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MEASURE-VALUED BRANCHING PROCESSES AND IMMIGRATION PROCESSES

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Abstract. This is a survey on the theory of measure-valued branching processes (Dawson-Watanabe superprocesses) and their associated immigration processes formulated by skew convolution semigroups. The following main topics are included: convergence of branching particle systems, basic regularities and limit theorems of superprocesses, non-linear differential equations, modifications of the branching models, skew convolution semigroups and entrance laws, construction of immigration processes from Kuznetsov processes.

Key words: branching process; measure-valued; particle system; immigration; skew convolution semigroup; entrance law; Kuznetsov measure

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1. Introduction

Suppose that E is a Lusin topological space, i.e., a homeomorphism of a Borel subset of a compact metric space, with the Borel σ-algebra $\mathcal{B}(E)$. Denote by $B(E)$ the set of bounded $\mathcal{B}(E)$-measurable functions on E, and $C(E)$ the subspace of $B(E)$ comprising continuous functions. The subsets of positive members of the function spaces are denoted by the superscript "+"; e.g., $B(E)^+$, $C(E)^+$. Let $M(E)$ be the totality of finite measures on $(E, \mathcal{B}(E))$. We topologize $M(E)$ by the weak convergence topology, so it also becomes a Lusin space. Put $M(E)^\circ = M(E) \setminus \{0\}$, where 0 denotes the null measure on E. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f \mathrm{d}\mu$. Suppose that X

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is a Markov process in $M(E)$ with transition semigroup $(Q_t)_{t\geq 0}$. It is natural to call X a *measure-valued branching process* (MB-process) provided

$$Q_t(\mu_1+\mu_2,\cdot) = Q_t(\mu_1,\cdot) * Q_t(\mu_2,\cdot), \quad t\geq 0,\ \mu_1,\mu_2 \in M(E), \tag{1.1}$$

where "$*$" denotes the convolution operation. For $f\in B(E)^+$ set

$$V_tf(x) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \quad t\geq 0, x\in E, \tag{1.2}$$

where δ_x denote the unit mass concentrated at $x\in E$. Throughout this paper we assume that, for every $l\geq 0$ and $f\in B(E)^+$, the function $V_tf(x)$ of (t,x) restricted to $[0,l]\times E$ is bounded. We call X a *regular* MB-process provided

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_tf)\right\}, \quad t\geq 0, \mu\in M(E). \tag{1.3}$$

When this is satisfied, the operators $(V_t)_{t\geq 0}$ form a semigroup which is called the *cumulant semigroup* of X. See e.g. Silverstein (1969) and Watanabe (1968). That an MB-process is not necessarily regular was shown in Dynkin et al. (1994). In the sequel of this paper all MB-processes are assumed regular.

Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a Borel right process in E with semigroup $(P_t)_{t\geq 0}$ and ϕ is a function on $E\times[0,\infty)$ given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u), \quad x\in E, z\geq 0, \tag{1.4}$$

where $b\in B(E)$, $c\in B(E)^+$ and $[u\wedge u^2]m(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$. From a general construction in Fitzsimmons (1988, 1992), the evolution equation

$$V_tf(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_sf(y)) P_{t-s}(x, \mathrm{d}y) = P_tf(x), \quad t\geq 0, x\in E, \tag{1.5}$$

defines the cumulant semigroup $(V_t)_{t\geq 0}$ of an MB-process, which is called a *Dawson-Watanabe superprocess* with parameters (ξ,ϕ), or simply a (ξ,ϕ)-*superprocess*. The (ξ,ϕ)-superprocesses constitute a rich class of infinite dimensional processes currently under rapid development. Such processes first arose as the high density limits of branching particle systems; see Feller (1951), Jiřina (1958, 1964), Watanabe (1968), etc. The development of this subject has been stimulated from different subjects including branching processes, interacting particle systems, stochastic partial differential equations and non-linear partial differential equations; see Dawson (1992, 1993). The study of MB-processes has also led to better understanding of some results in those subjects.

An MB-process describes the evolution of a population that evolves according to the law of chance. Typical examples of the model are biological populations in isolated regions, families of neutrons in nuclear reactions, cosmic ray showers and so on. If we consider a situation where there are some additional sources of population from which immigration occurs during the evolution, we need to consider measure-valued branching processes with immigration (MBI-processes). This type of modification is familiar from the branching process literature; see e.g. Arthreya and Ney (1972), Dawson and Ivanoff (1978), Ivanoff (1981), Kawazu and Watanabe (1971), Li (1992b) and Shiga (1990). From the view point of applications to physical and biological sciences, the immigration processes are clearly of great importance. For instance, a typical unadulterated branching process started with a finite initial state goes either extinction or explosion at large times, which is not desired for the transformation process of particles in a nuclear reactor, but the situation can be changed if we consider a subcritical branching process and support it with immigration.

A class of immigration processes associated with the MB-process may be formulated as follows. Let $(N_t)_{t\geq 0}$ be a family of probability measures on $M(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ if

$$N_{r+t} = (N_r Q_t) * N_t, \quad r, t \geq 0. \tag{1.6}$$

The relation (1.6) holds if and only if

$$Q_t^N(\mu, \cdot) := Q_t(\mu, \cdot) * N_t, \quad t \geq 0, \ \mu \in M(E), \tag{1.7}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. If Y is a Markov process in $M(E)$ having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *MBI-process*, or simply an *immigration process*, associated with X. The intuitive meaning of the immigration process is clear from (1.7), that is, $Q_t(\mu, \cdot)$ is the distribution of descendants of the people distributed as $\mu \in M(E)$ at time zero and N_t gives the distribution of descendants of the people immigrating to E in the time interval $(0, t]$. The definition (1.7) is similar to the construction of a Lévy's transition semigroup from the usual convolution semigroup. If $Q_t(\mu, \cdot) \equiv$ unit mass at μ, then $(N_t)_{t\geq 0}$ becomes a usual convolution semigroup. In this sense, the immigration process is a generalized form of the celebrated Lévy process.

The study of the immigration processes strongly depends on probabilistic potential theory. The skew convolution semigroup may be characterized in terms of an infinitely divisible probability entrance law for $(Q_t)_{t\geq 0}$. For the Dawson-Watanabe superprocess an infinitely divisible probability entrance law is determined uniquely by an infinitely divisible probability measure on the space of entrance laws for the underlying process. A general immigration process may be constructed using the Kuznetsov process determined by an entrance rule. The stationary distributions of immigration processes may be represented by excessive measures, and the abstract results in potential theory of excessive measures may be interpreted immediately in terms of stationary immigration processes. The MBI-processes involve more complicated trajectory structures than the processes without immigration. An immigration process associated with the Borel right superprocess does not always have a right continuous realization, and this irregularity is caused by the immigrants coming in from some boundary points of the underlying space E. For instance, if ξ is a minimal (absorbing barrier) Brownian motion in $(0, \infty)$, a non-right-continuous immigration process may be generated by cliques of immigrants with infinite mass entering from the origin. There are interesting central limit theorems for the stationary immigration processes, which give rise to a class of Ornstein-Uhlenbeck processes with distribution values.

The object of this paper is to give a brief introduction to the measure-valued branching processes and their associated immigration processes formulated by skew convolution semigroups. It is our hope that this would help the reader to pursue the lecture notes of Dawson (1993) and the extensive amount of original articles in the subject. The paper is written in English to use the TeX files of our previous papers. We hope that this will not bring much inconvenience to our Chinese readers. The paper is organized as follows. Section 2 contains some basic facts on classical branching processes. In section 3 we introduce the branching particle systems which is a natural generalization of the classical continuous time branching processes. The Dawson-Watanabe superprocesses arise as high density, small particle approximation of the branching particle systems. The basic regularities and path structures of superprocesses are discussed in section 4. In section 5 we describe the work of Dynkin and Le Gall on applications of superprocesses and stochastic snakes to non-linear differential equations. The ergodic theory and asymptotic behavior of superprocess are discussed in section 6. In section 7 we describe some modifications of the Dawson-Watanabe superprocess. The skew convolution semigroups associated with MB-processes and branching particle systems are defined in sections 8 and 9, respectively. In section 10 we construct the sample paths of general immigration processes using Kuznetsov processes and consider the a.s. behavior of the latter. In section 11, we discuss immigration processes over a minimal Brownian motion.

2. Classical branching processes

A discrete time and state branching process (Galton-Watson process) is an integer-valued Markov chain $\{Z_n : n = 0, 1, 2, \cdots\}$ with transition probabilities $P(i, j)$ determined in terms of a given probability distribution $\{p_k : k = 0, 1, 2, \cdots\}$ by

$$P(i,j) = \begin{cases} p_j^{*i} \text{ if } i \geq 1 \text{ and } j \geq 0, \\ \delta_{0j} \text{ if } i = 0 \text{ and } j \geq 0, \end{cases} \tag{2.1}$$

where $\{p_k^{*i} : k = 0, 1, 2, \cdots\}$ denotes the i-fold convolution of $\{p_k : k = 0, 1, 2, \cdots\}$.

The process $\{Z_n : n = 0, 1, 2, \cdots\}$ describes the evolution of a particle population. The population starts at time zero with Z_0 particles, each of which after a unit of time splits into a random number of offspring according to the law $\{p_k : k = 0, 1, 2, \cdots\}$. The total number Z_1 of offspring is then the sum of Z_0 random variables. These constitutes the first generation and go on to produce the second generation of Z_2 particles, and so on. It is assumed that the number of offspring produced by a single parent at any time is independent of the history of the population, and of other particles existing at present. The branching processes were first introduced to study the extinction of family names in the British peerage; see Watson and Galton (1874). Since then the study of these processes has gone a long history, interwoven with a number of applications in physical and biological sciences; see e.g. Harris (1963).

An important tool in the study of the branching process is the generating

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leq 1. \tag{2.2}$$

Define the iterates

$$f_0(s) = s \text{ and } f_n(s) = f(f_{n-1}(s)) \text{ for } n \geq 1. \tag{2.3}$$

Let $P_n(i,j)$ denote the n-step transition probabilities. Then we have

$$f_n(s) = \sum_{j=0}^{\infty} P_n(1,j)s^j, \quad |s| \leq 1, \tag{2.4}$$

Moreover, by (2.1) and the Markov property one finds easily

$$\sum_{j=0}^{\infty} P_n(i,j)s^j = \left[\sum_{j=0}^{\infty} P_n(1,j)s^j\right]^i, \quad |s| \leq 1, \tag{2.5}$$

for all $i, n = 0, 1, 2, \cdots$. The above equation characterizes the basic branching property of the process.

In the Galton-Watson process, the lifetime of each particle is one unit of time. A natural generalization is to allow these lifetimes to be random variables. An integer-valued Markov process $\{Z_t : t \geq 0\}$ is called a continuous time branching process if its transition probabilities $P_t(i,j)$ satisfy

$$\sum_{j=0}^{\infty} P_t(i,j)s^j = \left[\sum_{j=0}^{\infty} P_t(1,j)s^j\right]^i, \quad t \geq 0, |s| \leq 1, \tag{2.6}$$

for all $i = 0, 1, 2, \cdots$. From this property it follows that there exist

$$a > 0, \quad p_i \geq 0, \quad \sum_{i=0}^{\infty} p_i = 1,$$

such that as $t \to 0$

$$\begin{cases} P_t(i,j) = iap_{j-i+1}t + o(t) \text{ if } j \geq i-1 \text{ and } j \neq i, \\ P_t(i,i) = 1 - iat + o(t), \\ P_t(i,j) = o(t) \text{ if } j < i-1. \end{cases} \tag{2.7}$$

The transition probabilities $P_t(i,j)$ can be characterized by $\{a, p_k : k = 0, 1, 2, \cdots\}$ as solutions of the Kolmogorov forward and backward equations. The sequence $\{a, p_k : k = 0, 1, 2, \cdots\}$ has clear probabilistic interpretations in terms of the branching process $\{Z_t : t \geq 0\}$. If a particle is alive a a certain time, its additional life length is a random variable which is exponentially distributed with parameter $a > 0$. Upon its death it leaves $k \geq 0$ offspring with probability p_k. All the particles act independently of other particles, and of the history of the process.

For both the discrete and continuous time branching processes, the limit theorems constitute an important part of the theory. We refer the reader to Arthreya and Ney (1972) for a unified treatment of the limit theorems for discrete state branching processes and for a representive selection from the extensive amount of original articles; see also Pakes (1997) for some recent developments of the theory.

3. Branching particle systems

3.1. Let us describe a generalization of the continuous time branching process. Let $N(E)$ be the subspace of $M(E)$ comprising integer-valued measures and let $N(E)^\circ = N(E) \setminus \{0\}$. Suppose that $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\sigma)$ is a Markov process in $N(E)$ with transition semigroup $(Q_t)_{t\geq 0}$. We call X a *branching particle system* provided

$$Q_t(\sigma_1 + \sigma_2, \cdot) = Q_t(\sigma_1, \cdot) * Q_t(\sigma_2, \cdot), \quad t \geq 0,\ \sigma_1, \sigma_2 \in N(E). \tag{3.1}$$

The process X describes the evolution of a population of particles that migrate and propagate independently of each other in the space E. For $f \in B(E)^+$, let

$$U_t f(x) = -\log \int_{N(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \quad t \geq 0, x \in E. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} Q_t(\sigma, \mathrm{d}\nu) = \exp\{-\sigma(U_t f)\}, \quad t \geq 0, \sigma \in N(E). \tag{3.3}$$

The above formula can be regarded as a generalized form of (2.6). In the sequel, we shall always assume that for every $l \geq 0$ and $f \in B(E)^+$, the function $U_t f(x)$ of (t,x) restricted to $[0,l] \times E$ is bounded.

3.2. Let ξ be a Borel right process in E with conservative transition semigroup $(P_t)_{t\geq 0}$. Let $\gamma(\cdot) \in B(E)^+$ and $g(\cdot,\cdot) \in B(E \times [0,1])^+$. Suppose that for each fixed $x \in E$, $g(x,\cdot)$ coincides on $[0,1]$ with a probability generating function and that $g'_z(\cdot, 1^-) \in B(E)^+$. Set $\rho(r,t) = \exp\left\{-\int_r^t \gamma(\xi_s)\mathrm{d}s\right\}$. A branching particle system X is called a (ξ,γ,g)-*system* if its transition probabilities are determined by (3.3) with $u_t(x) = U_t f(x)$ being the unique positive solution to the evolute equation

$$\mathrm{e}^{-u_t(x)} = \mathbf{P}_x \rho(0,t) \mathrm{e}^{-f(\xi_t)} + \mathbf{P}_x \bigg\{ \int_0^t \rho(0,s) g(\xi_s, \exp\{-u_{t-s}(\xi_s)\}) \gamma(\xi_s) \mathrm{d}s \bigg\}. \tag{3.4}$$

The heuristic meaning of the (ξ, γ, g)-system is as follows. The particles in E move randomly according to the laws given by the transition probabilities of ξ. For a particle which is alive at time r and follows the path $\{\xi_s : s \geq r\}$, the conditional probability of survival during the time interval $[r, t]$ is $\rho(r, t)$. When the particle dies at a point $x \in E$, it gives birth to a random number of offspring according to the generating function $g(x, \cdot)$ and the offspring then move and propagate in E in the same fashion as their parents. It is assumed that the migrations, the life times and the branchings of the particles are independent of each other. The equation (3.4) follows as we think about that if a particle starts moving from point x at time zero, it follows a path of ξ and does not branch before time $t \geq 0$, or it splits at time $s \in (0, t]$. See Dynkin (1991a) for a vigorous construction of the (ξ, γ, g)-system.

It is easy to check that

$$\int_0^t \rho(s,t)\gamma(\xi_s)\mathrm{d}s = 1 - \rho(0,t), \quad t \geq 0. \tag{3.5}$$

Using (3.5) and the Markov property we have

$$\begin{aligned}
&\int_0^t \mathbf{P}_x\big\{\gamma(\xi_s)\mathbf{P}_{\xi_s}\big[\rho(0,t-s)\exp\{-f(\xi_{t-s})\}\big]\big\}\mathrm{d}s \\
=&\int_0^t \mathbf{P}_x\big[\gamma(\xi_s)\rho(s,t)\exp\{-f(\xi_t)\}\big]\mathrm{d}s \\
=&\mathbf{P}_x\big\{\big[1-\rho(0,t)\big]\exp\{-f(\xi_t)\}\big\}.
\end{aligned} \tag{3.6}$$

Similarly we have

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int_0^{t-s} \mathbf{P}_x\big\{\gamma(\xi_s)\mathbf{P}_{\xi_s}\big[\rho(0,r)g(\xi_r, \exp\{-u_{t-s-r}(\xi_r)\})\gamma(\xi_r)\big]\mathrm{d}r\big\} \\
=&\int_0^t \mathrm{d}s \int_0^{t-s} \mathbf{P}_x\big[\gamma(\xi_s)\rho(s,r+s)g(\xi_{r+s}, \exp\{-u_{t-s-r}(\xi_{r+s})\})\gamma(\xi_{r+s})\big]\mathrm{d}r \\
=&\int_0^t \mathrm{d}r \int_0^{r} \mathbf{P}_x\big[\gamma(\xi_s)\rho(s,r)g(\xi_r, \exp\{-u_{t-r}(\xi_r)\})\gamma(\xi_r)\big]\mathrm{d}s \\
=&\int_0^t \mathbf{P}_x\big\{\big[1-\rho(0,r)\big]\gamma(\xi_r)g(\xi_r, \exp\{-u_{t-r}(\xi_r)\})\big\}\mathrm{d}r.
\end{aligned} \tag{3.7}$$

Adding up both sides of (3.6) and (3.7) and using (3.4) we get

$$\begin{aligned}
\int_0^t \mathbf{P}_x\big[\gamma(\xi_{t-s})\exp\{-u_s(\xi_{t-s})\}\big]\mathrm{d}s = \mathbf{P}_x\exp\{-f(\xi_t)\} - \exp\{-u_t(x)\} \\
+ \int_0^t \mathbf{P}_x\left[\gamma(\xi_{t-s})g(\xi_{t-s}, \exp\{-u_s(\xi_{t-s})\})\right]\mathrm{d}s.
\end{aligned}$$

We shall simply write the above equation as

$$\mathrm{e}^{-u_t} = P_t \mathrm{e}^{-f} - \int_0^t P_{t-s}\left[\gamma\left(\mathrm{e}^{-u_s} - g(\mathrm{e}^{-u_s})\right)\right] \mathrm{d}s. \tag{3.8}$$

By Gronwall's inequality one sees that (3.8) has a unique solution, so it is an equivalent form of (3.4). See also Dawson (1993) and Dynkin (1991a). Let

$$J_t f(x) = 1 - \exp\{-U_t f(x)\}. \tag{3.9}$$

By (3.8) we have

$$J_t f(x) + \int_0^t \mathrm{d}s \int_E \varphi(y, J_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t\left(1 - \mathrm{e}^{-f}\right)(x), \tag{3.10}$$

where

$$\varphi(x, z) = \gamma(x)[g(x, 1-z) - (1-z)], \quad x \in E, 0 \leq z \leq 1, \tag{3.11}$$

The transition semigroup of the (ξ, γ, g)-system can also be determined by (3.3), (3.9) and (3.10). Clearly, this characterization of the system applies even for a non-conservative underlying semigroup $(P_t)_{t\geq 0}$.

3.3. Suppose we have a sequence of branching particle systems $\{X_t(k) : t \geq 0\}$ with parameters (ξ, γ_k, g_k), $k = 1, 2, \cdots$. Then $\{X_t^{(k)} := k^{-1} X_t(k) : t \geq 0\}$ is a Markov process in $M_k(E) := \{\sigma/k : \sigma \in N(E)\}$. By (3.3) and (3.5) the transition probabilities of $\{X_t^{(k)} : t \geq 0\}$ are determined by

$$\mathbf{Q}_\sigma^{(k)} \exp\big\{ - X_t^{(k)}(f)\big\} = \exp\big\{ - \sigma(k u_t^{(k)})\big\}, \tag{3.12}$$

where $u_t^{(k)}(x) \equiv u_t^{(k)}(x, f)$ is the solution to

$$\mathrm{e}^{-u_t^{(k)}} = P_t \mathrm{e}^{-f/k} - \int_0^t P_{t-s}\left[\gamma_k\left(\mathrm{e}^{-u_s^{(k)}} - g_k(\mathrm{e}^{-u_s^{(k)}})\right)\right] \mathrm{d}s. \tag{3.13}$$

Take $\mu \in M(E)$ and assume $X_0(k)$ is a Poisson random measure on E with intensity $k\mu$. Let $\mathbf{Q}_{(\mu)}^{(k)}$ denote the conditional law of $\{X_t^{(k)} : t \geq 0\}$. Then we have

$$\mathbf{Q}_{(\mu)}^{(k)} \exp\big\{ - X_t^{(k)}(f)\big\} = \exp\big\{ - \mu(v_t^{(k)})\big\}, \tag{3.14}$$

with $v_t^{(k)}(x) \equiv v_t^{(k)}(x, f)$ defined by

$$v_t^{(k)}(x) = k[1 - \exp\{-u_t^{(k)}(x)\}]. \tag{3.15}$$

From (3.13) it follows that

$$v_t^{(k)}(x) + \int_0^t P_{t-s}[\phi_k(v_s^{(k)})]\mathrm{d}s = P_t k[1 - \mathrm{e}^{-f/k}], \tag{3.16}$$

where

$$\phi_k(x,z) = k\gamma_k(x)[g_k(x, 1 - z/k) - (1 - z/k)], \quad 0 \le z \le k. \tag{3.17}$$

Note that transition probabilities of the sequence $\{X_t^{(k)} : t \ge 0\}$ can also be characterized by (3.12), (3.15) and (3.16), which are applicable even when $(P_t)_{t\ge 0}$ is non-conservative.

Lemma 3.1. (Li, 1991) *Assume that for each $l \ge 0$, on the set $E \times [0,l]$ of (x,z), the sequence $\phi_k(x,z)$ defined by (3.17) is uniformly Lipschitz in z and $\phi_k(x,z) \to \phi(x,z)$ uniformly as $k \to \infty$. Then $\phi(x,z)$ has the representation (1.4).*

Theorem 3.1. (Dynkin, 1991a) *Under the conditions of Lemma 3.1, both $v_t^{(k)}(x,f)$ and $ku_t^{(k)}(x,f)$ converge as $k \to \infty$ to the solution $V_t f(x)$ of (1.5) boundedly and uniformly on the set $[0,l] \times E$ of (t,x) for every $l \ge 0$. Thus the finite-dimensional distributions of $\{X_t^{(k)} : t \ge 0\}$ under $\mathbf{Q}_{(\mu)}^{(k)}$ converge to those of the (ξ,ϕ)-superprocess with initial state μ.*

4. Dawson-Watanabe superprocesses

4.1. The basic regularities of the general (ξ,ϕ)-superprocess were studied in Fitzsimmons (1988, 1992); see also Dynkin (1993b). Let W denote the space of all right continuous paths $\omega : [0,\infty) \to M(E)$ with the coordinate process denoted by $\{X_t(\omega) : t \ge 0\}$. Let $(\mathcal{G}^\circ, \mathcal{G}_t^\circ)$ denote the natural σ-algebras on W. Then we have

Theorem 4.1. (Fitzsimmons, 1988) *For each $\mu \in M(E)$ there is a unique probability measure $\mathbf{Q}_\mu$ on $(W, \mathcal{G}^\circ)$ such that $\mathbf{Q}_\mu\{X_0 = \mu\} = 1$ and $\{X_t : t \ge 0\}$ under $\mathbf{Q}_\mu$ is a Markov process with transition semigroup $(Q_t)_{t\ge 0}$. Furthermore, the system $(W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ is a Borel right process, where $(\mathcal{G}, \mathcal{G}_t)$ is the augmentation of $(\mathcal{G}^\circ, \mathcal{G}_{t+}^\circ)$ by the system $\{\mathbf{Q}_\mu : \mu \in M(E)\}$.*

Let A be the generator of $(P_t)_{t\ge 0}$. Then we may rewrite (1.5) into the equivalent differential form

$$\frac{\partial}{\partial t} V_t f(x) = A V_t f(x) - \phi(x, V_t f(x)), \quad V_0 f(x) = f(x), \quad t \ge 0, x \in E. \tag{4.1}$$

This leads through a formal calculation to the generator L of the superprocess:

$$\begin{aligned} LF(\mu) = &\int_E [AF'(\mu)(x) - b(x)F'(\mu,x)]\mu(\mathrm{d}x) + \int_E c(x)F''(\mu,x)\mu(\mathrm{d}x) \\ &+ \int_E \mu(\mathrm{d}x) \int_0^\infty [F(\mu + u\delta_x) - F(\mu) - uF'(\mu,x)]m(x,\mathrm{d}u) \end{aligned} \tag{4.2}$$

where

$$F'(\mu, x) = \lim_{r\downarrow 0} \frac{1}{r}[F(\mu + r\delta_x) - F(\mu)]$$

and $F''(\mu, x)$ is defined by the limit with $F(\mu)$ replaced by $F'(\mu, x)$. It was proved in Fitzsimmons (1988) that $\{\mathbf{Q}_\mu : \mu \in M(E)\}$ is the unique solution of the martingale problem associated with the generator L. Based on this martingale characterization Fitzsimmons (1988) obtained the following

Theorem 4.2. (Fitzsimmons, 1988) *Suppose that ξ is a conservative Hunt process and $m(x, \cdot) = 0$ for all $x \in E$. Then the process $\{X_t : t \geq 0\}$ is a.s. continuous under $\mathbf{Q}_\mu$ for every $\mu \in M(E)$.*

The *weighted occupation time* $\int_0^t X_s \mathrm{d}s$ is a powerful tool in the study of the (ξ, ϕ)-superprocess. For any $\mu \in M(E)$ and $f, g \in B(E)^+$ we have

$$\mathbf{Q}_\mu \exp\bigg\{ - X_t(f) - \int_0^t X_s(g)\mathrm{d}s\bigg\} = \exp\{-\mu(V_t(f, g))\}, \tag{4.3}$$

where $V_t(f, g)(x) \equiv u_t(x)$ is the unique bounded, positive solution to

$$u_t(x) + \int_0^t \mathrm{d}s \int_E \phi(x, u_s(y))P_{t-s}(x, \mathrm{d}y) = P_t f(x) + \int_0^t P_s g(x)\mathrm{d}s. \tag{4.4}$$

See e.g. Fitzsimmons (1988) and Iscoe (1986a). The joint distribution of X_t and $\int_0^t X_s \mathrm{d}s$ are characterized by (4.3) and (4.4).

4.2. In this paragraph we assume that E is a locally compact metric space, $(P_t)_{t\geq 0}$ is a conservative Feller semigroup and $\phi(x, z) \equiv b(x)z + c(x)z^2$. By Theorem 4.2, the (ξ, ϕ)-superprocess has a diffusion realization. This diffusion may be characterized by the martingale problem described as follows. Let $C([0, \infty), M(E))$ be the subspace of W of continuous paths. Then for each $\mu \in M(\mathbb{R}^d)$, $\mathbf{Q}_\mu$ is the unique probability measure on $C([0, \infty), M(E))$ such that, for any $f \in \mathcal{D}(A)$,

$$M_t(f) := X_t(f) - \mu(f) - \int_0^t X_s\big(Af - bf\big)\mathrm{d}s, \quad t \geq 0, \tag{4.5}$$

is a $\mathbf{Q}_\mu$-martingale starting at zero with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t X_s(cf^2)\mathrm{d}s, \quad t \geq 0. \tag{4.6}$$

See e.g. Fitzsimmons (1988) and Roelly-Coppoletta (1986). Since (4.5) is linear in $f \in \mathcal{D}(A)$, one can extend the system $\{M_t(f) : f \in \mathcal{D}(A), t \geq 0\}$ to a continuous orthogonal martingale measure $\{M_t(B) : B \in \mathcal{B}(E), t \geq 0\}$ with covariant measure $c(x)X_s(\mathrm{d}x)\mathrm{d}s$ in the sense of Walsh (1986). See also Méléard and Roelly (1993). Let

$M(\mathrm{d}s, \mathrm{d}x)$ denote the stochastic integral with respect to this martingale measure. By a standard argument, for any $t \geq 0$ and $f \in B(E)$ we have $\mathbf{Q}_\mu$-a.s.,

$$X_t(f) = X_0(P_t f) + \int_0^t \int_E P_{t-s} f(x) M(\mathrm{d}s, \mathrm{d}x) - \int_0^t X_s(b P_{t-s} f)\mathrm{d}s. \tag{4.7}$$

It was proved in Konno and Shiga (1988) and Reimers (1989) that for $E = \mathbb{R}$ and a large class of admissible ξ including the symmetric stable processes, $\{X_t : t > 0\}$ is absolutely continuous with respect to the Lebesgue measure with continuous density $\{X_t(x) : t > 0, x \in \mathbb{R}\}$ which may be given by

$$\begin{aligned} X_t(x) = \int_{\mathbb{R}} p_t(z, x) X_0(\mathrm{d}z) &+ \int_0^t \int_{\mathbb{R}} p_{t-s}(z, x) M(\mathrm{d}s, \mathrm{d}z) \\ &- \int_0^t \mathrm{d}s \int_{\mathbb{R}} b(z) p_{t-s}(z, x) X_s(\mathrm{d}z), \end{aligned}$$

where $p_t(z, x)$ is the transition density of ξ. In this case the martingale problem (4.5) and (4.6) can be reformulated into the stochastic partial differential equation

$$\frac{\partial}{\partial t} X_t(x) = \sqrt{c(x) X_t(x)}\dot{W}_t(x) + A^* X_t(x) - b(x) X_t(x), \tag{4.8}$$

where A^* is the adjoint of the generator A and $\dot{W}_t(x)$ is a time-space white noise defined on an extension of the original probability space. See also Zhao (1994a) for related work. The pointwise uniqueness for the equation (4.8) still remains open; see Dawson (1993).

In the case where ξ is a symmetric stable process in $E = \mathbb{R}^d$ with index α $(0 < \alpha \leq 2)$, $b(x) \equiv 0$ and $c(x) \equiv$ const, it was proved in Dawson and Hochberg (1979) and Zähle (1988) that for each $t > 0$ the Hausdorff dimension of the Borel support of X_t is almost surely $d \wedge \alpha$. Therefore, if $d > \alpha$, the random measure X_t is a.s. singular for each $t > 0$. (For $d = \alpha$ the singularity of X_t was proved in Dawson and Hochberg (1979) and Roelly-Coppoletta (1986) by different approaches.) Indeed, the singular measure X_t spreads its mass over its Borel supports in a very uniform manner simultaneously for all times $t > 0$. Let $\varphi_\alpha(z) = z^\alpha \log^+ \log^+(1/z)$ with $\log^+ = (0 \vee \log)$ and let φ_α-m denote the φ_α-Hausdorff measure. The following result was obtained in Perkins (1988): When $d > \alpha$, there are $0 < c(\alpha, d) \leq C(\alpha, d) < \infty$ and a set-valued process $\{A_t : t > 0\}$ such that a.s.

$$c(\alpha, d)\varphi_\alpha\text{-}m(\cdot \cap A_t) \leq X_t(\cdot) \leq C(\alpha, d)\varphi_\alpha\text{-}m(\cdot \cap A_t) \quad \text{for all } t > 0. \tag{4.9}$$

When $\alpha = 2$, Perkins (1989) proved that (4.9) holds with A_t replaced by $\mathrm{supp}(X_t)$, the closed support of X_t. But, this extension is false for $\alpha < 2$; see Perkins (1990). Using the "historical process" as a tool, Dawson and Perkins (1991) showed $c(\alpha, d) = C(\alpha, d)$ (for fixed $t > 0$) in the above results. One recent trend in this study is to analyze the multifractal structures of the superprocess; see e.g. Perkins and Taylor (1996).

4.3. Suppose that ξ is a Brownian motion in $I\!R^d$ generated by the Laplacian Δ and $\phi(x,z) \equiv z^2$. The (ξ,ϕ)-superprocess X becomes a critical *super Brownian motion.* Using a special case of the characterization give by (4.3) and (4.4), Iscoe (1988) proved that, if $\mathrm{supp}(X_0)$ is bounded, the process $\{X_t : t \geq 0\}$ spends its entire "life" within a bounded (random) set in $I\!R^d$. See Dawson et al (1989) for more complete results on the path properties and hitting of the super Brownian motion. From the results of Iscoe (1988) it follows that, if $X_0(\cdot) \geq 0$ is a continuous function on $I\!R$ with compact support, the non-negative solution $X_t(\cdot)$ to

$$\frac{\partial}{\partial t} X_t(x) = \sqrt{X_t(x)}\dot{W}_t(x) + \Delta X_t(x) \tag{4.10}$$

also has compact support for all $t > 0$. In other words, the compact support property of the solution to (4.10) propagates with the passage of time. The propagation of compact supports of more general stochastic differential equations was studied in Mueller (1991) and Shiga (1994). Indeed, as observed by Shiga (1994), this property is due to the degeneracy at zero of the coefficient of the noise-term in the equation. The results of Shiga (1994) are proved by reducing the general equation to (4.10) and using the results of Iscoe (1988) on non-linear partial differential equations with singular boundary conditions.

Some of the results in Iscoe (1988) have been generalized to super Brownian motions over Riemannian manifolds in Tang (1997ab). Consider the hyperbolic space $H^d = \{(x_1, \cdots, x_d, t) \in I\!R^{d+1} : t > 0, x_1^2 + \cdots + x_d^2 = t^2 - 1\}$ with the Riemannian metric $r(\cdot,\cdot)$ induced by the Lorentz metric in $I\!R^{d+1}$. Choose $x^0 = (0, \cdots, 0, 1) \in H^d$ as the pole and let $r(x) = r(x^0, x)$. For the super Brownian motion over H^d, Tang (1997a) proved that there is a constant $c = c(\varepsilon, d) > 0$ such that

$$\begin{aligned}\mathbf{Q}_{\delta_x}\{(X_t)_{t\geq 0} \text{ ever charges } \bar{B}(x^0;\varepsilon)\} &\sim 6r(x)^{-2}, \quad d = 1,\\ &\sim c\exp\{-(d-1)r(x)\}, \quad d \geq 2,\end{aligned}$$

extending a theorem of Iscoe (1988). See also Bao (1995) for some generalizations of the work of Iscoe (1988) to super Ornstein-Uhlenbeck processes.

5. Non-linear differential equations

5.1 The partial differential equations provide powerful tools for investigating the charging and hitting probabilities of superprocesses. On the other hand, the superprocesses can also be used to solve some problems on the differential equations. Let us describe some results on this topic. For simplicity we only consider the equation

$$\Delta u(x) = u(x)^2, \quad x \in D, \tag{5.1}$$

where D is an open set in $I\!R^d$. We assume that ξ is a Brownian motion in $I\!R^d$ generated by the Laplacian Δ and $\phi(x,z) \equiv z^2$. By modifying the construction of the super Brownian motion, we can obtain a family of random measures $\{X_\tau : \tau \in \mathcal{T}\}$, where $\mathcal{T}$

is a certain class of stopping times for the underlying Brownian motion including the first exit times from open sets. If τ is the first exit time from D, then X_τ is the mass distribution on ∂D obtained by freezing each particle at its first exit time from D; see Dynkin (1991ab).

Theorem 5.1. (Dynkin, 1993a) *Suppose that D is a bounded regular domain and f is a non-negative continuous function on ∂D. Then*

$$u(x) = -\log \mathbf{Q}_{\delta_x} \exp\{-X_\tau(f)\}, \quad x \in D, \tag{5.2}$$

defines the unique solution to (5.1) which satisfies the boundary condition

$$u(x) \to f(z) \text{ as } x \to z \in \partial D. \tag{5.3}$$

As observed by Loewer and Nirenberg (1974), (5.1) has the maximal solution v which tends to ∞ at ∂D. Let R denote the *range* of X, that is, the minimal closed set containing $\mathrm{supp}(X_t)$ for all $t \geq 0$. It was proved by Dynkin (1993a) that the maximal solution to (5.1) is given by

$$v(x) = -\log \mathbf{Q}_{\delta_x}\{R \subset D\}, \quad x \in D. \tag{5.4}$$

A set $B \subset \mathbb{R}^d$ is said to be *R-polar* if $\mathbf{Q}_{\delta_x}\{R \cap B \neq \emptyset\} = 0$ for all $x \notin B$. By the results of Brezis and Véron (1980), the maximal solution to (5.1) in $\mathbb{R}^d \setminus \{0\}$ is trivial if $d \geq 4$ and it is $2(4-d)/|x|^2$ if $d < 4$. It follows that, a singleton is R-polar if and only if $d \geq 4$.

5.2. The problem of describing all non-negative solutions to (5.1) has not been solved in general. Some significant progresses have been made by Le Gall (1993ab, 1995). We say a set K in $\mathbb{R}^d$ has *positive capacity* if $K \neq \emptyset$ when $d = 2$, and when $d \geq 3$ if K supports a non-trivial measure ν such that

$$\int_{\mathbb{R}^d} \nu(\mathrm{d}y) \int_{\mathbb{R}^d} |y-z|^{3-d} \nu(\mathrm{d}z) < \infty \text{ for } d \geq 4,$$
$$\int_{\mathbb{R}^d} \nu(\mathrm{d}y) \int_{\mathbb{R}^d} \log(|y-z|^{-1}) \nu(\mathrm{d}z) < \infty \text{ for } d = 3.$$

Otherwise, we say that K has *zero capacity.* The following result had been conjectured by Dynkin (1993a).

Theorem 5.2. (Le Gall, 1995) *Suppose that D is a bounded and sufficiently smooth domain. Then the non-negative solution u to (5.1) bounded above by a harmonic function is in 1-1 correspondence with the finite measure ν on ∂D that does not charge sets of zero capacity. The 1-1 correspondence is given by the equation*

$$u(x) = \int_{\partial D} P(x,y)\nu(\mathrm{d}y) - \frac{1}{2}\int_D G(x,y)u(y)^2 \mathrm{d}y, \quad x \in D, \tag{5.5}$$

where P is the Poisson kernel and G is the Green function of D. The first term on the right hand side of (5.5) gives the minimal harmonic function dominating u.

The proof of this result given by Le Gall (1995) is based on his path-valued process, or *Brownian snake*. In the special case where D is the unit disc in $I\!R^2$, Le Gall (1993a) gave a representation for *all* non-negative solutions to (5.1) using the Brownian snakes. In terms of the super Brownian motion, the result can be stated as follows. Let τ be the first exit time from the unit disc D in $I\!R^2$. Then X_τ is a.s. absolutely continuous with respect to the Lebesgue measure on ∂D having continuous density ρ_τ. For a closed subset K of ∂D let $Z_K = \infty$ if $R \cap K \neq \emptyset$ and $= 0$ if $R \cap K = \emptyset$. Then

$$u(x) = -\log \mathbf{Q}_{\delta_x} \exp\{-Z_K - \nu(\rho_\tau)\}, \quad x \in D; \tag{5.6}$$

determines a 1-1 correspondence between the set of all non-negative solutions to (5.1) and the set $\mathcal{G} = \{(K, \nu) : K \subset \partial D$ is closed and $\nu \in M(\partial D)$ satisfies $\nu(K) = 0\}$; see Dynkin (1994). See Overbeck (1993, 1994) and Zhao (1994b, 1996) for some other topics in the potential theory of superprocesses.

5.3. Let us give a brief description of the Brownian snake. A stopped path in $I\!R^d$ is a pair (w, ζ), where $\zeta \geq 0$ and w is a continuous mapping from $[0, \infty)$ into $I\!R^d$ that is constant over $[\zeta, \infty)$. Fix a starting point $x \in I\!R^d$ and denote by $\mathbf{W}_x$ the set of all stopped paths with initial points x. A metric d may be defined on $\mathbf{W}_x$ by

$$d((w, \zeta), (w', \zeta')) = |\zeta - \zeta'| + \sup\{|w(s) - w'(s)| : s \geq 0\}.$$

The *Brownian snake* starting at $x \in I\!R^d$ is the diffusion process $\{(B_t, \zeta_t) : t \geq 0\}$ in $\mathbf{W}_x$ whose distribution is characterized by the following properties.

(i) The process $\{\zeta_t : t \geq 0\}$ is a reflecting Brownian in $[0, \infty)$ with $\zeta_0 = 0$.

(ii) Given $\{\zeta_t : t \geq 0\}$, the process $\{B_t(\cdot) : t \geq 0\}$ is a time inhomogeneous Markov process such that, for any $t > r \geq 0$,

(a) $B_t(s) = B_r(s)$ for all $0 \leq s \leq m_{rt} := \inf\{\zeta_u : r \leq u \leq t\}$; and

(b) $\{B_t(m_{rt} + s) - B_t(m_{rt}) : s \geq 0\}$ is a standard Brownian motion in $I\!R^d$ stopped at $\zeta_t - m_{rt}$, independent of $\{B_r(s) : s \geq 0\}$.

Heuristically, the process $\{B_t(s) : s \geq 0\}$ is a standard Brownian motion stopped at a random time ζ_t. The life time $\{\zeta_t : t \geq 0\}$ evolves according to the law of a reflecting Brownian. When ζ_t decreases, the path $B_t(\cdot)$ is erased from its final point, and when ζ_t increases it is extended according to the law of the standard Brownian motion, independently of the past path. Le Gall (1995) showed that the finite measure ν mentioned in Theorem 5.2 determines a functional of the Brownian snake and represented u as the expectation of this functional with respect to an excursion law.

The connection of the Brownian snake and the super Brownian motion can be described as follows. It is well-known that there is a continuous two parameter process $\{l(t, s) : t \geq 0, s \geq 0\}$ such that a.s.

$$l(t, s) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[s, s+\varepsilon]}(\zeta_u) \mathrm{d}u, \quad t \geq 0, s \geq 0.$$

The process $\{l(t,s) : t \geq 0, s \geq 0\}$ is called the *local time* of the reflecting Brownian $\{\zeta_t : t \geq 0\}$. For any fixed $s \geq 0$, the process $\{l(t,s) : t \geq 0\}$ is a.s. non-decreasing and determines a random measure $l(\mathrm{d}t, s)$ on $[0, \infty)$ which is supported a.s. by $\{t \geq 0 : \zeta_t = s\}$. Fix $\gamma > 0$ and let $\alpha(\gamma) = \inf\{t \geq 0 : l(t, 0) \geq \gamma\}$. Define the measure-valued process $\{X_s : s \geq 0\}$ by

$$X_s(f) = \int_0^{\alpha(\gamma)} f(B_t(\zeta_t)) l(\mathrm{d}t, s), \quad s \geq 0, f \in B(I\!\!R^d). \tag{5.7}$$

Then $\{X_s : s \geq 0\}$ is the super Brownian motion starting from $\gamma\delta_x$; see Le Gall (1993a). Intuitively, $\{B_t(s) : 0 \leq t \leq \alpha(\gamma), 0 \leq s \leq \zeta_t\}$ constitute exactly the historical paths of $\{X_s : s \geq 0\}$. Indeed, $\{X_s : s \geq 0\}$ is the projection to $M(I\!\!R^d)$ of the process $\{H_s : s \geq 0\}$ in $M(\mathbf{W}_x)$ defined by

$$H_s(F) = \int_0^{\alpha(\gamma)} F(B_t) l(\mathrm{d}t, s), \quad s \geq 0, F \in B(\mathbf{W}_x), \tag{5.8}$$

which is the so-called *historical super Brownian motion*; see Dawson and Perkins (1991), Dynkin (1991b), Le Gall (1991), Watanabe (1997), etc.

6. Extension of the state space

6.1. The limit theorems constitute an important part of the branching process theory. Since Galton-Watson processes are unstable, people have derived limit theorems for them through devices such as modifying factors, conditioning, immigration, etc. A unified treatment of the limit theory of Galton-Watson processes is given in Arthreya and Ney (1972). Some of the above mentioned techniques have also been used in the measure-valued setting to get limit theorems for Dawson-Watanabe superprocesses. See e.g. Evans and Perkins (1990) and Krone (1995) for some limit theorems of the conditioned superprocesses. Indeed, the superprocess provides a richer source for limit theorems. A well-known result of Dawson (1977) is that, if the underlying motion is a transient symmetric stable process, the critical continuous superprocess started with the Lebesgue measure converges to a non-trivial steady state. It was also shown in Dawson (1977) that that the steady random measure has an interesting spatial central limit theorems which lead to Gaussian random fields. Some limit theorems for the weighted occupation time of the super stable process were proved in Iscoe (1986ab). Clearly, these results have no counterparts in Galton-Watson processes. To describe those limit theorems we need to extend the state space of the superprocess to include some infinite measures.

Let ξ be a Borel right process in E. Suppose $\beta > 0$ and $\rho \in C(E)^{++}$ is a β-excessive function for ξ. Let ϕ be a branching mechanism given by (1.4). Here we only assume $b \in B(E)^+$, $\rho c \in B(E)^+$ and

$$\sup_{x \in E} \int_0^\infty u \wedge [\rho(x) u^2] m(x, \mathrm{d}u) < \infty. \tag{6.1}$$

Let $B_\rho(E)^+$ be the totality of non-negative Borel functions on E bounded by $\rho \cdot \mathrm{const}$, and let $M_\rho(E)$ be the space of Borel measures μ on E satisfying $\mu(\rho) < \infty$. The topology on $M_\rho(E)$ is defined by the convention: $\mu_k \to \mu$ if and only if $\mu_k(f) \to \mu(f)$ for all continuous functions f dominated by $\rho \cdot \mathrm{const}$. Let W_ρ be the space of all right continuous paths $\omega : [0, \infty) \to M_\rho(E)$ with the coordinate process $\{X_t(\omega) : t \geq 0\}$, and let $(\mathcal{G}^\circ, \mathcal{G}_t^\circ)$ denote the natural σ-algebras on W_ρ. Suppose that $(V_t)_{t\geq 0}$ is defined by (1.5). Then we have the following

Theorem 6.1. *For each $\mu \in M_\rho(E)$ there is a unique probability measure $\mathbf{Q}_\mu$ on $(W_\rho, \mathcal{G}^\circ)$ such that $\mathbf{Q}_\mu\{X_0 = \mu\} = 1$ and $\{X_t : t \geq 0\}$ under $\mathbf{Q}_\mu$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$ defined by*

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_t f)\right\}, \quad \mu \in M_\rho(E), f \in B_\rho(E)^+. \tag{6.2}$$

Furthermore, the system $(W_\rho, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ is a Borel right process, where $(\mathcal{G}, \mathcal{G}_t)$ is the augmentation of $(\mathcal{G}^\circ, \mathcal{G}_{t+}^\circ)$ by the system $\{\mathbf{Q}_\mu : \mu \in M_\rho(E)\}$.

Proof. We use an argument suggested in El Karoui and Roelly (1991). Since $\rho \in C(E)^{++}$ is a β-excessive function for ξ, we may define the transition semigroup $(T_t)_{t\geq 0}$ of a Borel right process η on E by $T_t f(x) = \mathrm{e}^{-\beta t}\rho(x)^{-1}P_t(\rho f)(x)$. Let $\psi(x,z) = \phi(x,z) - \beta z$. Then

$$\psi(x,z) = [b(x) - \beta]z + c(x)\rho(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)\rho(x)^{-1} n(x, \mathrm{d}u), \tag{6.3}$$

where $n(x, \mathrm{d}u)$ is the image of $m(x, \mathrm{d}u)$ under the mapping $u \mapsto \rho(x)u$. Under our hypotheses, $(u \wedge u^2)\rho(x)^{-1}n(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. Let $(U_t)_{t\geq 0}$ be the solution to (1.5) with $(P_t)_{t\geq 0}$ and ϕ replaced by $(T_t)_{t\geq 0}$ and ψ, respectively. It is easy to check that $U_t f(x) = \rho(x)^{-1}V_t(\rho f)(x)$. Now Theorem 4.1 guarantee the existence of a Borel right (η, ψ)-superprocess Y with state space $M(E)$ and cumulant semigroup $(U_t)_{t\geq 0}$. Since $\mu(\mathrm{d}x) \mapsto \rho(x)^{-1}\mu(\mathrm{d}x)$ determines a homeomorphism between $M(E)$ and $M_\rho(E)$, the theorem follows immediately by Sharpe (1988: p75). $\square$

6.2. Let ξ be a symmetric stable process with index α $(0 < \alpha \leq 2)$. Let $h_p(x) = (1 + |x|^p)^{-1}$ for $x \in I\!\!R^d$ and $p > 0$. From the discussions in Iscoe (1986a) it can be deduced that h_p is a β-excessive function for the symmetric stable process for some $0 < \beta < \infty$. Write $M_p(I\!\!R^d)$ for $M_\rho(I\!\!R^d)$ with $\rho = h_p$. By Theorem 5.1, the state space of the super stable process can be extended to $M_p(I\!\!R^d)$. Now we can give the results of Dawson (1977); see also Dawson and Perkins (1991).

Theorem 6.2. (Dawson, 1977) *Let $\{X_t : t \geq 0\}$ be the super stable process with branching mechanism cz^2 ($c = \mathrm{const} > 0$) and state space $M_p(\mathbb{R}^d)$. We take $p > d$, so the Lebesgue measure λ belongs to $M_p(\mathbb{R}^d)$.*

(i) Suppose that the underlying process is recurrent, i.e. $\alpha \geq d$. If there is a constant $0 < \gamma < \infty$ such that $\mu(B) \leq \gamma\lambda(B)$ for all bounded cube centered at the origin, then $\lim_{t\to\infty} \mathbf{Q}_\mu\{X_t(K) > \varepsilon\} = 0$ for any $\varepsilon > 0$ and compact set $K \subset \mathbb{R}^d$.

(ii) Suppose that the underlying process is transient, i.e. $\alpha < d$. For any $0 < \theta < \infty$, the distribution of X_t under $\mathbf{Q}_{\theta\lambda}$ converges as $t \to \infty$ to a probability measure Q^θ on $M_p(\mathbb{R}^d)$ which is both a steady state for the super stable process and also invariant under spatial translation.

From Theorem 6.2 (i) it can be deduced that in low dimensions ($d \leq \alpha$) the only steady state with finite intensity for the super sable process X is the empty state. Indeed, Bramson et al (1994) showed that the empty state is the only steady state of the process without any restriction on the intensity. By Theorem 6.2 (ii), in high dimensions ($d > \alpha$), X has at least a family of steady states $\{Q^\theta : 0 < \theta < \infty\}$. An application of the results of Dynkin (1989) shows that every Q^θ is an extremal steady state of X and it has no other extremal steady states in $M_p(\mathbb{R}^d)$. The ergodic theory for X is interesting since it is quite similar to a wide class of processes such as branching particle systems, interacting diffusions, coupled random walk models, voter models, contact path processes, etc. See Bramson et al (1994), Cox and Griffeath (1986), Griffeath (1983) and the references therein. The renormalization theory for the steady states of X was also studied in Dawson (1977).

Theorem 6.3. (Dawson, 1977) *Suppose that $\alpha < d$. Let X_∞ be the steady state random measure of the super stable process with branching mechanism cz^2 ($c = \mathrm{const} > 0$). For $k > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$ let $X_\infty^{(k)}(B) = X_\infty(\{kx : x \in B\})$. Then there are constants a_k and b_k such that $(X_\infty^{(k)} - a_k)/b_k$ converges as $k \to \infty$ to a Gaussian random fields with convariance kernel given by the potential kernel of the underlying stable process.*

Some central limit theorems for the weighted occupation time process of the super stable process were given in Iscoe (1986a). Note that the central limit theorems of Dawson (1977) and Iscoe (1986a) only cover dimension numbers $d \geq 3$ in the case where ξ is a standard Brownian motion.

6.3. Let us consider the case where ξ is a diffusion process in $\mathbb{R}^d$ generated by the differential operator

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j}, \tag{6.4}$$

where (a_{ij}) is uniformly positive definite, bounded and continuous, and (b_j) is bounded and continuous. Let h_p be defined as in paragraph 6.2. Then we have

Lemma 6.1. *For some $\alpha \geq 0$, h_p is an α-excessive function for the diffusion process generated by the differential operator A.*

Proof. It is easy to check that for some $\alpha \geq 0$ we have $Ah_p(x) \leq \alpha h_p(x)$, so

$$\frac{\partial}{\partial t} P_t h_p(x) = P_t A h_p(x) \leq \alpha P_t h_p(x).$$

It follows that $P_t h_p(x) \leq \mathrm{e}^{\alpha t} P_t h_p(x)$. By the strong continuity of $(P_t)_{t\geq 0}$ we know that h_p is α-excessive. $\square$

The following theorem extends the results of Dawson (1977) and Dawson and Perkins (1991).

Theorem 6.4. (Wang, 1998) *Let ϕ be given by (1.4) with $b = 0$, $c \geq 0$ and $m(\mathrm{d}u)$ all independent of $x \in I\!\!R^d$ and $\int_0^\infty [u \vee u^2] m(\mathrm{d}u) < \infty$. Let X be the (ξ, ϕ)-superprocess. Suppose that $\mu \in M_p(I\!\!R^d)$ is an invariant measure for ξ and $\mu(\mathrm{d}x) = h(x)\lambda(\mathrm{d}x)$ for some $h \in C(I\!\!R^d)^+$.*

(i) If $d \leq 2$, then $\lim_{t\to\infty} \mathbf{Q}_\mu\{X_t(K) > \varepsilon\} = 0$ for any $\varepsilon > 0$ and compact set $K \subset I\!\!R^d$.

(ii) If $d \geq 3$, then the distribution of X_t under $\mathbf{Q}_\mu$ converges as $t \to \infty$ to a probability measure on $M_p(I\!\!R^d)$ which is a steady state for the superprocess X.

7. Modifications of the branching models

7.1. The particle system considered in section 3 only involves local branching, that is, all the offspring start migrating at the death sites of their parents. One may also consider the situation where the offspring are displaced randomly into the whole space, which can be formulated using a Markov kernel $\tau(x, \mathrm{d}\nu)$ from E to $N(E)$ in the place of the generating function $g(x, z)$. By considering the convergence of the generalized branching particle systems one obtains a superprocess X *with non-local branching.* In one typical case, the cumulant semigroup $(V_t)_{t\geq 0}$ of this superprocess is determined by

$$V_t f(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_E [\phi(y, V_{t-s} f(y)) - \varphi(y, V_{t-s} f)]\, P_s(x, \mathrm{d}y), \tag{7.1}$$

where ϕ is given by (1.4), φ is an operator on $B(E)^+$ having the representation

$$\varphi(x, f) = d(x, f) + \int_{M(E)^\circ} (1 - \mathrm{e}^{-\nu(f)}) n(x, \mathrm{d}\nu), \tag{7.2}$$

$d(x, \mathrm{d}y)$ is a bounded kernel on E and $[1 \wedge \nu(E)]n(x, \cdot)$ is a bounded kernel from E to $M(E)^\circ$. We may call X a (ξ, ϕ, φ)-*superprocess* if it is given by (1.3) and (7.1); see e.g. Dynkin (1993a). A class of multi-type Dawson-Watanabe superprocesses can be constructed by using the existence of a (ξ, ϕ, φ)-superprocess; see Gorostiza et al (1992) and Li (1992a, 1993). The multi-type superprocesses have also been studied in

Gorostiza and Lopez-Mimbela (1990), Gorostiza and Roelly (1990), Wang (1996), Ye (1995), etc.

7.2. Multi-level branching particle systems and superprocesses arise as mathematical models for hierarchically structured populations. Those models were introduced by Dawson and Hochberg (1991). Let X^1 be a (ξ, ϕ)-superprocess (one-level) and let $\phi^1(\cdot, \cdot)$ be a branching mechanism on $M(E)$. Then the (X^1, ϕ^1)-superprocess X^2 is called a *two-level superprocess*. We define the *aggregated process* Z associated with the two-level superprocess by

$$Z_t(B) = \int_{M(E)} \nu(B) X_t^2(\mathrm{d}\nu), \quad t \geq 0, B \in \mathcal{B}(E).$$

One remarkable feature of the multi-level models is that its critical dimensions which separate the persistence and extinction long time behaviors is of higher order than the one-level model; see Etheridge (1993), Gorostiza (1996) and Wu (1993, 1994).

7.3. A super Brownian motion *with a single point catalyst*, $X = \{X_t : t \geq 0\}$, over $\mathbb{R}$ was studied in detail in Dawson and Fleischmann (1994). The cumulant semigroup $(V_t)_{t\geq 0}$ of X is given by

$$V_t f(x) = P_t f(x) - \int_0^t [V_{t-s} f(z)]^2 p_s(x, z) \mathrm{d}s, \quad t \geq 0, x \in \mathbb{R}, \tag{7.3}$$

where $z \in \mathbb{R}$ is fixed, and $(P_t)_{t\geq 0}$ is the semigroup of the Brownian motion in $\mathbb{R}$ with density $p_t(\cdot, \cdot)$. In this model, the branching is allowed only at the point catalyst at $z \in \mathbb{R}$. The process X has a version such that a.s.

$$\int_0^t X_s(\mathrm{d}x)\mathrm{d}s = \eta(t, x)\mathrm{d}x, \quad t \geq 0, x \in \mathbb{R},$$

for a continuous process $\{\eta(t, x) : t \geq 0, x \in \mathbb{R}\}$, which is non-decreasing in $t \geq 0$. The *occupation density measures* of this process, $\lambda_x(\mathrm{d}t) := \mathrm{d}\eta(t, x)$, is a.s. absolutely continuous provided $x \neq z$. On the other hand, the measure $\lambda_z(\mathrm{d}t)$ at the location of the catalyst is a.s. singular with carrying Hausdorff dimension one; see Dawson and Fleischmann (1994).

For the study of branching models with catalysts; see also Dawson and Fleischmann (1992), Fleischmann (1994), Fleischmann and Le Gall (1995), etc.

7.4. Several kinds of interacting branching models have been constructed and studied as variations of the classical Dawson-Watanabe superprocesses. See e.g. Méléard and Roelly (1992, 1993), Perkins (1992). The processes studied in Méléard and Roelly (1993) involve mean field interactions in the sense that the migrating and branching of each particle is influenced by the entire population.

Recall that $C([0, \infty), M(\mathbb{R}^d))$ is the space of all continuous paths ω_t: $[0, \infty) \to M(\mathbb{R}^d)$ with the coordinate process $X_t(\omega) = \omega_t$. We fix two bounded, continuous functions $c(\cdot,\cdot)\geq 0$ and $b(\cdot,\cdot)$ on the space $M(I\!R^d)\times I\!R^d$. Let $\{P_t(\mu):\mu\in M(I\!R^d)\}$ be a family of conservative Feller semigroups on $C(I\!R^d)$ with generators $\{A(\mu):\mu\in M(I\!R^d)\}$. Assume that $\{A(\mu):\mu\in M(I\!R^d)\}$ have domains that all contain a vector space $\mathcal{D}$ independent of μ and dense in $C(I\!R^d)$. Furthermore, we assume that, for each $f\in\mathcal{D}$,

(4A) there is a constant $K(f)>0$ such that $\|A(\mu)f\|\leq K(f)\mu(1)$;

(4B) $\mu(A(\mu)f)$ is continuous in $\mu\in M(I\!R^d)$.

It follows from the construction in Méléard and Roelly (1993) that for each $\mu\in M(I\!R^d)$ there is a probability measure $\mathbf{Q}_\mu$ on $C([0,\infty),M(I\!R^d))$ such that, for any $f\in\mathcal{D}$,

$$M_t(f):=X_t(f)-\mu(f)-\int_0^t\mathrm{d}s\int_E[A(X_s)f(x)-b(X_s,x)f(x)]X_s(\mathrm{d}x),\quad t\geq 0,\tag{7.4}$$

is a $\mathbf{Q}_\mu$-martingale starting at zero with quadratic variation process

$$\langle M(f)\rangle_t=\int_0^t\int_{I\!R^d}c(X_s,x)f(x)^2X_s(\mathrm{d}x)\mathrm{d}s,\quad t\geq 0.\tag{7.5}$$

(See (4.5) and (4.6).) Let us call the process simply an *interacting superprocess* following Méléard and Roelly (1992, 1993). If A, b and c are all independent of μ, this degenerates to the non-interacting superprocess and $\{\mathbf{Q}_\mu:\mu\in M(I\!R^d)\}$ is uniquely determined by (7.4) and (7.5); see section 4. In general, the uniqueness of solutions to this martingale problem is still unknown.

We have mentioned that, when $d=1$, the non-interacting superprocess is absolutely continuous with respect to the Lebesgue measure on $I\!R$ for a large class of admissible generators A. The same result for the interacting superprocess was conjectured in Méléard and Roelly (1992). This has been proved by Zhao (1997); see also Li (1997c) and Liang and Li (1998).

The following results were proved in Méléard and Roelly (1993): If the underlying motion is a symmetric stable process with index α $(0<\alpha\leq 2)$ independent of μ, then for each $t>0$, the Hausdorff dimension of the Borel support of X_t is a.s. not less than $d\wedge\alpha$. Under the additional condition $c(\mu,x)\equiv\text{const}$, Méléard and Roelly (1993) proved that the carrying Hausdorff dimension of X_t is $d\wedge\alpha$ for all $t>0$ a.s. Compared with what we have known about the non-interacting superprocess, the interacting one is much less understood. See Wang and Zhao (1996) for more complete survey on measure-valued branching processes with interaction.

8. Skew convolution semigroups and entrance laws (I)

8.1. Let X be an MB-process with transition semigroup $(Q_t)_{t\geq 0}$. Recall that the family of probability measures $(N_t)_{t\geq 0}$ is called a skew convolution semigroup associated with X or $(Q_t)_{t\geq 0}$ if

$$N_{r+t}=(N_rQ_t)*N_t,\quad r,t\geq 0.\tag{8.1}$$

An immigration process Y associated with X is a Markov process in $M(E)$ with transition semigroup $(Q_t^N)_{t\geq 0}$ defined by

$$Q_t^N(\mu,\cdot) := Q_t(\mu,\cdot) * N_t, \quad t \geq 0,\ \mu \in M(E), \tag{8.2}$$

for a skew convolution semigroup $(N_t)_{t\geq 0}$. Here $(N_t)_{t\geq 0}$ determines the immigration structures of Y. A family of σ-finite measures $(K_t)_{t>0}$ on $M(E)$ is called an *entrance law* for X or its semigroup $(Q_t)_{t\geq 0}$ if $K_{r+t} = K_r Q_t$ for all $r,t > 0$. It is called a *probability entrance law* if each K_t is a probability measure on $M(E)$, an *infinitely divisible probability entrance law* if, in addition, each K_t is infinitely divisible.

It is well-known that a usual convolution semigroup on the Euclidean space is uniquely determined by an infinitely divisible probability measure. The next theorem characterizes the skew convolution semigroups associated with an MB-process in terms of its infinitely divisible probability entrance laws.

Theorem 8.1. (Li, 1996a) *The family of probability measures $(N_t)_{t\geq 0}$ on $M(E)$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ such that*

$$\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \int_0^t \left[\log \int_{M(E)} \mathrm{e}^{-\nu(f)} K_s(\mathrm{d}\nu)\right] \mathrm{d}s \tag{8.3}$$

for all $f \in B(E)^+$.

8.2. Let X be a (ξ,ϕ)-superprocess and let $\mathcal{K}^1(Q)$ denote the set of probability entrance laws $K = (K_t)_{t>0}$ for the semigroup $(Q_t)_{t\geq 0}$ such that

$$\int_0^1 \mathrm{d}s \int_{M(E)^\circ} \nu(E) K_s(\mathrm{d}\nu) < \infty. \tag{8.4}$$

Let $\mathcal{K}_m^1(Q)$ denote the subset of $\mathcal{K}^1(Q)$ comprising minimal elements. Denote by $(Q_t^\circ)_{t\geq 0}$ the restriction of $(Q_t)_{t\geq 0}$ to $M(E)^\circ$, and $\mathcal{K}(Q^\circ)$ the set of entrance laws K for $(Q_t^\circ)_{t\geq 0}$ satisfying (8.4). Let $\mathcal{K}(P)$ be the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for the underlying semigroup $(P_t)_{t\geq 0}$ that satisfy $\int_0^1 \kappa_s(E)\mathrm{d}s < \infty$. For $\kappa \in \mathcal{K}(P)$, set

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) \kappa_{t-s}(\mathrm{d}y). \tag{8.5}$$

Clearly, if $\kappa_t = \gamma P_t$ for some $\gamma \in M(E)$, then $S_t(\kappa, f) = \gamma(V_t f)$. The spaces $\mathcal{K}(P)$ and $\mathcal{K}_m^1(Q)$ are closely related:

Theorem 8.2. (Li, 1996b) *There is a one-to-one correspondence between $\kappa \in \mathcal{K}(P)$ and $K := l\kappa \in \mathcal{K}_m^1(Q)$, which is given by*

$$\kappa_t(f) = \lim_{r\downarrow 0} \int_{M(E)} \nu(P_{t-r}f) K_r(\mathrm{d}\nu), \tag{8.6}$$

and

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{-S_t(\kappa, f)\right\}. \tag{8.7}$$

If ξ is conservative, each $\kappa \in \mathcal{K}(P)$ is uniquely determined by a measure $\kappa_0 \in M(E_D)$, where E_D is the entrance space of ξ; see Sharpe (1988). In that case, Theorem 8.2 follows from a result of Fitzsimmons (1988). See also Dynkin (1989c) for the analogous results in the case where $\phi(x, z) \equiv c(x)z^2$ but ξ is allowed to be non-homogeneous and X is allowed to take values in a space of σ-finite measures.

We can give a description for infinitely divisible probability entrance laws for the (ξ, ϕ)-superprocess as follows.

Theorem 8.3. (Li, 1996b, 1997b) *The probability entrance law $K \in \mathcal{K}^1(Q)$ is infinitely divisible if and only if its Laplace functional has the representation*

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) \\ &= \exp\bigg\{ - S_t(\kappa, f) - \int_{\mathcal{K}(P)} \left(1 - \exp\left\{-S_t(\eta, f)\right\}\right) F(\mathrm{d}\eta)\bigg\},\end{aligned} \tag{8.8}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta) < \infty. \tag{8.9}$$

It follows by Theorems 8.1 and 8.3 that, under the first moment condition, the transition semigroup of a general immigration process associated with the (ξ, ϕ)-superprocess is given by

$$\begin{aligned}\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^N(\mu, \mathrm{d}\nu) = \exp\bigg\{ &- \mu(V_t f) \\ &- \int_0^t \bigg[S_r(\kappa, f) + \int_{\mathcal{K}(P)} \left(1 - \exp\left\{-S_r(\eta, f)\right\}\right) F(\mathrm{d}\eta)\bigg] \mathrm{d}r\bigg\},\end{aligned} \tag{8.10}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (8.9). Let us look at two examples of the immigration process; some other examples will be given latter.

Example 8.1. Let $a > 0$ and $d \geq 0$ be real constants. We consider the one-dimensional stochastic differential equation

$$\mathrm{d}Y_t = \sqrt{2a|Y_t|}\ \mathrm{d}B_t + d\mathrm{d}t, \tag{8.11}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion starting from zero. The equation defines a unique conservative diffusion process Y on $I\!R^+$ with generator $L^{a,d}$ such that

$$L^{a,d} f(x) = ax \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + d \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

and $\mathcal{D}(L^{a,d}) = C_0^2(I\!R^+)$, twice continuously differentiable functions on $I\!R^+$ vanishing at infinity. The transition semigroup $(Q_t^{a,d})_{t\geq 0}$ of Y is determined by

$$\int_0^\infty \mathrm{e}^{-\lambda y} Q_t^{a,d}(x, \mathrm{d}y) = \exp\bigg\{ - xv_t(\lambda) - \int_0^t dv_s(\lambda)\mathrm{d}s \bigg\},$$

where $v_t(\lambda) = \lambda/(at+1)$ is the solution to

$$\frac{\mathrm{d}v_t}{\mathrm{d}t}(\lambda) = -av_t(\lambda)^2, \quad v_0(\lambda) = \lambda.$$

Therefore Y is an MBI-process with the underlying space E degenerating to a singleton, which is known as a *continuous state branching process with immigration* (CBI-process) in the literature. See e.g. Ikeda and Watanabe (1989; p235) and Kawazu and Watanabe (1971). Let $\{Y_t(d) : t \geq 0\}$ be the solution to (8.11) with $a = 2$. Then $\{Y_t(d)^{1/2} : t \geq 0\}$ is a *Bessel diffusion process* with parameter d. That is, the Bessel diffusion is essentially a particular case of the CBI-process. This connection between the Bessel diffusion and the immigration process was first noticed by Shiga and Watanabe (1973). The work of Pitman and Yor (1982) on "quadratic functionals" of Bessel bridges is essentially based on this connection.

Example 8.2. Let us recall the Ray-Knight theorem on Brownian local times. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, B_t, \mathbf{P}_x)$ is a one dimensional Brownian motion with the local times $\{l(t, x) : t \geq 0, x \in I\!R\}$, which is a continuous two parameter process such that a.s.

$$2\int_A l(t,x)\mathrm{d}x = \int_0^t 1_A(B_s)\mathrm{d}s, \quad t \geq 0, A \in \mathcal{B}(I\!R).$$

For $b \geq 0$ and $\alpha \geq 0$, let $T_\alpha(-b) = \inf\{t >: l(t, -b) > \alpha\}$. Then the process $\{l(T_\alpha(-b), x) : x \in I\!R\}$ under $\mathbf{P}_0$ is an inhomogeneous Markov process with continuous paths and $l(T_\alpha(-b), -b) = \alpha$. There are three homogeneity intervals, that is, $\{l(T_\alpha(-b), x) : x \geq 0\}$ and $\{l(T_\alpha(-b), -b - x) : x \geq 0\}$ have the same generator $L^{1,0}$ and $\{l(T_\alpha(-b), -b + x) : 0 \leq x \leq b\}$ has the generator $L^{1,1}$. See e.g. Knight (1981; p137).

9. Skew convolution semigroups and entrance laws (II)

9.1. Recall that a branching particle system is a Markov process in $N(E)$, the space of integer-valued measures on E. In this section $(Q_t)_{t\geq 0}$ denotes the semigroup of such a system, and $(Q_t^\circ)_{t\geq 0}$ denotes the restriction of $(Q_t)_{t\geq 0}$ to the subspace $N(E)^\circ = N(E) \setminus \{0\}$. The notion of skew convolution semigroup can also be introduced for branching particle systems, which we shall not repeat here; see Li (1997a). We have the following analogue of Theorem 8.1 for a branching particle system.

Theorem 9.1. (Li, 1997a) *Suppose $(N_t)_{t\geq 0}$ is a family of probability measures on $N(E)$. Then $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is an entrance law $(H_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$ such that*

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \exp\left\{ -\int_0^t \mathrm{d}s \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu)\right\} \tag{9.1}$$

for all $f \in B(E)^+$.

The major difference between a (ξ, ϕ)-superprocess and a branching particle system is that, started with any deterministic state, the former is infinitely divisible and the latter, which can only be started with an integer-valued measure, is not. These cause some technical difficulties for the description of entrance laws for the particle system. Indeed, the characterization of all entrance laws for a general branching particle system still remains open although some partial results have been given in Li (1997a).

9.2. Suppose that D is a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D and closure $\bar{D}$. Let ξ be a minimal Brownian motion in D. Assume that both $g(x,z)$ and $[\mathrm{d}/\mathrm{d}z]g(x,z)$ can be extended to continuous functions on $\bar{D} \times [0,1]$. It is well-known that the transition density of ξ is continuously differentiable to the boundary ∂D; see e.g. Friedman (1984: p82). We use ∂ to denote the inward normal derivative operator at ∂D. In this paragraph, $\mathcal{K}(Q^\circ)$ denotes the space of entrance laws K for the (ξ, γ, g)-system satisfying (8.4) with $M(E)^\circ$ replaced by $N(D)^\circ$. Set $h(x) = \int_0^1 P_s 1(x) \mathrm{d}s$. Let $N_h(D)$ be the set of integer-valued measures σ on D satisfying $\sigma(h) < \infty$, and $N_h(\bar{D})$ the set of measures μ on $\bar{D}$ such that $\mu_D := \mu|_D \in N_h(D)$ and $\mu_\partial := \mu|_{\partial D} \in M(\partial D)$. Then we have the following

Theorem 9.2. (Li, 1997a) *In order that $(H_t)_{t>0} \in \mathcal{K}(Q^\circ)$ it is necessary and sufficient that its Laplace functional is given by*

$$\begin{aligned}&\int_{N(D)} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) \\ =&\gamma(\partial U_t f) + \int_{N_h(\bar{D})} \left(1 - \exp\left\{-\nu_D(U_t f) - \nu_\partial(\partial U_t f)\right\}\right) G(\mathrm{d}\nu),\end{aligned} \tag{9.2}$$

where $\gamma \in M(\partial D)$ *and* G *is a measure on* $N_h(\bar{D})$ *satisfying*

$$\int_{N_h(\bar{D})} [\nu(h) + \nu(\partial h)] G(\mathrm{d}\nu) < \infty.$$

10. Immigration processes and Kuznetsov processes

10.1. The measure-valued immigration processes may be constructed from Kuznetsov processes determined by entrance rules for the original MB-process. Let us review some basic facts in potential theory. A family of σ-finite measures $(J_t)_{t\in I\!R}$ is called an *entrance rule* for $(Q_t^\circ)_{t\geq 0}$ if $J_s Q_{t-s}^\circ \leq J_t$ for all $t > s \in I\!R$ and $J_s Q_{t-s}^\circ \uparrow J_t$ as $s \uparrow t$. Let $W(M(E))$ denote the space of paths $\{w_t : t \in I\!R\}$ that are $M(E)^\circ$-valued and right continuous on an open interval $(\alpha(w), \beta(w))$ and take the value of the null measure elsewhere. The path $[0]$ constantly equal to the null measure corresponds to (α, β) being empty. Set $\alpha([0]) = +\infty$ and $\beta([0]) = -\infty$. Let $(\mathcal{H}^\circ, \mathcal{H}_t^\circ)_{t\in I\!R}$ be the natural σ-algebras on $W(M(E))$ generated by the coordinate process. Then to each entrance rule $(J_t)_{t\in I\!R}$ for $(Q_t^\circ)_{t\geq 0}$, there corresponds a unique σ-finite measure $\mathbf{Q}^J$ on $(W(M(E)), \mathcal{H}^\circ)$ under which $\{w_t : t \in I\!R\}$ is a Markov process with one-dimensional distributions $(J_t)_{t\in I\!R}$ and semigroup $(Q_t^\circ)_{t\geq 0}$, that is, for any $t_1 < \cdots < t_n \in I\!R$, and $\nu_1, \cdots, \nu_n \in M(E)^\circ$,

$$\begin{aligned} &\mathbf{Q}^J\{\alpha < t_1, w_{t_1} \in \mathrm{d}\nu_1, w_{t_2} \in \mathrm{d}\nu_2, \cdots, w_{t_n} \in \mathrm{d}\nu_n, t_n < \beta\} \\ =& J_{t_1}(\mathrm{d}\nu_1) Q_{t_2 - t_1}^\circ(\nu_1, \mathrm{d}\nu_2) \cdots Q_{t_n - t_{n-1}}^\circ(\nu_{n-1}, \mathrm{d}\nu_n). \end{aligned} \tag{10.1}$$

The existence of this measure was proved by Kuznetsov (1974); see also Getoor and Glover (1987). The system $(W(M(E)), \mathcal{H}^\circ, \mathcal{H}_t^\circ, w_t, \mathbf{Q}^J)$ is commonly called the *Kuznetsov process* determined by $(J_t)_{t\in I\!R}$, and $\mathbf{Q}^J$ is called the *Kuznetsov measure*.

Recall that a probability measure F is infinitely divisible if and only if its Laplace functional has the canonical representation

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} F(\mathrm{d}\nu) = \exp\bigg\{ -\eta(f) - \int_{M(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(f)}\Big) H(\mathrm{d}\nu)\bigg\}, \tag{10.2}$$

where $\eta \in M(E)$ and $[1 \wedge \nu(E)] H(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. We write $F = I(\eta, H)$ if F is given by (10.2). From Theorem 8.1 it follows that, if $(N_t)_{t\geq 0}$ is a skew convolution semigroup, then $N_0 = \delta_0$ and each N_t is infinitely divisible.

Theorem 10.1. (Li, 1997d) *Suppose that* $(N_t)_{t\geq 0}$ *is a skew convolution semigroup with representation* $N_t = I(\gamma_t, G_t)$*. Define* $G_t = 0$ *for* $t < 0$*. Then* $(G_t)_{t\in I\!R}$ *is an entrance rule for* $(Q_t^\circ)_{t\geq 0}$*. Let* $N^G(\mathrm{d}w)$ *be a Poisson random measure on* $W(M(E))$ *with intensity* $\mathbf{Q}^G(\mathrm{d}w)$ *and define*

$$I_t^G = \int_{W(M(E))} w_t \, N^G(\mathrm{d}w), \quad t \geq 0. \tag{10.3}$$

Then $\{\gamma_t + I_t^G : t \geq 0\}$ *is an immigration process corresponding to* $(N_t)_{t\geq 0}$.

This shows that a general immigration process may be decomposed into two parts, one part is deterministic and the other part can be constructed from a Kuznetsov process. This type of constructions for immigration processes have been discussed in Li (1996b), Li and Shiga (1995) and Shiga (1990). See also Evans (1993) a similar, but different, construction for conditioned (ξ, ϕ)-superprocesses.

10.2. A natural and realistic problem one would raise is "For a given immigration process, what is the largest possible space where all the immigrants come from?" In view of Theorem 10.1, this problem may be answered by studying the behaviors of the Kuznetsov process $\{w_t : \alpha < t < \beta\}$ near the birth time $\alpha = \alpha(w)$. We shall see that almost all those paths start propagation in some extension of E which can be given explicitly as follows.

We consider a Doob's h-transform of the underlying semigroup $(P_t)_{t\geq 0}$. Set $h(x) = \int_0^1 P_s 1(x)\mathrm{d}s$. Since $h \in B(E)^+$ is an excessive function of $(P_t)_{t\geq 0}$,

$$T_t f(x) := h(x)^{-1} \int_E f(y)h(y)P_t(x, \mathrm{d}y) \tag{10.4}$$

defines a Borel right semigroup $(T_t)_{t\geq 0}$ with state space E. See e.g. Sharpe (1988). Let $(T_t^\partial)_{t\geq 0}$ be a conservative extension of $(T_t)_{t\geq 0}$ to $E^\partial := E \cup \{\partial\}$, where ∂ is the cemetery point. Let E_D^∂ be the entrance space of $(T_t^\partial)_{t\geq 0}$ and let $E_D^T = E_D^\partial \setminus \{\partial\}$. We endow E_D^∂ and E_D^T with the Ray topology of $(T_t^\partial)_{t\geq 0}$. Then we have

Theorem 10.2. (Li, 1997d) *Let* $(J_t)_{t\in I\!R}$ *be an entrance rule for* $(Q_t^\circ)_{t\geq 0}$ *such that*

$$\int_r^t \mathrm{d}s \int_{M(E)^\circ} \nu(E) J_s(\mathrm{d}\nu) < \infty, \quad r < t \in I\!R.$$

For $w \in W$ *define the* $M(E_D^T)$*-valued path* $\{h\bar{w}_t : t > 0\}$ *by*

$$h\bar{w}_t(E_D^T \setminus E) = 0 \text{ and } h\bar{w}_t(\mathrm{d}x) = h(x)w_t(\mathrm{d}x) \text{ for } x \in E. \tag{10.5}$$

Then for $\mathbf{Q}^J$*-a.a.* $w \in W(M(E))$, $\{h\bar{w}_t : t \in I\!R\}$ *is right continuous in* $M(E_D^T)^\circ$ *on the interval* $(\alpha(w), \beta(w))$ *and* $h\bar{w}_t \to$ *some* $h\bar{w}_\alpha \in M(E_D^T)$ *as* $t \downarrow \alpha(w)$. *Moreover, for* $\mathbf{Q}^J$*-a.a. paths* $w \in W(M(E))$ *with* $h\bar{w}_\alpha = 0$, *we have* $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ *for some* $x(w) \in E_D^T$ *as* $t \downarrow \alpha(w)$.

11. Immigration processes over the half line

11.1. Let us consider the case where E is the positive half line $H := (0, \infty)$. Suppose that ξ is the minimal Brownian motion in H. The transition semigroup $(P_t)_{t\geq 0}$ of ξ is determined by

$$P_t f(x) = \int_H [g_t(x - y) - g_t(x + y)] f(y)\mathrm{d}y, \tag{11.1}$$

where $g_t(x) = \exp\left\{-x^2/2t\right\}/\sqrt{2\pi t}$. We shall call the corresponding (ξ, ϕ)-superprocess X simply a *super minimal Brownian motion.* In this case, we may identify E_D^T as $\mathbb{R}^+$. Let $\kappa \in \mathcal{K}(P)$ be defined by $\kappa_t(f) = \partial_0 P_t f$, where ∂_0 denotes the upward derivative at the origin. Then $S_t(\kappa, f) = \partial_0 V_t f$. Let $M_h(H)$ be the set of Borel measures μ on H such that $\mu(h) < \infty$.

Lemma 11.1. (Li and Shiga, 1995) *For each $\eta \in \mathcal{K}(P)$, there exist a constant $q \geq 0$ and a measure $m \in M_h(H)$ such that $\eta_t = mP_t + q\kappa_t$ for all $t > 0$.*

If $\eta \in \mathcal{K}(P)$ is given as in the above lemma, then we have $S_t(\eta, f) = m(V_t f) + q\partial_0 V_t f$. Combining these with Theorems 8.1 and 8.3 gives a complete characterization of the immigration structures associated with the super minimal Brownian motion.

11.2. Let us consider an immigration process with transition semigroup $({}^{\kappa}Q_t)_{t\geq 0}$ defined by

$$\begin{aligned}&\int_{M(H)} \mathrm{e}^{-\nu(f)}\ {}^{\kappa}Q_t(\mu, \mathrm{d}\nu)\\ &= \exp\left\{ -\mu(V_t f) - \int_0^t (1 - \exp\{-\partial_0 V_s f\})\,\mathrm{d}s\right\}.\end{aligned} \tag{11.2}$$

Let $G_t = \int_0^t l\kappa_s \mathrm{d}s$, where $l\kappa$ is defined by (8.7). Then $h\bar{w}_t \to h'(0^+)\delta_0$ and hence $w_t(H) \to \infty$ as $t \downarrow \alpha$ for $\mathbf{Q}^G$-a.a. $w \in W(M(H))$, and the immigration process may be constructed by (10.3); see Li (1996b). This shows that the transformation $w_t \mapsto h\bar{w}_t$ in Theorem 10.2 is necessary if one hopes to get the limit $\lim_{t\downarrow\alpha} w_t$ for $w \in W(M(H))$ in some sense. Intuitively, the process is generated by cliques of immigrants with infinite mass coming in H from the original. The semigroup $({}^{\kappa}Q_t)_{t\geq 0}$ has no right continuous realization; see Li (1996b).

11.3. Let us consider the super Brownian motion over H with the branching mechanism $\phi(x, z) \equiv z^2/2$. Suppose that $\eta \in \mathcal{K}(P)$ is given as Lemma 11.1. Then

$$\begin{aligned}&\int_{M(H)} \mathrm{e}^{-\nu(f)} Q_t^{\eta}(\mu, \mathrm{d}\nu)\\ &= \exp\left\{ -\mu(V_t f) - \int_0^t [m(V_s f) + q\partial_0 V_s f]\mathrm{d}s\right\}\end{aligned} \tag{11.3}$$

defines the transition semigroup $(Q_t^{\eta})_{t\geq 0}$ of an immigration diffusion process $Y = (W, \mathcal{G}, \mathcal{G}_t, Y_t, \mathbf{Q}_{\mu}^{\eta})$. Indeed, it was proved in Li and Shiga (1995) that any immigration diffusion process associated with the super Brownian motion has semigroup in the form (11.3).

Theorem 11.2. (Li and Shiga, 1995) *The process* $\{Y_t(\mathrm{d}x) : t > 0\}$ *is* $\mathbf{Q}_\mu^\eta$*-a.s. absolutely continuous relative to the Lebesgue measure on* H *having continuous density* $\{Y_t(x) : t > 0, x > 0\}$ *which satisfies* $Y_t(0^+) \equiv 2q$ *and solves the following stochastic partial differential equation with singular drift term:*

$$\frac{\partial}{\partial t} Y_t(x) = \sqrt{Y_t(x)}\,\dot{W}_t(x) + \frac{1}{2}\Delta Y_t(x) + \dot{m}(x) + q d_0, \tag{11.4}$$

where $\dot{W}_t(x)$ *is a time-space white noise,* Δ *is the Laplacian on* H *with Dirichlet boundary condition,* $\dot{m}(x)$ *is the generalized function given by the measure* $m(\mathrm{d}x)$*, and* $-d_0$ *is the derivative of the Dirac function at the origin. More precisely,* $\langle \dot{m}, f\rangle = m(f)$ *and* $\langle d_0, f\rangle = f'(0^+)$ *for all* $f \in C_{00}^2(\mathbb{R}^+)$*, twice continuously differentiable functions on* $\mathbb{R}^+$ *vanishing at zero and infinity.*

Theorem 11.3. (Li and Shiga, 1995) *Suppose that the closed supports of* $\mu \in M(H)$ *and* $m \in M_h(H)$ *are bounded. Then* $\mathbf{Q}_\mu^\eta$*-a.s.* $\{Y_t(\mathrm{d}x) : t \geq 0\}$ *have bounded closed supports. Let* $R_t = \sup\{x > 0 : x \in \mathrm{supp}(X_s)$ *for some* $0 \leq s \leq t\}$*. Then the distribution of* $t^{-1/3}R_t$ *converges as* $t \to \infty$ *to the Fréchet distribution given by* $F(z) = \mathrm{e}^{-\alpha/z^3}$ $(z > 0)$*, where*

$$\alpha = \frac{1}{18}\left(\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)}\right)^3 \left(q + \int_H x\, m(\mathrm{d}x)\right).$$

Some central limit theorems for the above immigration process were given in Li and Shiga (1995). See also Li et al (1993) and Ye (1993) for related results.

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