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MEASURE-VALUED IMMIGRATION DIFFUSIONS AND GENERALIZED ORNSTEIN-UHLENBECK DIFFUSIONS

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Three different kinds of fluctuation limits (high density fluctuation, small branching fluctuation and large scale fluctuation) of the measure-valued immigration diffusion process are studied, which lead to the generalized Ornstein-Uhlenbeck diffusion defined by a Langevin equation of the type of Holley and Stroock (1978). The fluctuation limit theorems cover all dimension numbers and give physical interpretations to the parameters appearing in the equation.

Key words: Langevin equation; Ornstein-Uhlenbeck diffusion; branching; measure-valued diffusion; immigration; fluctuation limit

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1. Introduction

Let A be a bounded linear operator on $\mathcal{S}(I\!\!R^d)$ which admits a non-positive definite self-adjoint extension on $L^2(I\!\!R^d)$. Assume that A generates a strongly continuous semigroup $(P_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{S}(I\!\!R^d)$. Let B be a bounded linear operator on $L^2(I\!\!R^d)$. From the results of Holley and Stroock (1978) we know that there is a diffusion process $\{Z_t : t \geq 0\}$ in $\mathcal{S}'(I\!\!R^d)$ that solves the Langevin equation:

$$\mathrm{d}Z_t = A^* Z_t \mathrm{d}t + \mathrm{d}W_t, \qquad t \geq 0, \tag{1.1}$$

where $\{W_t : t \geq 0\}$ is a white noise. For any testing function $\phi \in \mathcal{S}(I\!\!R^d)$ the process $\{W_t(\phi) : t \geq 0\}$ is a one-dimensional Brownian motion with quadratic variation $\langle W(\phi)\rangle_t = t\lambda(|B\phi|^2)$, where λ is the Lebesgue measure on $I\!\!R^d$. The process $\{Z_t : t \geq 0\}$ defined by (1.1) is called a *generalized Ornstein-Uhlenbeck diffusion*. A nice interpretation of the generalized Ornstein-Uhlenbeck diffusion was given by Holley and Stroock (1978), who showed that in the particular case where $A = \Delta/2$ and $B = I$ the process may arise as the large scale fluctuation limit of a critical branching Brownian particle system. The limit theorem of Holley and Stroock (1978) covers dimension numbers

$d \geq 3$. The fluctuation limits of interacting particle systems have also been studied by others; see e.g. Bojdecki and Gorostiza (1986, 1991), Chang and Yau (1992), Dawson (1981), Dittrich (1987, 1988), Gorostiza (1988, 1996ab), Walsh (1986) and the references therein. Three different kinds of fluctuation limit theorems for branching particle systems have been proved in the literature, that is, large scale fluctuation, high density fluctuation and small branching fluctuation. All of those have lead to better understanding of the particle systems as well as the Langevin type equations. We would particularly like to mention that Bojdecki and Gorostiza (1986) and Gorostiza (1988) have considered the fluctuations of branching models with immigration which are related to the work of this paper.

The main purpose of this paper is to study the fluctuation limits of some subcritical branching measure-valued immigration diffusion processes formulated by skew convolution semigroups; see Li (1996ab, 1998) and Li and Shiga (1995). An immigration process may be constructed by picking up the paths of a measure-valued branching diffusion by a Poisson random process. The advantage of considering subcritical branching immigration processes is that they are comparatively easier to handle than critical branching processes without immigration, and this makes it possible to carry out the arguments in more general settings. In our context, A is the generator of a diffusion process ξ describing the underlying motion in $\mathbb{R}^d$ of the particles, B is a multiplication operator giving the branching rate, and λ is a general excessive measure for ξ determining the rate of immigration. We shall see that the high density fluctuation and the small branching fluctuation for an immigration diffusion process are essentially equivalent and can be proved for all dimension numbers. The large scale fluctuation in the spatially homogeneous situation is in fact a special form of the high density fluctuation. As in Holley and Stroock (1978), the large scale fluctuation limit theorem is only available for the dimension numbers $d \geq 3$. This restriction is a consequence of the scaling property of the Lebesgue measure, which is evident from our derivation of the limit theorem.

2. Measure-valued branching diffusions

In this section, we catalogue some facts about the measure-valued branching diffusions. Let E be a locally compact metric space. Let $C_0(E)$ denote the set of continuous functions on E that vanish at infinity. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a Markov process with strongly continuous Feller transition semigroup $(P_t)_{t\geq 0}$ on $C_0(E)$. Let A be the strong generator of $(P_t)_{t\geq 0}$ with domain $D(A) \subset C_0(E)$. We choose a strictly positive reference function $\rho \in D(A)$ satisfying $A\rho \in C_\rho(E)$, where $C_\rho(E)$ denote the set of functions $f \in C_0(E)$ satisfying $|f| \leq \mathrm{const} \cdot \rho$. Then there is some $\beta \geq 0$ such that ρ is a β-excessive function for the semigroup $(P_t)_{t\geq 0}$. Let $D_\rho(A)$ be the set of functions $f \in D(A) \cap C_\rho(E)$ with $Af \in C_\rho(E)$. Subsets of non-negative elements of the function spaces are indicated by the superscript '$^+$', e.g. $C_\rho(E)^+$. Throughout the paper we fix a bounded, positive, continuous function $c = c(\cdot)$ on E which is also bounded away from zero.

Let $M_\rho(E)$ be the space of σ-finite measures μ on $(E, \mathcal{B}(E))$ such that $\mu(f) := \int_E f \mathrm{d}\mu < \infty$ for all $f \in C_\rho(E)$. We equip $M_\rho(E)$ with the topology defined by the convention: $\mu_k \to \mu$ if and only if $\mu_k(f) \to \mu(f)$ for all $f \in C_\rho(E)$. Then there is a transition semigroup $(Q_t)_{t\geq 0}$ on $M_\rho(E)$ that is determined by

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_t f)\right\}, \quad f \in C_\rho(E)^+, \mu \in M_\rho(E), \tag{2.1}$$

where $V_t f$ denotes the unique positive solution of the evolution equation

$$V_t f(x) + \frac{1}{2}\int_0^t \mathrm{d}s \int_E c(y) V_s f(y)^2 P_{t-s}(x, \mathrm{d}y) = P_t f(x), \quad t \geq 0, x \in E. \tag{2.2}$$

Any Markov process X with transition semigroup $(Q_t)_{t\geq 0}$ is called a *measure-valued branching diffusion process* (Dawson-Watanabe superprocess) with parameters (A, c). The equation (1.1) defines a semigroup $(V_t)_{t\geq 0}$ of nonlinear operators on $C_\rho(E)^+$, which is called the *cumulant semigroup* of X. Observe that we may rewrite the above equation into the following differential form:

$$\begin{aligned} \frac{\partial}{\partial t} V_t f(x) &= A V_t f(x) - \frac{1}{2} c(x) V_t f(x)^2, \\ V_0 f(x) &= f(x), \qquad t \geq 0, x \in E. \end{aligned} \tag{2.3}$$

For any $f \in D(E)^+$ the two equations (2.2) and (2.3) are equivalent. The measure-valued branching diffusion arises as a high-density limit of branching particle systems. We simply refer the reader to Dawson (1993) and the references therein for the construction and basic properties of the measure-valued branching diffusion.

Lemma 2.1. *For $t \geq 0$ and $f \in C_\rho(E)^+$ let*

$$V_t^{(n)} f = (-1)^{n-1} \frac{\partial^n}{\partial \theta^n} V_t(\theta f)\big|_{\theta=0}, \quad n = 1, 2, \cdots. \tag{2.4}$$

Then we have $V_t^{(1)} f = P_t f$ and

$$V_t^{(n)} f = \sum_{k=1}^{n-1} \binom{n-1}{k} \int_0^t P_{t-s}(c V_s^{(k)} f V_s^{(n-k)} f) \mathrm{d}s, \quad n \geq 2.$$

Proof. These follow immediately by differentiating both sides of the equation (2.2); see Konno and Shiga (1988). $\square$

Let $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ be a realization of the measure-valued branching diffusion process with parameters (A, c). A general expression for the moments of the process is recalled in the following

Lemma 2.2. *Fix $\mu \in M_\rho(E)$. For $t \geq 0$ and $f \in C_\rho(E)$ let*

$$M_t^{(n)}(f) = \mathbf{Q}_\mu \left\{ X_t(f)^n \right\}, \quad n = 1, 2, \cdots . \tag{2.5}$$

Then we have $M_t^{(1)}(f) = \mu(P_t f)$ and

$$M_t^{(n)}(f) = \sum_{k=0}^{n-1} \binom{n-1}{k} \mu(V_t^{(n-k)} f) M_t^{(k)}(f), \quad n \geq 2.$$

In particular,

$$M_t^{(2)}(f) = \mu(P_t f)^2 + \int_0^t \mu P_{t-s}(c(P_s f)^2) \mathrm{d}s.$$

Proof. For $t \geq 0$ and $f \in C_\rho(E)^+$, differentiating in $\theta > 0$ both sides of the relation

$$\mathbf{Q}_\mu \exp\left\{-\theta X_t(f)\right\} = \exp\left\{-\mu(V_t(\theta f))\right\},$$

we get

$$-\mathbf{Q}_\mu X_t(f) \exp\left\{-\theta X_t(f)\right\} = -\frac{\partial}{\partial\theta} \mu(V_t(\theta f)) \mathbf{Q}_\mu \exp\left\{-\theta X_t(f)\right\}.$$

Continuing differentiating in $\theta > 0$ and letting $\theta \downarrow 0$ yields the desired results for $f \in C_\rho(E)^+$, which can be extended to all $f \in C_\rho(E)$ by induction in $n \geq 1$; see Konno and Shiga (1988). $\square$

Then we give some estimates for the moments of the measure-valued branching diffusion, which will be used in the subsequent discussions.

Lemma 2.3. *Let $q > 0$ and let $f \in D_\rho(A)$. Then there exists a constant $C(\rho, q, \|f/\rho\|) > 0$ such that for $0 \leq t \leq q$ and $\mu \in M_\rho(E)$ we have*

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(P_t f)]^2 \right\} \leq C(\rho, q, \|f/\rho\|) \mu(\rho) t,$$

and

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(P_t f)]^4 \right\} \leq C(\rho, q, \|f/\rho\|) \left[\mu(\rho) + \mu(\rho)^2 \right] t^2.$$

Proof. It is well-known that for any $f \in D_\rho(A)$ the process

$$M_t(f) := X_t(f) - \mu(f) - \int_0^t X_s(Af) \mathrm{d}s, \quad t \geq 0, \tag{2.6}$$

is a $\mathbf{Q}_\mu$-martingale starting at zero with quadratic variation process

$$\langle M(f) \rangle_t = \int_0^t \int_E c(x) f(x)^2 X_s(\mathrm{d}x) \mathrm{d}s, \quad t \geq 0. \tag{2.7}$$

These define a continuous martingale measure $\{M_t : t \geq 0\}$ on E with covariant measure $c(x)X_t(\mathrm{d}x)\mathrm{d}t$ in the sense of El Karoui and Méléard (1990) and Walsh (1986). Moreover, for any $f \in C_\rho(E)$ we have a.s.

$$X_t(f) = \mu(P_t f) - \int_E \int_0^t P_{t-s} f(x) M(\mathrm{d}x, \mathrm{d}s), \quad t \geq 0.$$

By a moment inequality for continuous martingales we may find a universal constant $C > 0$ such that

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(P_t f)]^2 \right\} \leq C \mathbf{Q}_\mu \bigg\{ \int_0^t X_s(P_{t-s} f) \mathrm{d}s \bigg\},$$

and

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(P_t f)]^4 \right\} \leq C \mathbf{Q}_\mu \bigg\{ \bigg[\int_0^t X_s(c P_{t-s} f) \mathrm{d}s \bigg]^2 \bigg\}.$$

Then the results follows by Lemma 2.2 and Hölder's inequality. $\square$

Lemma 2.4. *Let $q > 0$ and let $f \in D_\rho(A)$. Then there is a constant $C(\rho, q, \|f/\rho\|) > 0$ such that for $0 \leq t \leq q$ and $\mu \in M_\rho(E)$ we have*

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(f)]^2 \right\} \leq C(\rho, q, \|f/\rho\|)(1 + \|Af/\rho\|^2) \left[\mu(\rho) + \mu(\rho)^2 \right] t,$$

and

$$\mathbf{Q}_\mu \left\{ [X_t(f) - \mu(f)]^4 \right\} \leq C(\rho, q, \|f/\rho\|)(1 + \|Af/\rho\|^4) \left[\mu(\rho) + \mu(\rho)^4 \right] t^2.$$

Proof. Note that we have $\|Af/\rho\| < \infty$ for any $f \in D_\rho(A)$. Clearly,

$$|P_t f(x) - f(x)| \leq \int_0^t P_s |Af|(x) \mathrm{d}s \leq \|Af/\rho\| \int_0^t P_s \rho(x) \mathrm{d}s.$$

Recall that there is some $\beta \geq 0$ such that ρ is a β-excessive function for $(P_t)_{t \geq 0}$. Then the estimates follow from Lemma 2.3 by the elementary c_r-inequality. $\square$

3. Measure-valued immigration diffusions

A class of measure-valued immigration processes can be formulated by skew convolution semigroup associated with measure-valued branching processes; see Li (1996ab, 1998). Let us consider a special form of the immigration process in our present setting. Suppose that $(V_t)_{t \geq 0}$ is given by (2.2). Let $\mathcal{K}_\rho(P)$ denote the totality of all entrance laws for the semigroup $(P_t)_{t \geq 0}$ such that $\int_0^1 \kappa_s(\rho) \mathrm{d}s < \infty$. For any $\kappa \in \mathcal{K}_\rho(P)$ let

$$S_t(\kappa, f) = \kappa_t(f) - \frac{1}{2} \int_0^t \kappa_{t-s}(c(V_s f)^2) \mathrm{d}s, \quad t > 0, f \in C_\rho(E)^+. \tag{3.1}$$

By simple modifications of the results of Li (1996b) and Li and Shiga (1995) we see that the formula

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\bigg\{ -\mu(V_t f) - \int_0^t S_r(\kappa, f)\mathrm{d}r \bigg\} \tag{3.2}$$

defines a transition semigroup $(Q_t^\kappa)_{t\geq 0}$ on $M_\rho(E)$. A Markov process with semigroup $(Q_t^\kappa)_{t\geq 0}$ is called a *measure-valued immigration process* with parameters (A, c, κ). The immigration process can be constructed by picking up measure-valued paths in a Poisson process as follows.

Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of the Markov semigroup $(Q_t)_{t\geq 0}$ to $M_\rho(E)^\circ := M_\rho(E) \setminus \{0\}$. Then

$$\int_{M_\rho(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(f)}\Big) L\kappa_t(\mathrm{d}\nu) = S_t(\kappa, f) \tag{3.3}$$

defines an entrance law $L\kappa = (L\kappa_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$; see Dynkin (1989) or Li and Shiga (1995). Let W_{0+} be the space of continuous paths $w : (0, \infty) \to M_\rho(E)$ such that $w(t) = 0$ for all $t \geq \tau(w) := \inf\{s > 0 : w(s) = 0\}$, furnished with the σ-algebra generated by the coordinate process. By the theory of Markov processes, there is a unique σ-finite measure $\mathbf{Q}_{L\kappa}$ on W_{0+} which satisfies the following form of Markov property: for $0 < t_1 < \cdots < t_n$ and $\nu_1, \cdots, \nu_n \in M_\rho(E)^\circ$ we have

$$\begin{aligned} &\mathbf{Q}_{L\kappa}\{w_{t_1} \in \mathrm{d}\nu_1, w_{t_2} \in \mathrm{d}\nu_2, \cdots, w_{t_n} \in \mathrm{d}\nu_n\} \\ &\quad = L\kappa_{t_1}(\mathrm{d}\nu_1) Q_{t_2-t_1}^\circ(\nu_1, \mathrm{d}\nu_2) \cdots Q_{t_n-t_{n-1}}^\circ(\nu_{n-1}, \mathrm{d}\nu_n). \end{aligned} \tag{3.4}$$

Let $N(\mathrm{d}w, \mathrm{d}s)$ be a Poisson random measure on $W_{0+} \times [0, \infty)$ with intensity $\mathbf{Q}_{L\kappa}(\mathrm{d}w)\mathrm{d}s$. Define a measure-valued process $\{Y_t : t \geq 0\}$ by

$$Y_t = \int_{W_{0+}} \int_0^t w_{t-s} N(\mathrm{d}w, \mathrm{d}s), \quad t \geq 0, \tag{3.5}$$

where $w_t = 0$ for $t \leq 0$ by convention. Using (3.4) and (3.5), it is easy to check that $\{Y_t : t \geq 0\}$ is a Markov process in $M_\rho(E)$ with semigroup $(Q_t^\kappa)_{t\geq 0}$. The construction (3.5) gives the physical interpretation for the measure-valued immigration process. We refer the reader to Li (1998) for description of the particle pictures of measure-valued immigration processes.

Lemma 3.1. *For any $t > 0$ and $f \in C_\rho(E)$ let*

$$S_t^{(n)}(\kappa, f) = (-1)^{n-1} \frac{\partial^n}{\partial \theta^n} S_t(\kappa, \theta f)\big|_{\theta=0}, \quad n = 1, 2, \cdots. \tag{3.6}$$

Then we have $S_t^{(1)}(\kappa, f) = \kappa_t(f)$ and

$$S_t^{(n)}(\kappa, f) = \sum_{k=1}^{n-1} \binom{n-1}{k} \int_0^t \kappa_{t-s}(cV_s^{(k)} f V_s^{(n-k)} f)\mathrm{d}s, \quad n \geq 2,$$

where $V_s^{(k)} f$ is defined by (2.4).

Proof. The equalities can be proved by differentiating both sides of (3.1). $\square$

Theorem 3.2. *For $q > 0$ and $f \in D_\rho(A)$, there is a constant $C(\rho, q, \kappa, \|f/\rho\|) > 0$ such that for all $0 \le r < t \le q$,*

$$\mathbf{E}\left\{\left(Y_t(f) - Y_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s\right)^4\right\} \le C(\rho, q, \kappa, \|f/\rho\|)(1 + \|Af/\rho\|^4)(t-r)^2.$$

Consequently, the process $\{Y_t : t \ge 0\}$ has a continuous modification.

Proof. Replacing f with θf in (3.3), differentiating both sides with respect to $\theta > 0$ and using Lemma 3.1 we get

$$\int_{M_\rho(E)^\circ} \nu(f) L\kappa_t(\mathrm{d}\nu) = S_t^{(1)}(\kappa, f) = \kappa_t(f). \tag{3.7}$$

Then it is easy to see that

$$Y_t(f) - Y_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s = \int_{W_{0+}} \int_0^t [w_{t-s}(f) - w_{r-s}(f)]\tilde{N}(\mathrm{d}w, \mathrm{d}s). \tag{3.8}$$

Define the random signed measure $\tilde{N}(\mathrm{d}w, \mathrm{d}s) := N(\mathrm{d}w, \mathrm{d}s) - \mathbf{Q}_{L\kappa}(\mathrm{d}w)\mathrm{d}s$. By a moment calculation of Poisson random measures we have

$$\begin{aligned}
&\mathbf{E}\left\{\left(Y_t(f) - Y_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s\right)^4\right\} \\
=&3\left[\int_0^t \mathbf{Q}_{L\kappa}\left\{|w_{t-s}(f) - w_{r-s}(f)|^2\right\}\mathrm{d}s\right]^2 \\
&+ \int_0^t \mathbf{Q}_{L\kappa}\left\{|w_{t-s}(f) - w_{r-s}(f)|^4\right\}\mathrm{d}s
\end{aligned} \tag{3.9}$$

Using the Markov property (3.4) we get

$$\begin{aligned}
&\int_0^t \mathbf{Q}_{L\kappa}\left\{|w_{t-s}(f) - w_{r-s}(f)|^2\right\}\mathrm{d}s \\
=&\int_0^r \mathrm{d}s \int_{M_\rho(E)} \mathbf{Q}_\nu\left\{|X_{t-r}(f) - \nu(f)|^2\right\} L\kappa_{r-s}(\mathrm{d}\nu) \\
&+ \int_r^t \mathrm{d}s \int_{M_\rho(E)} \nu(f)^2 L\kappa_{t-s}(\mathrm{d}\nu),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \mathbf{Q}_{L\kappa}\left\{|w_{t-s}(f) - w_{r-s}(f)|^4\right\}\mathrm{d}s \\
=&\int_0^r \mathrm{d}s \int_{M_\rho(E)} \mathbf{Q}_\nu\left\{|X_{t-r}(f) - \nu(f)|^4\right\} L\kappa_{r-s}(\mathrm{d}\nu) \\
&+ \int_r^t \mathrm{d}s \int_{M_\rho(E)} \nu(f)^4 L\kappa_{t-s}(\mathrm{d}\nu).
\end{aligned}$$

Incorporating those into (3.9) and using Lemmas 2.2 and 2.4 and the c_r-inequality we get

$$\begin{aligned}
&\mathbf{E}\left\{\left(Y_t(f)-Y_r(f)-\int_r^t \kappa_s(f)\mathrm{d}s\right)^4\right\}\\
\leq &C_1(\rho,q,\|f/\rho\|)C_2(\rho,q,\kappa)(1+\|Af/\rho\|^4)(t-r)^2 \qquad (3.10)\\
&\quad+6\left[\int_0^{t-r}\mathrm{d}s\int_{M_\rho(E)}\nu(f)^2L\kappa_s(\mathrm{d}\nu)\right]^2+\int_0^{t-r}\mathrm{d}s\int_{M_\rho(E)}\nu(f)^4L\kappa_s(\mathrm{d}\nu),
\end{aligned}$$

for all $0\leq r\leq t\leq q$, where $C(\rho,q,\|f/\rho\|)>0$ is a constant depending on $(\rho,q,\|f/\rho\|)$ and

$$\begin{aligned}
C_2(\rho,q,\kappa)=&\int_0^q\mathrm{d}s\int_{M_\rho(E)}[\nu(\rho)+\nu(\rho)^4]L\kappa_s(\mathrm{d}\nu)\\
&+\left[\int_0^q\mathrm{d}s\int_{M_\rho(E)}[\nu(\rho)+\nu(\rho)^2]L\kappa_s(\mathrm{d}\nu)\right]^2.
\end{aligned}$$

Using (3.3) and Lemmas 2.1 and 3.1 one can compute that

$$\begin{aligned}
&\int_{M_\rho(E)^\circ}\nu(f)^2L\kappa_t(\mathrm{d}\nu)=S_t^{(2)}(\kappa,f)\leq\|cf\|\kappa_t(f)t,\\
&\int_{M_\rho(E)^\circ}\nu(f)^4L\kappa_t(\mathrm{d}\nu)=S_t^{(4)}(\kappa,f)\leq 3\|cf\|^3\kappa_t(f)t^3.
\end{aligned} \qquad (3.11)$$

Combining (3.10) and (3.11) we get the desired estimate. Since there exists a convergence determining sequence $\{f_n\}\subset D_\rho(A)$ for the topology of $M_\rho(E)$, the second assertion is immediate. □

Indeed, it can be proved as in Li and Shiga (1995) that the process $\{Y_t: t\geq 0\}$ constructed by (3.5) is a.s. continuous. By the above theorem, $(Q_t^\kappa)_{t\geq 0}$ is the semigroup of a diffusion process in $M_\rho(E)$.

4. Fluctuation limits of immigration diffusions

Let ξ_θ be a diffusion process in the Euclidean space $\mathbb{R}^d$ with generator A_θ depending on a parameter $\theta>0$. For the sake of concreteness, we assume $A_\theta=L-b_\theta$, where L is a differential operator and $b_\theta\in C(\mathbb{R}^d)^+$ is bounded away from zero for each $\theta>0$. We consider the measure-valued branching diffusion process with parameters $(L-b_\theta,c)$. Then the cumulant semigroup $(V_t^\theta)_{t\geq 0}$ is determined by the equation

$$V_t^\theta f(x)+\frac{1}{2}\int_0^t\mathrm{d}s\int_E c(y)V_s^\theta f(y)^2P_{t-s}^\theta(x,\mathrm{d}y)=P_t^\theta f(x),\quad t\geq 0, x\in\mathbb{R}^d, \qquad (4.1)$$

where $(P_t^\theta)_{t\geq 0}$ is the transition semigroup of ξ_θ. In this special case, we may fix any $p>d$ and take $\rho(x)=(1+|x|^p)^{-1}$. We shall write $M_p(\mathbb{R}^d)$ for $M_\rho(\mathbb{R}^d)$ and write

$C_p(I\!R^d)$ for $C_\rho(I\!R^d)$. Note that the Lebesgue measure λ on $I\!R^d$ is included in $M_\rho(I\!R^d)$. We take an excessive measure $\gamma \in M_p(I\!R^d)$ for the diffusion process with generator L. Then it is a purely excessive measure for the semigroup $(P_t^\theta)_{t\geq 0}$ generated by A_θ, and hence $\gamma = \int_0^\infty \kappa_s^\theta \mathrm{d}s$ for some $\kappa^\theta \in \mathcal{K}_\rho(P^\theta)$. Now (3.1) becomes

$$S_t(\kappa^\theta, f) = \kappa_t^\theta(f) - \frac{1}{2}\int_0^t \kappa_{t-s}^\theta(c(V_s^\theta f)^2)\mathrm{d}s, \quad t > 0, f \in C_p(I\!R^d)^+. \tag{4.2}$$

Let $\{Y_t^\theta : t \geq 0\}$ be the subcritical branching immigration process with parameters $(L - b_\theta, c, \theta\kappa^\theta)$ and with $Y_0 = \theta\gamma$. Let $\mathcal{S}(I\!R^d)$ be the space of infinitely differentiable, rapidly decreasing functions whose all derivatives are also rapidly decreasing. Let $\mathcal{S}'(I\!R^d)$ denote the dual space of $\mathcal{S}(I\!R^d)$. We define the distribution-valued process $\{Z_t^\theta : t \geq 0\}$ by

$$Z_t^\theta(f) = \frac{1}{\sqrt{\theta}}\big[Y_t^\theta(f) - \theta\gamma(f)\big], \quad t \geq 0, f \in \mathcal{S}(I\!R^d). \tag{4.3}$$

Observe that by (4.1) we have

$$\begin{aligned}
&\int_0^\infty \kappa_{r+t}^\theta(\theta^{1/2} f)\mathrm{d}r - \int_0^\infty \kappa_r^\theta(\theta V_t^\theta(f/\sqrt{\theta}))\mathrm{d}r \\
=&\frac{1}{2}\int_0^\infty \mathrm{d}r \int_0^t \kappa_{r+t-s}^\theta(\theta c[V_s^\theta(f/\sqrt{\theta})]^2)\mathrm{d}s \\
=&\frac{1}{2}\int_0^t \mathrm{d}s \int_{t-s}^\infty \kappa_u^\theta(\theta c[V_s^\theta(f/\sqrt{\theta})]^2)\mathrm{d}u.
\end{aligned}$$

On the other hand, by (4.2) it follows that

$$\begin{aligned}
&\int_0^t \kappa_r^\theta(\theta^{1/2} f)\mathrm{d}r - \int_0^t S_r(\theta\kappa^\theta, f/\sqrt{\theta})\mathrm{d}r \\
=&\frac{1}{2}\int_0^t \mathrm{d}r \int_0^r \kappa_{r-s}^\theta(\theta c[V_s^\theta(f/\sqrt{\theta})]^2)\mathrm{d}s \\
=&\frac{1}{2}\int_0^t \mathrm{d}s \int_0^{t-s} \kappa_u^\theta(\theta c[V_s^\theta(f/\sqrt{\theta})]^2)\mathrm{d}u.
\end{aligned}$$

Combining the above two equations we have

$$\begin{aligned}
\gamma(\theta^{1/2} f)-\gamma(\theta V_t^\theta(f/\sqrt{\theta})) &- \int_0^t S_r(\theta\kappa^\theta, f/\sqrt{\theta})\mathrm{d}r \\
&=\frac{1}{2}\int_0^t \gamma(\theta c[V_s^\theta(f/\sqrt{\theta})]^2)\mathrm{d}s.
\end{aligned} \tag{4.4}$$

By Sharpe (1988; p75), it follows that

$$\mathbf{E}\exp\{-Z_t^\theta(f)\} = \exp\bigg\{\frac{1}{2}\int_0^t \gamma(c(V_s^{\{\theta\}}f)^2)\mathrm{d}s\bigg\}, \tag{4.5}$$

and

$$\begin{aligned}&\mathbf{E}\left[\exp\{-Z_{r+t}^\theta(f)\}|Z_s^\theta : 0 \le s \le r\right]\\ &= \exp\bigg\{-Z_r^\theta(V_t^{\{\theta\}}f) + \frac{1}{2}\int_0^t \gamma(c(V_s^{\{\theta\}}f)^2)\mathrm{d}s\bigg\},\end{aligned} \tag{4.6}$$

where $V_t^{\{\theta\}}f = \sqrt{\theta}V_t^\theta(f/\sqrt{\theta})$. Now we prove the following

Theorem 4.1. *Suppose that $b_\theta \to 0$ boundedly as $\theta \to \infty$. Then the process $\{Z_t^\theta : t \ge 0\}$ defined by (4.3) converges weakly in $C([0,\infty), \mathcal{S}'(I\!R^d))$ to a diffusion process $\{Z_t : t \ge 0\}$ with $Z_0 = 0$ and with semigroup $(R_t)_{t\ge 0}$ given by*

$$\int_{\mathcal{S}'(I\!R^d)} \mathrm{e}^{i\nu(f)}R_t(\mu,\mathrm{d}\nu) = \exp\bigg\{i\mu(P_tf) - \frac{1}{2}\int_0^t \gamma(c(P_sf)^2)\mathrm{d}s\bigg\}, \tag{4.7}$$

where $(P_t)_{t\ge 0}$ is the semigroup of linear operators on $C(I\!R^d)$ generated by L.

Proof. Let $\{X_t : t \ge 0\}$ be a measure-valued branching process with parameters $(L - b_\theta, c)$ and with $X_0 = \theta\gamma$. Let $\{Y_t : t \ge 0\}$ be an immigration process parameters $(L - b_\theta, c, \theta\kappa^\theta)$ and with $Y_0 = 0$. Assume that $\{X_t : t \ge 0\}$ and $\{Y_t : t \ge 0\}$ are independent. Then $\{Y_t^\theta : t \ge 0\}$ is equivalent to the process $\{X_t + Y_t : t \ge 0\}$. Observe that for any $f \in \mathcal{S}(I\!R^d)$ we have

$$\mathbf{E}\left\{[Z_t^\theta(f) - Z_r^\theta(f)]^4\right\} = \theta^{-2}\mathbf{E}\left\{[Y_t^\theta(f) - Y_r^\theta(f)]^4\right\}. \tag{4.8}$$

Recall that $\gamma = \int_0^\infty \kappa_s^\theta \mathrm{d}s$. By the c_r-inequality one sees that, up to a constant multiplication, the value in (4.8) is upper bounded by

$$\mathbf{Q}_{\theta\gamma}\left\{[X_t(f) - X_r(f) - \theta\gamma(P_tf - P_rf)]^4\right\} + \mathbf{Q}_0^{\theta\kappa^\theta}\bigg\{\bigg[Y_t(f) - Y_r(f) - \int_r^t \theta\kappa_s^\theta(f)\mathrm{d}s\bigg]^4\bigg\}.$$

Applying Lemma 2.3 and Theorem 3.2 we reach the estimate

$$\mathbf{E}\left\{[Z_t^\theta(f) - Z_r^\theta(f)]^4\right\} \le C(\rho, q, \kappa, \|f/\rho\|)(1 + \|A_\theta f/\rho\|^4)(t-r)^2.$$

By Kolmogorov's criterion, $\{Z_t^\theta(f) : t \ge 0\}$ is a tight family in $C([0,\infty), I\!R)$, hence $\{Z_t^\theta : t \ge 0\}$ is tight in $C([0,\infty), \mathcal{S}'(I\!R^d))$ by the result of Mitoma (1983). Let $\{Z_t : t \ge 0\}$ be a limit point of $\{Z_t^\theta : t \ge 0\}$. For $0 \le t_1 < \cdots < t_n$ and $f_1, \cdots, f_n \in \mathcal{S}(I\!R^d)$ let

$$h_j^{\{\theta\}} = f_j + V_{t_{j+1}-t_j}^{\{\theta\}}(f_{j+1} + \cdots + V_{t_n-t_{n-1}}^{\{\theta\}}f_n).$$

Using (4.6) inductively we get

$$\mathbf{E}\exp\Big\{-\sum_{j=1}^{n} Z^{\theta}_{t_j}(f_j)\Big\} = \exp\Big\{\frac{1}{2}\sum_{j=1}^{n}\int_0^{t_j-t_{j-1}}\gamma(c(V^{\{\theta\}}_s h^{\{\theta\}}_j)^2)\mathrm{d}s\Big\}. \tag{4.9}$$

It is well-known that (4.1) is equivalent to the equation

$$V^{\theta}_t f(x) + \frac{1}{2}\int_0^t \mathrm{d}s\int_E [c(y)V^{\theta}_s f(y)^2 - b_{\theta}(y)V^{\theta}_s f(y)]P_{t-s}(x,\mathrm{d}y) = P_t f(x),\ t\geq 0, x\in E.$$

Based on this it can be proved inductively that

$$h^{\{\theta\}}_j \to h_j := f_j + P_{t_{j+1}-t_j}(f_{j+1}+\cdots+P_{t_n-t_{n-1}}f_n)$$

boundedly as $\theta\to\infty$. Returning to (4.9),

$$\lim_{\theta\to\infty}\mathbf{E}\exp\Big\{-\sum_{j=1}^{n} Z^{\theta}_{t_j}(f_j)\Big\} = \exp\Big\{\frac{1}{2}\sum_{j=2}^{n}\int_0^{t_j-t_{j-1}}\gamma(c(P_s h_j)^2)\mathrm{d}s\Big\}. \tag{4.10}$$

Therefore, $\{Z_t : t\geq 0\}$ is a Markov process in $\mathcal{S}'(\mathbb{R}^d)$ with $Z_0 = 0$ and with transition semigroup $(R_t)_{t\geq 0}$, and the finite-dimensional distributions of $\{Z^{\theta}_t : t\geq 0\}$ converge to those of $\{Z_t : t\geq 0\}$. By the tightness proved above, $\{Z^{\theta}_t : t\geq 0\}$ converges to $\{Z_t : t\geq 0\}$ weakly in $C([0,\infty),\mathcal{S}'(\mathbb{R}^d))$. $\square$

Observe that if $\{Z_t : t\geq 0\}$ is given by the above theorem, then $\{Z_t + \mu P_t : t\geq 0\}$ is a Markov process with transition semigroup $(R_t)_{t\geq 0}$ starting at $\mu\in\mathcal{S}'(\mathbb{R}^d)$. By Bojdecki and Gorostiza (1991; p1139), $\{Z_t : t\geq 0\}$ is a generalized Ornstein-Uhlenbeck diffusion which solves the following Langevin equation:

$$\mathrm{d}Z_t = L^* Z_t\mathrm{d}t + \mathrm{d}W_t, \tag{4.11}$$

where L^* is the adjoint of L and $\{W_t : t\geq 0\}$ is a distribution-valued martingale with independent increments given by

$$\mathbf{E}\exp\{i[W_t(f) - W_r(f)]\} = \exp\big\{-(t-r)\gamma(cf^2)/2\big\}, \quad t\geq r\geq 0. \tag{4.12}$$

Therefore, $\{W_t : t\geq 0\}$ is a white noise (Wiener process) with intensity $c(x)\gamma(\mathrm{d}x)\mathrm{d}t$ in the sense of Walsh (1986).

The generalized Ornstein-Uhlenbeck diffusion process can also be obtained by considering fluctuation limit of the immigration diffusion process with small branching rate. Let $\{Y^{(\theta)}_t : t\geq 0\}$ be an immigration process with parameters $(L - b_{\theta}, \theta c, \kappa^{\theta})$ and with $Y_0 = \gamma$. Obviously, the degenerate case $b_\theta = \theta = 0$ corresponds to the constant deterministic process with value γ. We define the fluctuation process $\{Z_t^{(\theta)} : t \geq 0\}$ by

$$Z_t^{(\theta)} = \frac{1}{\sqrt{\theta}}\big[Y_t^{(\theta)} - \gamma\big], \quad t \geq 0, f \in \mathcal{S}(I\!R^d). \tag{4.13}$$

By a similar argument as for the process $\{Z_t^\theta : t \geq 0\}$ we get

$$\mathbf{E}\,\exp\{-Z_t^{(\theta)}(f)\} = \exp\bigg\{\frac{1}{2}\int_0^t \gamma(c\theta[V_s^{(\theta)}(f/\sqrt{\theta})]^2)\mathrm{d}s\bigg\}, \tag{4.14}$$

and

$$\begin{aligned}&\mathbf{E}\,[\exp\{-Z_{r+t}^{(\theta)}(f)\}|Z_s^{(\theta)} : s \leq r]\\ =&\exp\bigg\{ - Z_r^{(\theta)}(\sqrt{\theta}V_t^{(\theta)}(f/\sqrt{\theta}) + \frac{1}{2}\int_0^t \gamma(c\theta([V_s^{(\theta)}(f/\sqrt{\theta})]^2)\mathrm{d}s\bigg\},\end{aligned} \tag{4.15}$$

where $(V_t^{(\theta)})_{t\geq 0}$ is the unique solution to (2.2) with $c(y)$ replaced by $\theta c(y)$. It is not difficult to check that $\sqrt{\theta}V_t^{(\theta)}(f/\sqrt{\theta})$ and $V_t^\theta(\sqrt{\theta}f)/\sqrt{\theta}$ solve the same equation, so they are equal. Now the proof of Theorem 4.1 implies that, if $b_\theta \to 0$ as $\theta \to 0$, then $\{Z_t^{(\theta)} : t \geq 0\}$ converges as $\theta \to 0$ to the generalized Ornstein-Uhlenbeck process $\{Z_t : t \geq 0\}$ with semigroup $(R_t)_{t\geq 0}$. A similar phenomenon for measure-valued branching diffusions without immigration has been studied by Gorostiza (1996b).

Let us now discuss the large scale fluctuation limit of the measure-valued process as in Holley and Stroock (1978). We consider the measure-valued branching process with parameters $(\Delta/2 - b_\theta, c)$ for constants $b_\theta > 0$ and $c > 0$. Let $(V_t^\theta)_{t\geq 0}$ be the cumulant semigroup defined by

$$\begin{aligned}\frac{\partial}{\partial t}V_t^\theta f(x) &= \frac{1}{2}\Delta V_t^\theta f(x) - b_\theta V_t^\theta f(x) - \frac{1}{2}cV_t^\theta f(x)^2,\\ V_0^\theta f(x) &= f(x), \qquad t \geq 0, x \in I\!R^d,\end{aligned} \tag{4.16}$$

Recall that λ denotes the Lebesgue measure on $I\!R^d$. Then $\kappa_t^\theta := b_\theta \mathrm{e}^{-b_\theta t}\lambda$ defines an entrance law $\kappa^\theta \in \mathcal{K}(P^\theta)$ and $\lambda = \int_0^\infty \kappa_s^\theta \mathrm{d}s$. It is clear that $S_t(\kappa^\theta, f) = b_\theta\lambda(V_t^\theta f)$. Let $\{Y_t^{[\theta]} : t \geq 0\}$ be an immigration process with parameters $(\Delta/2 - b_\theta, c, \kappa^\theta)$ and with $Y_0 = \lambda$. For $f \in C_0(I\!R^d)$ let $k_\theta f(x) = f(x/\theta)/\theta^2$ and let $k_\theta^{-1}f(x) = \theta^2 f(\theta x)$. We define the operators K_θ on $M_p(I\!R^d)$ by $K_\theta\mu(f) = \mu(k_\theta f)$. Then we have $K_\theta\lambda = \theta^{d-2}\lambda$. Let

$$Z_t^{[\theta]}(f) = \frac{1}{\theta^{(d-2)/2}}\big[K_\theta Y_{\theta^2 t}^{[\theta]}(f) - \theta^{d-2}\lambda(f)\big], \quad t \geq 0, f \in \mathcal{S}(I\!R^d). \tag{4.17}$$

Theorem 4.2. *Suppose that $d \geq 3$ and $\theta^2 b_\theta \to 0$ boundedly as $\theta \to \infty$. Then the process $\{Z_t^{[\theta]} : t \geq 0\}$ defined by (4.17) converges weakly in $C([0,\infty), \mathcal{S}'(\mathbb{R}^d))$ to the Ornstein-Uhlenbeck diffusion process $\{Z_t : t \geq 0\}$ with $Z_0 = 0$ and with semigroup $(R_t)_{t\geq 0}$ defined by (4.7), where $(P_t)_{t\geq 0}$ is the semigroup of a standard Brownian motion.*

Proof. By (4.16) it is not hard to check that $u(t,x) := k_\theta^{-1} V_{\theta^2 t}^\theta (k_\theta f)$ is the solution to

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{1}{2}\Delta u(t,x) - \theta^2 b_\theta u(t,x) - \frac{1}{2} c u(t,x)^2, \\ u(0,x) &= f(x), \qquad t \geq 0, x \in \mathbb{R}^d. \end{aligned}$$

By this scaling property, we see that $\{K_\theta Y_{\theta^2 t}^{[\theta]} : t \geq 0\}$ is a immigration process with parameters $(\Delta/2 - \theta^2 b_\theta, c, \theta^{d-2}\eta^\theta)$ and with $Y_0 = \theta^{d-2}\lambda$, where η^θ is an entrance law for the process with generator $\Delta/2 - \theta^2 b_\theta$ and $\int_0^\infty \eta_s^\theta \mathrm{d}s = \lambda$. Then the result follows from Theorem 4.1. $\square$

We conclude this section by remarking that both the high density and the small branching fluctuation limits considered above can be generalized to the case where L is a differential operator in a domain of $\mathbb{R}^d$.

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