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★**Measure-valued branching Markov processes.**
Probability and its Applications (New York).
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The super-Brownian motion, or Dawson-Watanabe process, is the fundamental measure-valued random process. With close links to spatially distributed branching models, research on the Dawson-Watanabe process and related topics has been a tremendously active field for several decades. With a blend of many mathematical topics that involve classical branching and diffusion processes, nonlinear partial differential equations, functional analysis, random trees, martingale theory, and stochastic differential equations, as well as many interesting connections to applied models in population biology, interacting systems, etc., this area keeps attracting both established researchers and newcomers to the field. Today general superprocesses and measure-valued branching processes represent a very rich class of stochastic models for which a diverse collection of methods and techniques have been developed and many deep results are known.

Several monographs and lecture notes devoted to measure-valued processes have appeared over the past years which summarize and expand the current state of knowledge, and these have been highly appreciated by the research community. Yet the present book appears to be the first monograph in textbook format that provides a rigorous treatment of general theory for a wide class of measure-valued processes. The approach is analytic rather than probabilistic, in the sense that the basic existence results are obtained as unique solutions of the relevant integral evolution equations. Only afterwards are the various parameters given their interpretations in terms of spatial motion and branching, the two basic mechanisms of superprocesses. One chapter of the book focuses on the intuitive interpretations of superprocesses by giving limit results for branching particle systems under rescaling. It is probably fair to say, however, that it is not a main priority in the book to provide probabilistic training and insight, but rather to supply powerful and general methods for existence, regularity and construction results. A selection of good examples throughout the book helps clarify the results.

The first part of the book develops in a general setting the analysis of log-Laplace functionals and cumulant equations for measure-valued Markov branching processes and the corresponding particle models. The technical setting of the general model allows for spatial motions to be Borel right processes in Lusin topological space and for general branching mechanisms, not necessarily decomposable in local and non-local parts. These assumptions are shown to be convenient as the regularity properties are then essentially inherited by the resulting superprocesses. A number of variations of these models are studied, such as occupation times and multitype models, and further tools and results are developed in detail including Girsanov transformation and martingale problems for superprocesses.

The main theme of the second part of the book is immigration models. In comparison to earlier accounts of measure-valued processes, where immigration mechanisms tend to be peripheral, this book argues that the addition of immigration from outside sources is of great importance and physical appeal. Based on a detailed development of entrance laws and excursion laws, and the notion of skew convolution semigroups, two chapters of the book are devoted to independent immigration structures and state-dependent immigration structures, respectively. Building on these, the book concludes with sections on generalized Ornstein-Uhlenbeck processes and on fluctuation limit theorems in the limit of small branching.

A nice additional inclusion in the book is that the classical one-dimensional models are revisited, and new results obtained, by shrinking the underlying space to a single point. To give an isolated example of the usefulness of this approach, this reviewer was grateful to be reminded that the mean-reverting diffusion process (ξ_t), which solves the stochastic differential equation

$$d\xi_t = \delta(\gamma - \xi_t)\,dt + \sqrt{2\delta\xi_t}\,dB_t, \quad \delta > 0,\ \gamma > 1,$$

perhaps most well-known as the "CIR diffusion process" in mathematical finance, is also a continuous-state branching process with immigration. Hence, given $\xi_0 = x$,

$$-\log E_x(e^{-\theta\xi_t}) = xv_t(\theta) + \gamma\delta\int_0^t v_s(\theta)\,ds \to \gamma\ln(1+\theta), \quad t\to\infty,$$

where $v_t(\theta) = \theta e^{-\delta t}/(1+(1-e^{-\delta t})\theta)$ is the solution of $v_t' + \delta v_t + \delta v_t^2 = 0$, $v_0 = \theta$, and so the gamma distribution $\Gamma(\gamma, 1)$ is a stationary steady-state for (ξ_t). Now we combine this with the concept of a weighted occupation time for continuous state branching processes with immigration to obtain

$$-\log E_x(e^{-\theta\int_0^t \xi_s\,ds}) = xu_t(\theta) + \gamma\delta\int_0^t u_s(\theta)\,ds,$$

where $u_t' + \delta u_t + \delta u_t^2 = \theta$, $u_0 = 0$. In equilibrium, with $\xi_0 \sim \Gamma(\gamma, 1)$,

$$-\log E(e^{-\theta\int_0^t \xi_s\,ds}) = \gamma\log g_t(\theta),$$

$$g_t(\theta) = (1+u_t(\theta))\exp\Big\{\delta\int_0^t u_s(\theta)\,ds\Big\}.$$

But now it is easy to check that $g_t(\theta)$ solves the linear, second-order, homogeneous differential equation

$$g_t'' + \delta g_t' - \delta\theta g_t = 0, \quad g_0 = 1,\ g_0' = \theta.$$

Thus, for $\theta \geq 0$, putting $\widehat{\delta} = \sqrt{\delta^2 + 4\delta\theta}$,

$$E(e^{-\theta\int_0^t \xi_s\,ds}) = \Big(\frac{2\widehat{\delta}}{(\widehat{\delta}+\delta+2\theta)e^{(\widehat{\delta}-\delta)t/2} + (\widehat{\delta}-\delta-2\theta)e^{-(\widehat{\delta}+\delta)t/2}}\Big)^{\gamma}.$$

This is a well-written, concise and dedicated book. It will serve excellently as a platform and reference for the next phase of development of superprocesses and measure-valued branching processes.

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