u(t, 0), \quad \frac{\partial}{\partial \theta}u(0, 0) = 0.$$

We can solve the above equation to get

$$\frac{\partial}{\partial\theta}u(t,0) = b^{-1}(1 - \mathrm{e}^{-bt}), \tag{9.4}$$

where $b^{-1}(1-\mathrm{e}^{-bt}) = t$ for $b = 0$ by convention. Similarly we have

$$\frac{\partial}{\partial\theta}u_r(t,0) = b_r^{-1}(1 - \mathrm{e}^{-b_r t}). \tag{9.5}$$

By Theorem 9.1 it follows that

$$\mathbf{P}_x\Big\{\max_{0<s\le t}\Delta x(s) > r\Big\} = 1 - \exp\{-xu_r(t, m(r,\infty))\},$$

For $r > q > 0$, we have obviously $\phi \le \phi_r \le \phi_q$. By Corollary 4.6 we see

$$u_q(t, m(r,\infty)) \le u_r(t, m(r,\infty)) \le u(t, m(r,\infty)).$$

It follows that

$$\begin{aligned} 1 - \exp\{-xu_q(t, m(r,\infty))\} \ &\le\ \mathbf{P}_x\Big\{\max_{0<s\le t}\Delta x(s) > r\Big\} \\ &\le\ 1 - \exp\{-xu(t, m(r,\infty))\}. \end{aligned}$$

By (9.4) and (9.5), as $r \to \infty$ we have

$$\begin{aligned} 1 - \exp\{-xu(t, m(r,\infty))\} \ &\sim\ xu(t, m(r,\infty)) \\ &\sim\ xb^{-1}(1 - \mathrm{e}^{-bt})m(r,\infty), \end{aligned}$$

and

$$\begin{aligned} 1 - \exp\{-xu_q(t, m(r,\infty))\} \ &\sim\ xu_q(t, m(r,\infty)) \\ &\sim\ xb_q^{-1}(1 - \mathrm{e}^{-b_q t})m(r,\infty). \end{aligned}$$

The proof is completed as we notice $\lim_{q\to\infty} b_q = b$. □

We can also give some characterizations of the global maximal jump of the CB-process. Let $\phi_r^{-1}(\theta) := \inf\{z \ge 0 : \phi_r(z) > \theta\}$ for $\theta \ge 0$. It is easy to see that $\phi_r^{-1}(m(r,\infty)) \to 0$ as $r \to \infty$ if and only if $b \ge 0$. Let $\phi'(\infty)$ be given by (3.2). By Theorems 4.8 and 9.1 we have:

Corollary 9.3 *Suppose that $\phi'(\infty) > 0$. Then for any $r > 0$ with $m(r,\infty) > 0$ we have*

$$\mathbf{P}_x\Big\{\sup_{s>0}\Delta x(s) \le r\Big\} = \exp\{-x\phi_r^{-1}(m(r,\infty))\}.$$

Corollary 9.4 *Suppose that $b > 0$ and the measure $m(\mathrm{d}u)$ has unbounded support. Then as $r \to \infty$ we have*

$$\mathbf{P}_x\Big\{\sup_{s>0}\Delta x(s) > r\Big\} \sim xb^{-1}m(r,\infty).$$

The results on local maximal jumps obtained above can be generalized to the case of a CBI-process. Let (ϕ, ψ) be the branching and immigration mechanisms given respectively by (2.13) and (5.9) with $\nu(\mathrm{d}u)$ satisfying (5.11). Let $\{y(t) : t \geq 0\}$ be the CBI-process defined by (8.1) with $y(0) = x \geq 0$. For $r > 0$ let

$$\psi_r(z) = \beta z + \int_0^r (1 - \mathrm{e}^{-zu})\nu(\mathrm{d}u).$$

Based on Theorem 8.8, the following theorem can be proved by modifying the arguments in the proof of Theorem 9.1:

Theorem 9.5 *Let $\Delta y(t) = y(t) - y(t-)$ for $t \geq 0$. Then for any $r > 0$ we have*

$$\begin{aligned}
&\mathbf{P}_x\Big\{\max_{0<s\leq t} \Delta y(s) \leq r\Big\} \\
&\quad = \exp\Big\{ - xu_r(t) - \nu(r,\infty)t - \int_0^t \psi_r(u_r(s))\mathrm{d}s\Big\},
\end{aligned}$$

where $u_r(t) = u_r(t, m(r,\infty))$.

The results given in this section were adopted from He and Li (2016). We refer the reader to Bernis and Scotti (2020) and Jiao et al. (2017) for more careful analysis of the jumps of CBI-processes. In particular, the distributions of the numbers of large jumps in intervals were characterized in Jiao et al. (2017). The analysis is important for the study in mathematical finance as it allows one to describe in a unified way several recent observations on the bond markets such as the persistency of low interest rates together with the presence of large jumps.

10 A coupling of CBI-processes

In this section, we give some characterizations of a coupling of CBI-processes constructed by the stochastic equation (8.5). Using this coupling we prove the strong Feller property and the exponential ergodicity of the CBI-process under suitable conditions. We shall follow the arguments of Li and Ma (2015). Suppose that (ϕ, ψ) are the branching and immigration mechanisms given respectively by (2.13) and (5.9) with $\nu(\mathrm{d}u)$ satisfying (5.11). Let $(P_t)_{t\geq 0}$ be the transition semigroup of the corresponding CBI-process defined by (2.19) and (5.10).

Theorem 10.1 *If* $\{x(t) : t \geq 0\}$ *and* $\{y(t) : t \geq 0\}$ *are positive solutions to* (8.5) *with* $\mathbf{P}\{x(0) \leq y(0)\} = 1$*, then* $\{y(t) - x(t) : t \geq 0\}$ *is a CB-process with branching mechanism* ϕ*.*

Proof. By Theorem 8.4 we have $\mathbf{P}\{x(t) \leq y(t) \text{ for all } t \geq 0\} = 1$. Let $z(t) = y(t)-x(t)$. From (8.5) we have

$$\begin{aligned} z(t) &= z(0) + \sqrt{2c}\int_0^t \int_{x(s-)}^{y(s-)} W(\mathrm{d}s, \mathrm{d}u) - b\int_0^t z(s-)\mathrm{d}s \\ &\quad + \int_0^t \int_0^\infty \int_{x(s-)}^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &= z(0) + \sqrt{2c}\int_0^t \int_0^{z(s-)} W(\mathrm{d}s, x(s-) + \mathrm{d}u) - b\int_0^t z(s-)\mathrm{d}s \\ &\quad + \int_0^t \int_0^\infty \int_0^{z(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, x(s-) + \mathrm{d}u), \end{aligned}$$

where $W(\mathrm{d}s, x(s-) + \mathrm{d}u)$ is a Gaussian time-space white noise with intensity $\mathrm{d}s\mathrm{d}u$ and $M(\mathrm{d}s, \mathrm{d}z, x(s-)+\mathrm{d}u)$ is a Poisson time-space random measure with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$. That shows $\{z(t)\}$ is a weak solution to (8.5) with $\eta(t) \equiv 0$. Then it is a CB-process with branching mechanism ϕ. □

For $x \geq 0$ and $y \geq 0$, let $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ be the positive strong solutions to (8.5) with $x(0) = x$ and $y(0) = y$. This construction gives a natural *coupling* of the CBI-processes. Let $\tau(x, y) = \inf\{t \geq 0 : x(t) = y(t)\}$ be the *coalescence time* of the coupling. The distribution of this stopping time is given in the following theorem.

Theorem 10.2 *Suppose that Condition 3.5 holds. Then for any* $t \geq 0$ *we have*

$$\mathbf{P}\{\tau(x, y) \leq t\} = \mathbf{P}\{y(t) = x(t)\} = \exp\{-|x - y|\bar{v}_t\}, \tag{10.1}$$

where $t \mapsto \bar{v}_t$ *is the unique solution to* (3.19) *with singular initial condition* $\bar{v}_{0+} = \infty$*.*

Proof. It suffices to consider the case of $y \geq x \geq 0$. By Theorem 10.1 the difference $\{y(t) - x(t) : t \geq 0\}$ is a CB-process with branching mechanism ϕ. By Theorem 3.4 the probability $\mathbf{P}\{\tau(x, y) \leq t\} = \mathbf{P}\{y(t) = x(t)\}$ is given by (10.1). □

Theorem 10.3 *Suppose that Condition 3.5 holds. Then for $t > 0$ and $x, y \geq 0$ we have*

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\mathrm{var}} \leq 2(1 - \mathrm{e}^{-\bar{v}_t|x-y|}) \leq 2\bar{v}_t|x - y|, \tag{10.2}$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the total variation norm.

Proof. Let $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ be given as above. Since $\{y(t) - x(t) : t \geq 0\}$ is a CB-process with branching mechanism ϕ, for any bounded Borel function f on $[0, \infty)$, we have

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= \big|\mathbf{P}[f(x(t))] - \mathbf{P}[f(y(t))]\big| \\
&\leq \mathbf{P}\big[|f(x(t)) - f(y(t))| 1_{\{y(t) \neq x(t)\}}\big] \\
&\leq \mathbf{P}\big[(|f(x(t))| + |f(y(t))|) 1_{\{y(t) \neq x(t)\}}\big] \\
&\leq 2\|f\| \mathbf{P}\{y(t) - x(t) \neq 0\} \\
&= 2\|f\|(1 - \mathrm{e}^{-\bar{v}_t|x-y|}).
\end{aligned}$$

where the last equality follows by Theorem 10.2. Then we get (10.2) by taking the supremum over f with $\|f\| \leq 1$. □

By Theorem 10.3, for each $t > 0$ the operator P_t maps any bounded Borel function on $[0, \infty)$ into a bounded continuous function, that is, the transition semigroup $(P_t)_{t \geq 0}$ satisfies the *strong Feller property*.

Theorem 10.4 *Suppose that $b > 0$. Then the transition semigroup $(P_t)_{t \geq 0}$ has a unique stationary distribution η given by*

$$L_\eta(\lambda) = \exp\bigg\{ - \int_0^\infty \psi(v_s(\lambda))\mathrm{d}s \bigg\} = \exp\bigg\{ - \int_0^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z \bigg\}, \tag{10.3}$$

and $P_t(x, \cdot) \to \eta$ weakly on $[0, \infty)$ as $t \to \infty$ for every $x \geq 0$. Moreover, we have

$$\int_{[0,\infty)} y \eta(\mathrm{d}y) = b^{-1}\psi'(0), \tag{10.4}$$

where $\psi'(0)$ is given by (5.13)*.*

Proof. Since $b > 0$, we have $\phi(z) \geq 0$ for all $z \geq 0$, so $t \mapsto v_t(\lambda)$ is decreasing. By Corollary 3.2 we have $\lim_{t \to \infty} v_t(\lambda) = 0$. From (3.6) it follows that

$$\int_0^t \psi(v_s(\lambda))\mathrm{d}s = \int_{v_t(\lambda)}^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z.$$

In view of (5.10), we have

$$\begin{aligned}
\lim_{t \to \infty} \int_{[0,\infty)} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) &= \exp\bigg\{ - \int_0^\infty \psi(v_s(\lambda))\mathrm{d}s \bigg\} \\
&= \exp\bigg\{ - \int_0^\lambda \frac{\psi(z)}{\phi(z)} \mathrm{d}z \bigg\}.
\end{aligned}$$

By Theorem 1.2 there is a probability measure η on $[0, \infty)$ defined by (10.3). It is easy to show that η is the unique stationary distribution for $(P_t)_{t \geq 0}$. The expression (10.4) for its first moment follows by differentiating both sides of (10.3) at $\lambda = 0$. □

Theorem 10.5 *Suppose that $b > 0$ and Condition 3.5 holds. Then for any $x \geq 0$ and $t \geq r > 0$ we have*

$$\|P_t(x,\cdot) - \eta(\cdot)\|_{\mathrm{var}} \leq 2[x + b^{-1}\psi'(0)]\bar{v}_r\, \mathrm{e}^{b(r-t)}, \tag{10.5}$$

where η is given by (10.3).

Proof. Since η is a stationary distribution for $(P_t)_{t\geq 0}$, by Theorem 10.3 one can see

$$\begin{aligned} \|P_t(x,\cdot) - \eta(\cdot)\|_{\mathrm{var}} &= \bigg\| \int_{[0,\infty)} [P_t(x,\cdot) - P_t(y,\cdot)]\eta(\mathrm{d}y) \bigg\|_{\mathrm{var}} \\ &\leq \int_{[0,\infty)} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{var}} \eta(\mathrm{d}y) \\ &\leq 2\bar{v}_t \int_{[0,\infty)} |x - y| \eta(\mathrm{d}y) \\ &\leq 2\bar{v}_t \int_{[0,\infty)} (x + y) \eta(\mathrm{d}y) \\ &= 2[x + b^{-1}\psi'(0)]\bar{v}_t, \end{aligned}$$

where the last equality follows by (10.4). The semigroup property of $(v_t)_{t\geq 0}$ implies $\bar{v}_t = v_{t-r}(\bar{v}_r)$ for any $t \geq r > 0$. By (3.4) we see $\bar{v}_t = v_{t-r}(\bar{v}_r) \leq \mathrm{e}^{b(r-t)}\, \bar{v}_r$. Then (10.5) holds. □

The result of Theorem 10.1 was used to construct flows of CBI-processes in Dawson and Li (2012). Clearly, the right-hand side of (10.5) decays exponentially fast as $t \to \infty$. This property is called the *exponential ergodicity* of the transition semigroup $(P_t)_{t\geq 0}$. It has played an important role in the study of asymptotics of the estimators for the α-stable CIR-model in Li and Ma (2015).

References

Aliev, S.A. (1985): A limit theorem for the Galton–Watson branching processes with immigration. *Ukrainian Math. J.* **37**, 535–438.

Aliev, S.A. and Shchurenkov, V.M. (1982): Transitional phenomena and the convergence of Galton–Watson processes to Jiřina processes. *Theory Probab. Appl.* **27**, 472–485.

Athreya, K.B. and Ney, P.E. (1972): *Branching Processes*. Springer, Berlin.

Bernis, G. and Scotti, S. (2020): *Clustering effects through Hawkes processes*. A Chapter in: *From Probability to Finance, Mathematical Lectures from Peking University*, pp. 145–181. Edited by Y. Jiao. Springer, Singapore.

Bertoin, J. and Le Gall, J.-F. (2006): Stochastic flows associated to coalescent processes III: Limit theorems. *Illinois J. Math.* **50**, 147–181.

Bienaymé, I.J. (1845): De la loi de multiplication et de la durée des families. *Soc. Philomat. Paris Extraits* **5**, 37–39.

Chung, K.L. (1982). *Lectures from Markov Processes to Brownian Motion*. Springer, Heidelberg.

Cox, J.; Ingersoll, J. and Ross, S. (1985): A theory of the term structure of interest rate. *Econometrica* **53**, 385–408.

Dawson, D.A. and Li, Z. (2003): Construction of immigration superprocesses with dependent spatial motion from one-dimensional excursions. *Probab. Theory Related Fields* **127**, 37–61.

Dawson, D.A. and Li, Z. (2006): Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34**, 1103–1142.

Dawson, D.A. and Li, Z. (2012): Stochastic equations, flows and measure-valued processes. *Ann. Probab.* **40**, 813–857.

Dellacherie, C. and Meyer, P.A. (1982): *Probabilities and Potential*. Chapters V–VIII. North-Holland, Amsterdam.

Duffie, D.; Filipović, D. and Schachermayer, W. (2003): Affine processes and applications in finance. *Ann. Appl. Probab.* **13**, 984–1053.

Duquesne, T. and Labbé, C. (2014): On the Eve property for CSBP. *Electron. J. Probab.* **19**, Paper No. 6, 1–31.

El Karoui, N. and Méléard, S. (1990): Martingale measures and stochastic calculus. *Probab. Theory Related Fields* **84**, 83–101.

El Karoui, N. and Roelly, S. (1991): Propriétés de martingales, explosion et representation de Lévy–Khintchine d'une classe de processus de branchement à valeurs mesures. *Stochastic Process. Appl.* **38**, 239–266.

Ethier, S.N. and Kurtz, T.G. (1986): *Markov Processes: Characterization and Convergence*. Wiley, New York.

Feller, W. (1951): Diffusion processes in genetics. In: *Proceedings 2nd Berkeley Symp. Math. Statist. Probab.*, 1950, 227–246. Univ. of California Press, Berkeley and Los Angeles.

Fu, Z. and Li, Z. (2010): Stochastic equations of nonnegative processes with jumps. *Stochastic Process. Appl.* **120**, 306–330.

Galton, F. and Watson, H.W. (1874): On the probability of the extinction of families. *J. Anthropol. Inst. Great Britain and Ireland* **4**, 138–144.

Grey, D.R. (1974): Asymptotic behaviour of continuous time, continuous state-space branching processes, *J. Appl. Probab.* **11**, 669–677.

Grimvall, A. (1974): On the convergence of sequences of branching processes. *Ann. Probab.* **2**, 1027–1045.

Harris, T.E. (1963): *The Theory of Branching Processes*. Springer, Berlin.

He, X. and Li, Z. (2016): Distributions of jumps in a continuous-state branching process with immigration. *J. Appl. Probab.* **53**, 1166–1177.

Hewitt, E. and Stromberg, K. (1965): *Real and Abstract Analysis*. Springer, Heidelberg.

Ikeda, N. and Watanabe, S. (1989): *Stochastic Differential Equations and Diffusion Processes*. 2nd Ed. North-Holland, Amsterdam; Kodansha, Tokyo.

Jacod, J. and Shiryaev, A.N. (2003): *Limit Theorems for Stochastic Processes*. 2nd Ed. Springer, Heidelberg.

Jiao, Y.; Ma, C. and Scotti, S. (2017): Alpha-CIR model with branching processes in sovereign interest rate modeling. *Finance Stochastics* **21**, 789–813.

Kawazu, K. and Watanabe, S. (1971): Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36–54.

Keller-Ressel, M.; Schachermayer, W. and Teichmann, J. (2011): Affine processes are regular. *Probab. Theory Related Fields* **151**, 591–611.

Kyprianou, A.E. (2014): *Fluctuations of Lévy Processes with Applications*. 2nd Ed. Springer, Heidelberg.

Lambert, A. (2005): The branching process with logistic growth. *Ann. Appl. Probab.* **15**, 1506–1535.

Lamberton, D. and Lapeyre, B. (1996): *Introduction to Stochastic Calculus Applied to Finance*. Chapman and Hall, London.

Lamperti, J. (1967a): The limit of a sequence of branching processes. *Z. Wahrsch. verw. Geb.* **7**, 271–288.

Lamperti, J. (1967b): Continuous state branching processes. *Bull. Amer. Math. Soc.* **73**, 382–386.

Li, Z. (1996): Immigration structures associated with Dawson–Watanabe superprocesses. *Stochastic Process. Appl.* **62**, 73–86.

Li, Z. (2000): Asymptotic behavior of continuous time and state branching processes. *J. Austral. Math. Soc. Ser. A* **68**, 68–84.

Li, Z. (2003): Skew convolution semigroups and related immigration processes. *Theory Probab. Appl.* **46**, 274–296.

Li, Z. (2006): A limit theorem for discrete Galton–Watson branching processes with immigration. *J. Appl. Probab.* **43**, 289–295.

Li, Z. (2011): *Measure-Valued Branching Markov Processes*. Springer, Heidelberg.

Li, Z. (2014): Path-valued branching processes and nonlocal branching superprocesses. *Ann. Probab.* **42**, 41–79.

Li, Z. and Ma, C. (2008): Catalytic discrete state branching models and related limit theorems. *J. Theoret. Probab.* **21**, 936–965.

Li, Z. and Ma, C. (2015): Asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model. *Stochastic Process. Appl.* **125**, 3196–3233.

Li, Z. and Shiga, T. (1995): Measure-valued branching diffusions: Immigrations, excursions and limit theorems. *J. Math. Kyoto Univ.* **35**, 233–274.

Overbeck, L. and Rydén, T. (1997): Estimation in the Cox–Ingersoll–Ross model. *Econometric Theory* **13**, 430–461.

Pardoux, E. (2016): *Probabilistic Models of Population Evolution: Scaling Limits, Genealogies and Interactions*. Springer, Switzerland.

Parthasarathy, K.R. (1967): *Probability Measures on Metric Spaces*. Academic Press, New York.

Pitman, J. and Yor, M. (1982): A decomposition of Bessel bridges. *Z. Wahrsch. verw. Geb.* **59**, 425–457.

Situ, R. (2005): *Theory of Stochastic Differential Equations with Jumps and Applications*. Springer, Heidelberg.

Walsh, J.B. (1986): An introduction to stochastic partial differential equations. In: *Ecole d'Eté de Probabilités de Saint-Flour* XIV-1984, 265–439. Lecture Notes Math. **1180**. Springer, Berlin.

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